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## Homoclinic Orbits on Non-Compact Riemannian Manifolds for Second Order Hamiltonian Systems.

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**ABSTRACT** - We consider second order Hamiltonian systems on non-compact Riemannian manifolds. We prove the existence of a nontrivial homoclinic orbit under conditions related to the global superquadratic condition of Rabinowitz in [17].

### 0. Introduction.

In this paper, we study the existence of homoclinic orbits on a complete connected noncompact Riemannian manifold  $M$  of class  $C^3$ . For a given function  $V(t, x) \in C^2(\mathbf{R} \times M, \mathbf{R})$ , we consider the second order Hamiltonian system:

$$(0.1) \quad D_t \dot{x}(t) + \text{grad}_x V(t, x(t)) = 0 \quad \text{in } \mathbf{R}$$

where  $\dot{x}(t)$  denotes the derivative of  $x(t)$  with respect to  $t$ ,  $D_t \dot{x}(t)$  the covariant derivative of  $\dot{x}(t)$  and  $\text{grad}_x V(t, x)$  the gradient of  $V(t, x)$  with respect to the variable  $x$ .

Let  $x_0 \in M$  be a point such that

$$V(t, x_0) = 0, \quad \text{grad}_x V(t, x_0) = 0 \quad \text{for all } t \in \mathbf{R}.$$

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We say a solution  $x(t)$  of (0.1) is a homoclinic orbit emanating from  $x_0$  if and only if

$$(0.2) \quad x(t) \rightarrow x_0, \quad \dot{x}(t) \rightarrow 0 \quad \text{as } t \rightarrow \pm \infty.$$

Our aim is to derive conditions which ensure the existence of a homoclinic orbit emanating from  $x_0$ .

This kind of problems has recently been extensively studied via variational methods. See [1, 2, 5, 7, 11, 16, 17, 18, 22, 24] for homoclinic orbits or heteroclinic orbits in  $\mathbf{R}^N$  and [3, 4, 8, 9, 12] for homoclinic orbits on Riemannian manifolds. See also [6, 10, 19-21, 23] for first order Hamiltonian systems.

To our knowledge, so far the existence of homoclinic orbits on Riemannian manifold has been studied under two types of conditions.

In [8, 9, 12],  $V(t, x)$  is periodic in  $t$  and satisfies

$$\begin{aligned} V(t, x) &\leq 0, \quad \text{for all } t, x, \\ V(t, x) &= 0, \quad \text{if and only if } x = x_0, \end{aligned}$$

while in [3],  $V(t, x)$  verifies the above assumptions but it is time independent.

In [4] (cf. [1, 18]), the potential is time independent and  $x_0 \in M$  is a local maximum of  $V(x)$  such that  $V(x_0) = 0$ . The existence of a homoclinic orbit emanating from  $x_0$  is proved under the assumptions that the set  $\Omega = \{x \in M; V(x) < 0\} \cup \{x_0\}$  is open and bounded and  $\text{grad } V(x) \neq 0$  for all  $x \in \partial\Omega$ .

In this paper, we consider the existence of a homoclinic orbit in the situation where  $V(t, x)$  depends on  $t$  and changes sign on  $M$ . As far as we know, such a situation is studied only in the case  $M = \mathbf{R}^N$  and the following global superquadratic condition is assumed ([7, 17])

(GSQ)  $V(t, x)$  is a periodic function in  $t$  of the form

$$V(t, x) = -\frac{1}{2} (L(t)x, x) + W(t, x) \quad \text{for all } t, x,$$

where  $L(t)$  is a positive definite symmetric matrix depending on  $t$  continuously and  $W(t, x)$  satisfies, for some  $\mu > 2$ ,

$$W(t, 0) \equiv 0,$$

$$0 < \mu W(t, x) \leq (W_x(t, x), x) \quad \text{for all } t, x \neq 0.$$

Our main purpose is to show the existence of a homoclinic orbit on a Riemannian manifold under a condition which is a generalization of (GSQ). We also show the existence of a homoclinic orbit in  $\mathbf{R}^N$  under

weaker condition than (GSQ). See Remark 0.2 and Example 0.4 below.

