Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 93 (1995), p. 1-6
<http://www.numdam.org/item?id=RSMUP_1995__93__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 93 (1995)

On Deficient Products in Infinite Groups (*).

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1. Introduction.

In [2] groups with the deficient squares property were completely characterized. It was shown in [3] that a group G has the deficient squares property if and only if it does not contain an infinite fully-independent subset.

In this paper we investigate infinite groups with deficient products properties. To make this more precise, let G be an infinite group and let $n, k \in \mathbb{N}$, $k \geq 2$. A subset of G with n elements will be called an *n-set* of G. We say that $G \in DP(n, k)$ if all k-tuples $X_1, X_2, \ldots, X_k$ of n-sets in G satisfy

$$UP(X_1, \ldots, X_k) =_{\text{def}} |\cup \{X_i X_j \,|\, 1 \leq i, j \leq k, i \neq j\}| < (k^2 - k) n^2 .$$

In particular, $G \in DP(n)$ stands for $G \in DP(n, 2)$. Finally, we say that $G \in DP$ if $G \in DP(n, k)$ for some positive integers $n, k \in \mathbb{N}$, $k \geq 2$. Our main results are expressed in the following theorems.

THEOREM 1. *Let G be an infinite group and let $n \in \mathbb{N}$. Then $G \in DP(n)$ if and only if G is abelian.*

This theorem follows immediately from

(*) This article was written during the first author's visit to the University of Padova. He would like to thank the Department of Mathematics for the invitation and for their kind hospitality.

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THEOREM 2. *Let G be an infinite non-abelian group. Then G contains two infinite product-independent subsets.*

Two subsets A and B of G are *product-independent* if whenever $a, a' \in A$ and $b, b' \in B$, then $ab \neq b'a'$ and $ab = a'b'$ or $ba = b'a'$ only if $a = a'$ and $b = b'$.

We say that a group G satisfies $G \in FIZ$ if its center is of finite index. With respect to the DP property we prove

THEOREM 3. *Let G be an infinite group. Then $G \in DP$ if and only if $G \in FIZ$.*

Theorem 3 follows easily from

THEOREM 4. *Let G be an infinite group. Then G contains $\aleph_0$ mutually product-independent infinite subsets if and only if $G \notin FIZ$.*

The proofs of Theorems 1, 2, 3 and 4 will be presented in Section 2. It is easy to see that similar proofs yield the following generalizations of Theorems 1 and 3. In order to state these generalizations, we need some additional notation. For each $k \in \mathbb{N}$, $k \geq 2$ we define $((n), k) = (n_1, n_2, \ldots, n_k)$, where $n_1, n_2, \ldots, n_k \in \mathbb{N}$. We say that $G \in DP((n), k)$ if all k-tuples $X_1, X_2, \ldots, X_k$ of subsets of G with $(|X_1|, |X_2|, \ldots, |X_k|) = (n_1, n_2, \ldots, n_k)$ satisfy

$$UP(X_1, X_2, \ldots, X_k) < \sum\{n_i n_j \mid 1 \leq i, j \leq k, i \neq j\}.$$

Finally, we say that $G \in DP^*$ if $G \in DP((n), k)$ for some $((n), k)$, $k \geq 2$. The generalized theorems are:

THEOREM 1$'$. *Let G be an infinite group. Then $G \in DP((n), 2)$ if and only if G is abelian.*

THEOREM 3$'$. *Let G be an infinite group. Then $G \in DP^*$ if and only if $G \in FIZ$.*

Section 3 deals with a related, but different, topic. Following [4] we say that $G \in P_n$ if $XY = YX$ for all n-sets X, Y in G and $G \in P_n^*$ if every infinite set of distinct n-sets of G contains a pair X, Y of distinct members such that $XY = YX$. In [4] it was shown that $G \in P_n$ if and only if G is abelian and in [5] the second author showed that $G \in P_n^*$ for $n > 1$ if and only if G is abelian. Theorems 1, 3, 1$'$ and 3$'$ are generalizations of the first mentioned result and we extend the second result by considering groups $G \in P^*$, which satisfy the property that every infinite set of distinct finite subsets of G of specified sizes not less than two contains a pair of distinct members X, Y such that $XY = YX$. We prove

THEOREM 5. *Let G be an infinite group. Then $G \in P^*$ if and only if G is abelian.*

The following notation and definitions will be used in this paper. The letter G denotes an infinite group and $\mathbb{N}$ denotes the set of positive integers. The letters $i, j, l, m, n, \lambda, \sigma, \rho, \tau$ will denote positive integers. A subset S of G will be called a *Sidon set* if whenever $x, y, z, w \in S$ and $|\{x, y, z, w\}| \geqslant 3$, then $xy \neq zw$.