To state our result, we need some notations: let  $\langle \cdot, \cdot \rangle_x$  be the Riemannian structure of  $M$ . For  $W(x) \in C^2(M, \mathbf{R})$ ,  $\text{grad } W(x)$  and  $H^W(x)[v, v]$  will denote the Riemannian gradient and the Riemannian Hessian of  $W$ , i.e.,

$$\langle \text{grad } W(x), v \rangle_x = \left. \frac{d}{ds} \right|_{s=0} W(\gamma(s)),$$

$$H^W(x)[v, v] = \left. \frac{d^2}{ds^2} \right|_{s=0} W(\gamma(s)) \quad \text{for } v \in T_x M,$$

where  $\gamma(s)$  is the geodesic such that  $\gamma(0) = x$ ,  $\dot{\gamma}(0) = v$ . In case  $W(t, x) \in C^2(\mathbf{R} \times M, \mathbf{R})$  also depends on  $t \in \mathbf{R}$ , we denote by  $\text{grad}_x W(t, x)$  and  $H_x^W(t, x)[v, v]$  the Riemannian gradient and the Riemannian Hessian of  $W$  respect to  $x$ .

We assume

(V0)  $V \in C^2(\mathbf{R} \times M, \mathbf{R})$  is 1-periodic in  $t$ ,

(V1)  $V(t, x_0) = 0$ ,  $\text{grad}_x V(t, x_0)$  for all  $t \in \mathbf{R}$  and

$$H_x^V(t, x_0)[v, v] < 0 \quad \text{for all } v \in T_{x_0} M \setminus \{0\},$$

(V2) the set  $\Omega(t) = \{x \in M; V(t, x) \leq 0\}$  is compact for all  $t$  and  $\text{grad}_x V(t, x) \neq 0$  for all  $x \in \partial\Omega(t)$  and  $t \in \mathbf{R}$ ,

(V3)  $\liminf_{d(x, x_0) \rightarrow \infty} \inf_{t \in \mathbf{R}} V(t, x) > 0$ .

(V4)  $V(t, x)$  is of the form:

$$V(t, x) = -\psi(t, x) + W(t, x),$$

where  $\psi(t, x), W(t, x) \in C^2(\mathbf{R} \times M, \mathbf{R})$  are 1-periodic in  $t$ . Moreover there is a function  $\varphi(x) \in C^2(M, \mathbf{R})$  such that

( $\varphi 1$ )  $\text{grad } \varphi(x_0) = 0$ ,

( $\varphi 2$ ) there exists  $\mu > 0$ ,  $c_0 \in (0, 1/2)$ ,  $c'_0 > 0$  such that for all  $x \in M$  and  $v \in T_x M$ ,

$$\frac{1}{\mu} H^\varphi(x)[v, v] \leq \left( \frac{1}{2} - c_0 \right) \langle v, v \rangle_x,$$

$$|H^\varphi(x)[v, v]| \leq c'_0 \langle v, v \rangle_x,$$

( $\varphi 1$ )  $\psi(t, x) \geq 0$  for all  $t \in \mathbf{R}$  and  $x \in M$ ,

( $\varphi 2$ ) there exists  $c_0'' \in (0, \mu)$  such that

$$\langle \text{grad}_x \psi(t, x), \text{grad} \varphi(x) \rangle_x \leq (\mu - c_0'') \psi(t, x)$$

for all  $t \in \mathbf{R}$  and  $x \in M$ ,

(w)  $\langle \text{grad}_x W(t, x), \text{grad} \varphi(x) \rangle_x \geq \max \{ \mu W(t, x), 0 \}$  for all  $t \in \mathbf{R}$  and  $x \in M$ .

Now we state our main result.

**THEOREM 0.1.** *Let  $M$  be as above and assume (V0)-(V4). Then (0.1)-(0.2) has at least one non-trivial homoclinic orbit emanating from  $x_0$ .*

The following observations clarify the meaning of conditions (V0)-(V4).

**REMARK 0.2.** Let  $M = \mathbf{R}^N$  with a standard Euclidean metric and assume  $V(t, x)$  satisfies condition (GSQ). Setting  $\varphi(x) = 1/2 |x|^2$  and  $\psi(t, x) = 1/2 (L(t)x, x)$ , we see conditions (V0)-(V4) are satisfied. We obtain conditions (V0)-(V4) as a first trial to generalize condition (GSQ) of Rabinowitz [17]. We hope that they may be improved.

**REMARK 0.3.** (i) In the condition (V4), we can take  $\varphi(t, x) \equiv 0$ . In this case,  $V(t, x) = W(t, x)$  and if we take a  $\varphi(x) \in C^2(M, \mathbf{R})$  such that

$$\text{supp grad } \varphi(x) \subset \bigcup_{t \in \mathbf{R}} \{x \in M; \text{dist}(x, M \setminus \Omega(t)) < \delta\} \equiv A$$

for some  $\delta > 0$ , the condition (V4)-(w) is clearly satisfied for  $x \in M \setminus A$  and it can be regarded as a condition on the set  $A$ , that is, a condition on the behavior of  $V(t, x)$  in a neighborhood of  $\Omega(t)^c$ . See Example 0.4 below.

(ii) By the condition (V4)-(w),  $W(t, x)$  cannot take a positive maximum. Thus (V4)-(w) is satisfied only on non-compact manifolds.

EXAMPLE 0.4. Let  $M = \mathbf{R}^N$  with a standard Euclidean metric. Let  $\bar{\varphi}(r) \in C^2([0, \infty), \mathbf{R})$  be a function such that for some  $\delta \in (0, 1/2)$

$$\bar{\varphi}(r) = \begin{cases} \frac{1}{2} r^2, & \text{for } r \geq 1 - \delta/2, \\ \text{constant}, & \text{for } r \leq 1 - \delta, \end{cases}$$

$$\bar{\varphi}'(r) \geq 0.$$

Suppose  $V(t, x) \in C^2(\mathbf{R} \times \mathbf{R}^N, \mathbf{R})$  is 1-periodic in  $t$  and satisfies

$$V(t, x) = \begin{cases} \frac{1}{\mu} (|x|^\mu - 1), & \text{for } |x| \geq 1 - \delta, \\ -\frac{1}{2} |x|^2, & \text{for } |x| \leq \delta, \\ \text{negative}, & \text{for } |x| \in (\delta, 1 - \delta). \end{cases}$$

Then  $V(t, x)$  satisfies (V0)-(V4) for  $\varphi(x) = \bar{\varphi}(|x|)$  and  $\psi(t, x) \equiv 0$  provided that

$$\mu > 2 \max_{x \in \mathbf{R}^N} |\varphi''(x)|.$$

We remark in the set  $\{\delta < |x| < 1 - \delta\}$  conditions (V0)-(V4) are satisfied if  $V(t, x)$  is negative and 1-periodic. In general, (GSQ) is not satisfied in this case.

REMARK 0.5. In the above example, putting a «handle» in the set  $\{\rho < |x| < 1 - \rho\}$ , we see that there exists a couple  $(M, V)$  satisfying conditions (V0)-(V4) with the Riemannian manifold  $M$  not being diffeomorphic to  $\mathbf{R}^N$ .

In the following sections, we prove Theorem 0.1. First we consider the problem on bounded intervals  $[-n, n]$  of  $\mathbf{R}$  and second we take a limit as  $n \rightarrow \infty$ .

## 1. Preliminaries.

Let  $(M, \langle \cdot, \cdot \rangle)$  be a complete, connected finite dimensional Riemannian manifold of class  $C^3$ . By the well-known Nash embedding theorem ([13]),  $M$  can be embedded in  $\mathbf{R}^N$  (with a standard Euclidean metric) for sufficiently large  $N$ . Thus we may assume  $M$  is a submanifold of  $\mathbf{R}^N$  whose Riemannian structure is induced from the standard Euclidean metric on  $\mathbf{R}^N$ . We may also assume  $x_0 \in M$  is corresponding to  $0 \in \mathbf{R}^N$  by the embedding.