2. Groups with deficient products.

In this section we shall prove Theorems 1, 2, 3 and 4. Our proofs rely on the following important results of B. H. Neumann and of Babai and Sós.

THEOREM A (B. H. Neumann [6]). *Let G be a group. Then $G \in FIZ$ if and only if G does not contain an infinite subset U satisfying $uv \neq vu$ whenever $u, v \in U$ and $u \neq v$.*

THEOREM B (L. Babai and V. T. Sós [1]). *If U is an infinite subset of a group G, then U contains an infinite Sidon set.*

We proceed with our

LEMMA 1. *Let G be an infinite group and let k denote a positive integer or $\aleph_0$. Then there exist k infinite sequences $A_i = (a_1^i, a_2^i, \ldots)$, $i = 1, 2, \ldots, k$ of elements of G such that*

$$A_\lambda A_\sigma \cap A_\rho A_\tau = \emptyset \quad \text{if } |\{\lambda, \sigma, \rho, \tau\}| \geqslant 3 \tag{2.1}$$

and

$$a_i^\lambda a_j^\sigma = a_k^\lambda a_l^\sigma \quad \text{for } \lambda \neq \sigma \text{ if and only if } i = k \text{ and } j = l. \tag{2.2}$$

If $G \notin FIZ$, we may choose the A_i in such a way, that in addition to the above mentioned properties they will satisfy

$$A_\lambda A_\sigma \cap A_\sigma A_\lambda = \emptyset \quad \textit{for } \lambda \neq \sigma. \tag{2.3}$$

PROOF. Suppose, first, that $G \notin FIZ$. By Theorem A there exists an infinite sequence $B = (g_1, g_2, \ldots)$ of distinct elements of G such that

$g_i g_j \neq g_j g_i$ if $i \neq j$. By Theorem B there exists an infinite subsequence $U = (u_1, u_2, \ldots)$ of B such that $u_i u_j = u_k u_l$ implies that either $i = k$ and $j = l$ (i.e. $u_i u_j = u_i u_j$) or $i = j$ and $k = l$ (i.e. $u_i^2 = u_k^2$). Partition the set U into k disjoint infinite subsequences $A_i = (a_1^i, a_2^i, \ldots)$, $i = 1, 2, \ldots, k$. Then (2.1), (2.2) and (2.3) hold. The proof in the case $G \notin FIZ$ is complete.

If $G \in FIZ$, then $Z(G)$ is of infinite order and by Theorem B $Z(G)$ contains an infinite sequence $U = (u_1, u_2, \ldots)$ of distinct elements such that $u_i u_j = u_k u_l$ implies that either $i = k$ and $j = l$ (i.e. $u_i u_j = u_i u_j$) or $i = j$ and $k = l$ (i.e. $u_i^2 = u_k^2$) or $i = l$ and $j = k$ (i.e. $u_i u_j = u_j u_i$). Construct the sequences $A_i = (a_1^i, a_2^i, \ldots)$ for $i = 1, 2, \ldots, k$ as before. Then (2.1) and (2.2) hold, as claimed. ■

We are ready now to prove Theorem 2.

Proof of Theorem 2. If $G \notin FIZ$, then Theorem 2 follows from Lemma 1, with $k = 2$. So suppose that $G \in FIZ$. Then $Z(G)$ is infinite, G' is finite and $Z(G)/H$ is infinite, where $H = Z(G) \cap G'$. Let T be a transversal of H in $Z(G)$ and let U be an infinite subset of T such that $\{uH \mid u \in U\}$ is a Sidon set in $Z(G)/H$. This means that if $u, v, w, t \in U$ satisfy $uvH = wtH$ then $|\{uH, vH, wH, tH\}| < 3$, hence also $|\{u, v, w, t\}| < 3$. Now split U into two infinite disjoint subsets R and S; fix elements $x, y \in G$ with $xy \neq yx$ and finally set $A = xR$, $B = yS$. If $a = xr$, $a' = xr'$, $b = ys$ and $b' = ys'$, with $r, r' \in R$ and $s, s' \in S$, satisfy $ab = b'a'$, then $xrys = ys'xr'$, which implies $[x, y]\ rs = s'r'$ and $[x, y] \in H$. But then $rsH = s'r'H$ and it follows that $|\{r, s, r', s'\}| < 3$ and $r = r'$, $s = s'$, which implies $[x, y] = 1$, a contradiction. Thus whenever $a, a' \in A$ and $b, b' \in B$, then $ab \neq b'a'$. Suppose, now, that $ab = a'b'$ (or $ba = b'a'$). Then $xrys = xr'ys'$ (or $ysxr = ys'xr'$), which implies $rs = r's'$ (or $sr = s'r'$) and as above $r = r'$, $s = s'$, yielding $a = a'$ and $b = b'$. Hence A and B are infinite product-independent subsets of G and the proof of Theorem 2 is complete. ■

Theorem 1 follows immediately from Theorem 2.