In what follows, we denote by  $|\cdot|$  the Euclidean norm, by  $\langle \cdot, \cdot \rangle$  the Euclidean s-calar product, by  $\llbracket - \rrbracket$  the difference in  $\mathbf{R}^N$ , by  $d(\cdot, \cdot)$  the distance on  $M$  indiced by the Riemannian structure, and we also write

$$B(x, r) = \{y \in M; d(x, y) < r\} \quad \text{for all } x \in M \text{ and } r > 0.$$

For a technical reason, we consider the Hamiltonian system (0.1) on bounded intervals first. For  $n \in N$ , we define

$$\begin{aligned} W_0^{1,2}(n) &= W_0^{1,2}([-n, n], \mathbf{R}^N) = \\ &= \{x(t): [-n, n] \rightarrow \mathbf{R}^N; x(t), \dot{x}(t) \in L^2([-n, n], \mathbf{R}^N), x(\pm n) = 0\}, \\ \Omega_n^1 &= \Omega^1([-n, n], M, x_0) = \\ &= \{x(t) \in W_0^{1,2}(n); x(t) \in M \quad \text{for all } t \in [-n, n]\}. \end{aligned}$$

It is well-known (e.g. Palais [15]) that  $\Omega_n^1$  is a Hilbert manifold of class  $C^2$  and its tangent space at  $x \in \Omega_n^1$  is given by

$$T_x \Omega_n^1 = \{v \in W_0^{1,2}(n); v(t) \in T_{x(t)} M \quad \text{for all } t \in \mathbf{R} \text{ and } v(\pm n) = 0\}.$$

We define a scalar product and a norm on  $T_x \Omega_n^1$  by

$$\langle v, v \rangle_n = \int_{-n}^n \langle D_t v, D_t v \rangle dt, \quad \|v\|_n = \langle v, v \rangle_n^{1/2},$$

where  $D_t$  is the covariant derivative along the curve  $x(t) \in \Omega_n^1$ . We also define the distance  $d_{\Omega_n^1}(x, y)$  between  $x(t) \in \Omega_n^1$  and  $y(t) \in \Omega_n^1$  by

$$d_{\Omega_n^1}(x, y) = \inf \left\{ \int_0^1 \left\| \frac{\partial h}{\partial s}(s) \right\|_n ds; h \in C^1([0, 1], \Omega_n^1), h(0) = x, h(1) = y \right\}.$$

We have the following relation between  $d_{\Omega_n^1}$  and  $d$ .

**LEMMA 1.1.** *For any  $x(t), y(t) \in \Omega_n^1$ ,*

$$d(x(t), y(t)) \leq \min \{ |t - n|^{1/2}, |t + n|^{1/2} \} d_{\Omega_n^1}(x, y)$$

*for all  $t \in [-n, n]$ .*

PROOF. For any  $h \in C^1([0, 1], \Omega_n^1)$  with  $h(0) = x$ ,  $h(1) = y$ ,

$$d(x(t), y(t)) \leq \int_0^1 \left| \frac{\partial h}{\partial s}(s)(t) \right| ds \quad \text{for all } t.$$

Since  $h \in C^1([0, 1], \Omega_n^1)$ , we have  $(\partial h / \partial s)(s)(t) \in C([0, 1], W_0^{1,2}(n))$  with  $(\partial h / \partial s)(s)(t) \in T_{h(s)(t)}M$  for all  $t, s$  and  $\partial h / \partial s(s)(\pm n) = 0$ . Now from

$$\begin{aligned} \left| \frac{d}{d\tau} \left| \frac{\partial h}{\partial s}(s)(\tau) \right| \right| &= \\ &= \frac{|\langle D_\tau (\partial h / \partial s)(s)(\tau), \partial h / \partial s(s)(\tau) \rangle|}{|\partial h / \partial s(s)(\tau)|} \leq \left| D_\tau \frac{\partial h}{\partial s}(s)(\tau) \right|, \end{aligned}$$

we have

$$\begin{aligned} \left| \frac{\partial h}{\partial s}(s)(t) \right| &\leq \int_{-n}^t \left| D_\tau \frac{\partial h}{\partial s}(s)(\tau) \right| d\tau \leq \\ &\leq |t + n|^{1/2} \left( \int_{-n}^n \left| D_\tau \frac{\partial h}{\partial s}(s)(\tau) \right|^2 d\tau \right)^{1/2} = |t + n|^{1/2} \left\| \frac{\partial h}{\partial s}(s) \right\|_n. \end{aligned}$$

Therefore

$$d(x(t), y(t)) \leq |t + n|^{1/2} \int_0^1 \left\| \frac{\partial h}{\partial s}(s) \right\|_n ds.$$

Since  $h(s)$  is arbitrary, we get

$$d(x(t), y(t)) \leq |t + n|^{1/2} d_{\Omega_n^1}(x, y).$$

Similarly,

$$d(x(t), y(t)) \leq |t - n|^{1/2} d_{\Omega_n^1}(x, y).$$

Thus we get the conclusion. ■

We consider on  $\Omega_n^1$  the functional

$$I_n(x) = \int_{-n}^n \left[ \frac{1}{2} |\dot{x}(t)|^2 - V(t, x(t)) \right] dt: \Omega_n^1 \rightarrow \mathbf{R}.$$

We can establish the following lemma in a standard way.

**LEMMA 1.2.**  $I_n(x) \in C^2(\Omega_n^1, \mathbf{R})$  and  $x(t) \in \Omega_n^1$  is a critical point of  $I_n(x)$  if and only if  $x(t)$  solves

$$(1.1) \quad D_t \dot{x} + \text{grad}_x V(t, x(t)) = 0, \quad \text{in } (-n, n),$$

$$(1.2) \quad x(-n) = x(n) = x_0 (= 0).$$

Here  $D_t$  is the covariant derivative along the curve  $x(t)$ . ■

We also remark that

$$I'_n(x)[v] = \int_{-n}^n [\langle \dot{x}(t), D_t v(t) \rangle - \langle \text{grad}_x V(t, x), v \rangle] dt$$

for all  $x(t) \in \Omega_n^1$  and  $v(t) \in T_x \Omega_n^1$ .

**REMARK 1.3.** For  $x \in M$ , let  $P(x)(\cdot)$  be the orthogonal projection from  $\mathbf{R}^N$  onto  $T_x M$  and let  $Q(x)v = v - P(x)v$ . Then we can write

$$D_t v(t) = P(x(t)) \dot{v}(t) = \dot{v}(t) - Q(x(t)) \dot{v}(t) \quad \text{for all } v(t) \in T_x \Omega_n^1.$$

Since  $Q(x(t))v(t) = 0$  for  $v(t) \in T_x M$ , we have

$$Q(x(t)) \dot{v}(t) = -dQ(x(t))[\dot{x}(t)]v(t).$$

Thus

$$(1.3) \quad D_t v(t) = \dot{v}(t) + dQ(x(t))[\dot{x}(t)]v(t).$$

## 2. The mountain pass structure of $I_n \in C^1(\Omega_n^1, \mathbf{R})$ .

In this section, we prove that  $I_n \in C^1(\Omega_n^1, \mathbf{R})$  satisfies the assumptions of the Mountain Pass Theorem.

LEMMA 2.1. *There exist  $\delta_0, \rho_0 > 0$  independent of  $n \in N$  such that if  $x(t) \in \Omega_n^1$  satisfies*

$$(2.1) \quad \int_{-n}^n [|\dot{x}(t)|^2 + |x(t)|^2] dt = \delta_0,$$

*then*

$$I_n(x) \geq \rho_0 > 0.$$

*Moreover*

$$I_n(x) \geq 0$$

*for  $x(t) \in \Omega_n^1$  satisfying*

$$\int_{-n}^n [|\dot{x}(t)|^2 + |x(t)|^2] dt \leq \delta_0.$$

PROOF. By the assumption (V1), we can choose  $\delta > 0$  and  $a > 0$  such that

$$-V(t, x) \geq a|x|^2 \quad \text{for all } x \in B(x_0, \delta) = B(0, \delta).$$

Now taking  $\delta_0 > 0$  sufficiently small so that (2.1) implies

$$x(t) \in B(0, \delta) \quad \text{for all } t \in [-n, n],$$

for such a  $\delta_0 > 0$ , clearly it follows

$$I_n(x) \geq \int_{-n}^n \left[ \frac{1}{2} |\dot{x}|^2 + a|x|^2 \right] dt \geq \min \left\{ \frac{1}{2}, a \right\} \delta_0 \equiv \rho_0 > 0.$$

We can deduce the second assertion in a similar way. ■

Next we take a point  $x_\infty \in M$  such that  $V(t, x_\infty) > 0$  for all  $t$  and choose a curve  $q_1(t) \in \Omega_1^1$  such that

$$q_1(0) = x_\infty.$$

(Note that the existence of  $x_\infty$  follows from (V3).)