We proceed with a proof of Theorem 4, from which Theorem 3 follows easily.

Proof of Theorem 4. If $G \notin FIZ$, then G contains $\aleph_0$ mutually product-independent infinite subsets by Lemma 1. On the other hand, if $G \in FIZ$, then $|G : Z(G)| = n$ for some $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_{n+1}$ be infinite subsets of G. Then there exist $s \in G$ and $i, j \in \mathbb{N}$ such that $1 \leqslant i, j \leqslant n + 1, i \neq j, A_i \cap sZ(G) \neq \emptyset$ and $A_j \cap sZ(G) \neq \emptyset$. It follows that

A_i and A_j are not product-independent, and in particular G does not contain $\aleph_0$ mutually product-independent infinite subsets. ∎

Finally we prove Theorem 3, in which the DP-groups are characterized.

PROOF OF THEOREM 3. Suppose, first, that $G \in FIZ$ and $|G: Z(G)| = m$. Then $G \in DP(1, m+1)$ since given $m+1$ elements $g_1, g_2, \ldots, g_{m+1}$ of G, at least two of them belong to the same coset of $Z(G)$ in G and therefore commute with each other. This implies that

$$UP(\{g_1\}, \{g_2\}, \ldots, \{g_{m+1}\}) < (m+1)^2 - (m+1)$$

and hence $G \in DP(1, m+1)$, as claimed. But this implies that $G \in DP$, thus completing the proof in one direction.

Suppose, now, that $G \in DP$. Then $G \in DP(n, k)$ for some positive integers n, k with $k \geqslant 2$. It follows immediately by Theorem 4 that $G \in FIZ$. ∎

3. Sequences with equal products.

In this section we shall prove Theorem 5.

PROOF OF THEOREM 5. If G is abelian then clearly $G \in P^*$. So suppose that $G \in P^*$ and G is non-abelian. We shall reach a contradiction from our assumptions. Let $(n_1, n_2, \ldots)$ be an infinite sequence of integers greater or equal to two. Pick $x, y \in G$ such that $xy \neq yx$.

If $G \notin FIZ$ then by Lemma 1 applied to G and $k = \aleph_0$, the subsequences $U_i = (a_1^i, a_2^i, \ldots, a_{n_i}^i)$ of the infinite sequences $A_i = (a_1^i, a_2^i, \ldots)$, $i = 1, 2, \ldots$ satisfy $U_i U_j \cap U_j U_i = \emptyset$ for $i \neq j$ and hence $G \notin P^*$, a contradiction.

So suppose that $G \in FIZ$. Then $Z(G)$ is infinite and by Lemma 1 applied to $Z(G)$ and $k = 3$, there exist three disjoint infinite sequences $(a_1, a_2, \ldots)$, $(b_1, b_2, \ldots)$ and $(t_1, t_2, \ldots)$ of distinct elements of $Z(G)$ such that $a_i b_j \neq a_j b_i$ whenever $i \neq j$. Define the following infinite sequence of sequences:

$$U_1 = (t_1, t_2, \ldots, t_{n_1 - 2}, a_1 x, b_1 y),$$

$$U_2 = (t_{n_1 - 1}, t_{n_1}, \ldots, t_{n_1 + n_2 - 4}, a_2 x, b_2 y),$$

$$U_3 = (t_{n_1 + n_2 - 3}, t_{n_1 + n_2 - 2}, \ldots, t_{n_1 + n_2 + n_3 - 6}, a_3 x, b_3 y)$$

and so on. Suppose that for some $i \neq j$ we have $U_i U_j = U_j U_i$. Since $t_k, a_l, b_m \in Z(G)$ for all $k, l, m \in \mathbb{N}$, x, y, $xy \notin Z(G)$ and $a_i b_j \neq a_j b_i$, we must have $a_i x b_j y = b_j y a_i x$, which implies $xy = yx$, a contradiction. Thus $G \notin P^*$, a final contradiction. ∎

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Manoscritto pervenuto in redazione l'1 aprile 1993.