For  $n \in N$ , we set

$$q_n(t) = \begin{cases} q_1(t - n + 1), & \text{for } t \in [n - 1, n], \\ q_1(t + n - 1), & \text{for } t \in [-n, -n + 1], \\ x_\infty, & \text{for } t \in [-n + 1, n - 1]. \end{cases}$$

Then we can see easily

$$I_n(q_n) = I_1(q_1) - 2(n - 1) \int_0^1 V(t, x_\infty) dt \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

Thus there exists an  $n_0 \in N$  such that

$$(2.2) \quad (i) \quad \int_{-n_0}^{n_0} [|\dot{q}_{n_0}|^2 + |q_{n_0}|^2] dt > \delta_0, \text{ where } \delta_0 > 0 \text{ is defined in Lemma 2.1.}$$

$$(2.3) \quad (ii) \quad I_{n_0}(q_{n_0}) < 0.$$

We fix such an  $n_0 \in N$  and write  $\bar{q}_0(t) = q_{n_0}(t)$ . Setting

$$\bar{q}_0(t) = x_0 (= 0) \quad \text{in } [-n, n] \setminus [-n_0, n_0],$$

we regard

$$\bar{q}_0(t) \in \Omega_n^1 \quad \text{for } n \geq n_0.$$

We define for  $n \geq n_0$

$$\Gamma_n = \{\gamma \in C([0, 1], \Omega_n^1); \gamma(0)(t) = x_0, \gamma(1)(t) = \bar{q}_0(t) \text{ for all } t\},$$

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{s \in [0, 1]} I_n(\gamma(s)).$$

By Lemma 2.1, and formulas (2.2), (2.3), we can see the following

LEMMA 2.2.

$$b_{n_1} \geq b_{n_2} \geq \rho_0 \quad \text{for } n_2 \geq n_1 \geq n_0,$$

where  $\rho_0 > 0$  is given in Lemma 2.1. In particular the limit

$$b_\infty = \lim_{n \rightarrow \infty} b_n \in [\rho_0, \infty)$$

exists.

PROOF. Since we can regard  $\gamma_{n_1} \in \Gamma_{n_1} \subset \Gamma_{n_2}$  for  $n_2 \geq n_1 \geq n_0$ , the conclusion of Lemma 2.2 follows from the definition of  $b_n$ . ■

We can also verify the Palais-Smale compactness condition and we can see  $b_n$  is a critical value of  $I_n(x)$ , that is, there exists a non-trivial solution  $x_n(t)$  of (1.1)-(1.2). One may expect after suitable shifts in time  $\bar{x}_n(t) = x_n(t - t_n)$  ( $t_n \in \{-n+1, \dots, 0, \dots, n-1\}$ ) converges to a homoclinic orbit. However, there is a possibility

$$n - t_n \rightarrow s_0^+ \neq \infty \quad \text{or} \quad -n - t_n \rightarrow s_0^- \neq -\infty$$

and  $\bar{x}_n(t)$  may converge to a non-trivial solution of (0.1) in  $(-\infty, s_0^+)$  with

$$\begin{cases} x(-\infty) = x_0, & \dot{x}(-\infty) = 0, \\ x(s_0) = x_0, \end{cases}$$

or

$$\begin{cases} x(\infty) = x_0, & \dot{x}(\infty) = 0, \\ x(s_0) = x_0. \end{cases}$$

To overcome this problem, we will get a homoclinic orbit as a limit of a special sequence of approximate solutions of (1.1)-(1.2) in the following section. For this we need.

**LEMMA 2.3.** *There exist constants  $\delta_1, C_0 > 0$  independent of  $n \in N$  such that if  $x(t) \in \Omega_n^1$  satisfies for  $\varepsilon \in (0, 1]$*

$$(2.4) \quad (i) \quad x([-n, n]) \subset B(x_0, \delta_1),$$

$$(2.5) \quad (ii) \quad \|I'_n(x)\| \leq \varepsilon, \text{ that is, } |\langle I'_n(x), v \rangle| \leq \varepsilon \|v\|_n \text{ for all } v \in T_x \Omega_n^1.$$

Then

$$(2.6) \quad I_n(x) \leq C_0 \varepsilon^2.$$

**PROOF.** By the assumption (V1), there exist  $\delta_2 > 0, C_1, C_2 > 0$  such that

$$(2.7) \quad \langle -\text{grad}_x V(t, x), P(x)x \rangle \geq C_1 |x|^2,$$

$$(2.8) \quad -V(t, x) \leq C_2 |x|^2$$

for all  $t$  and  $x \in B(0, \delta_2)$ .

For  $x(t) \in \Omega_n^1$ , we define

$$v(t) = P(x(t))x(t) \in T_x \Omega_n^1.$$

Then by (1.3)

$$\begin{aligned} D_t v(t) &= \dot{v}(t) + dQ(x(t))[\dot{x}(t)]v(t) = P(x(t)) \dot{x}(t) + \\ &+ dP(x(t))[\dot{x}(t)]x(t) + dQ(x(t))[\dot{x}(t)]P(x(t))x(t) = \dot{x}(t) + f(x(t)) \dot{x}(t), \end{aligned}$$

where  $f(x): M \rightarrow \mathbf{R}^{N^2}$  is an  $N \times N$  matrix-valued function satisfying

$$\lim_{x \rightarrow 0} |f(x)| = 0.$$

Therefore there exist  $\delta_3 > 0$  and  $C'_1, C'_2 > 0$  such that

$$(2.9) \quad C'_1 |\dot{x}|^2 \leq \langle \dot{x}, D_t(P(x(t))x(t)) \rangle,$$

$$(2.10) \quad |D_t(P(x(t))x(t))|^2 \leq C'_2 |\dot{x}(t)|^2$$

for all  $x(t) \in \Omega_n^1$  with  $x([-n, n]) \in B(0, \delta_3)$ .

Now we set  $\delta_1 = \min\{\delta_2, \delta_3\} > 0$  and we assume  $x(t) \in \Omega_n^1$  satisfies (2.4) and (2.5). By (2.5), we have

$$\langle I'_n(x), P(x)x \rangle \leq \varepsilon \|P(x)x\|_n,$$

that is,

$$\begin{aligned} \int_{-n}^n [\langle \dot{x}, D_t(P(x)x) \rangle - \langle \text{grad}_x V(t, x), P(x)x \rangle] dt &\leq \\ &\leq \varepsilon \left( \int_{-n}^n |D_t(P(x)x)|^2 dt \right)^{1/2}. \end{aligned}$$

Thus using (2.7), (2.9), (2.10), we get

$$\int_{-n}^n |\dot{x}|^2 dt \leq C_3 \varepsilon^2, \quad \int_{-n}^n |x|^2 dt \leq C_4 \varepsilon^2,$$

where  $C_3, C_4 > 0$  are independent of  $n$  and  $\varepsilon$ . Using these bounds, we get by (2.8)

$$I_n(x) = \int_{-n}^n \left[ \frac{1}{2} |\dot{x}|^2 - V(t, x) \right] dt \leq \left( \frac{C_3}{2} + C_2 C_4 \right) \varepsilon^2 \equiv C_0 \varepsilon^2. \quad \blacksquare$$

### 3. Approximate solutions and their estimates.

For  $k \in \mathbf{N}$ , by Lemma 2.2, we can choose  $n_k \geq n_0$  and  $\gamma_k \in \Gamma_{n_k}$  such that

- 1)  $b_\infty \leq b_{n_k} \leq b_\infty + 1/2k$ ,
- 2)  $\max_{s \in [0, 1]} I_{n_k}(\gamma_k) \leq b_{n_k} + 1/2k$ .

For  $\varepsilon_0 \equiv \delta_1/2 > 0$ , where  $\delta_1 > 0$  is given in Lemma 2.3, we set

$$l_k = [\varepsilon_0^2 k],$$

where  $[a]$  denotes the integer part of  $a \in \mathbf{R}$ . We regard  $\gamma_k \in \Gamma_{n_k} \subset \Gamma_{n_k + l_k}$  and by Lemma 2.2 and 1), 2), we can see that

$$\max_{s \in [0, 1]} I_{n_k + l_k}(\gamma_k(s)) \leq b_{n_k + l_k} + \frac{1}{k}.$$

Now we apply the following proposition, which is a consequence of the usual deformation argument.

**PROPOSITION 3.1.** *Suppose  $J(x) \in C^1(\Omega_n^1, \mathbf{R})$  satisfies*

$$J(x_0) = 0, \quad J(x_1) \leq 0$$

*for some  $x_0, x_1 \in \Omega_n^1$  and set*

$$\Gamma = \{\gamma \in C([0, 1], \Omega_n^1); \gamma(0) = x_0, \gamma(1) = x_1\},$$

$$b = \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} J(\gamma(s)).$$

*Assume  $b > 0$  and there exists a  $\gamma \in \Gamma$  such that for some  $\varepsilon > 0$*

$$\max_{s \in [0, 1]} J(\gamma(s)) \leq b + \varepsilon.$$

*Then there exists  $x(t) \in \Omega_n^1$  such that*

- (i)  $J(x) \in [b - \varepsilon, b + \varepsilon]$ ,
- (ii)  $\|J'(x)\| \leq \sqrt{\varepsilon}$ ,
- (iii)  $\min_{s \in [0, 1]} d_{\Omega_n^1}(\gamma(s), x) \leq \sqrt{\varepsilon}$ . ■

We apply the above proposition to the case  $n = n_k + l_k$ .  $J = I_{n_k + l_k}$ ,  $\gamma = \gamma_k$ ,  $\varepsilon = 1/k$  and obtain the existence of  $x_k \in \Omega_{n_k + l_k}^1$  such that

$$(3.1) \quad (i) \quad I_{n_k + l_k}(x_k) \in \left[ b_{n_k + l_k} - \frac{1}{k}, b_{n_k + l_k} + \frac{1}{k} \right],$$

$$(3.2) \quad (ii) \quad \|I_{n_k + l_k} + (x_k)\| \leq \frac{1}{\sqrt{k}}, \text{ i.e.,}$$

$$\left| \int_{-n_k - l_k}^{n_k + l_k} [\langle \dot{x}_k, D_t v \rangle - \langle \text{grad}_x V(t, x_k), v \rangle] dt \right| \leq \frac{1}{\sqrt{k}} \|v\|_{n_k + l_k}$$

for all  $v \in T_{x_k} \Omega_{n_k + l_k}^1$ ,

$$(3.3) \quad (iii) \quad \min_{s \in [0, 1]} d_{\Omega_{n_k + l_k}^1}(\gamma_k(s), x_k) \leq \frac{1}{\sqrt{k}}.$$

We derive some properties of  $x_k(t)$ .

LEMMA 3.2.

$$d(x_k(t), x_0) \leq \varepsilon_0 \quad \text{for all } t \in [-n_k - l_k, n_k + l_k] \setminus [-n_k, n_k].$$

PROOF. Since  $\gamma_s(t)(t) = 0$  for all  $s \in [0, 1]$  and  $t \in [-n_k - l_k, n_k + l_k] \setminus [-n_k, n_k]$ , we can see from (3.3) and Lemma 1.1 that

$$d(x_k(t), x_0) \leq \sqrt{l_k} d_{\Omega_{n_k + l_k}^1}(x_k, \gamma_k(s))$$

for all  $s \in [0, 1]$  and  $t \in [-n_k - l_k, n_k + l_k] \setminus [-n_k, n_k]$ . Thus

$$d(x_k(t), x_0) \leq \varepsilon_0. \quad \blacksquare$$

PROPOSITION 3.3. *There exist constants  $C_1, C_2 > 0$  independent of  $k \in N$  such that*

$$\int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k|^2 dt \leq C_1, \quad \int_{-n_k - l_k}^{n_k + l_k} |V(t, x_k)| dt \leq C_2.$$

PROOF. Here we use the assumption (V4). We write

$$W_+(t, x) = \max\{W(t, x), 0\}, \quad W_-(t, x) = \max\{-W(t, x), 0\}.$$

Then  $W(t, x) = W_+(t, x) - W_-(t, x)$ . By (V4)-(w) we have

$$(3.4) \quad \int_{n_k - l_k}^{n_k + l_k} \mu W_+(t, x_k) dt \leq \int_{-n_k - l_k}^{n_k + l_k} \langle \text{grad}_x W(t, x_k), \text{grad } \varphi(x_k) \rangle dt.$$

Setting  $v = \text{grad } \varphi(x_k) \in T_{x_k} \Omega_{n_k + l_k}^1$  in (3.2), we see

$$\begin{aligned}
 (3.5) \quad & \int_{-n_k - l_k}^{n_k + l_k} \langle \text{grad}_x W(t, x_k), \text{grad } \varphi(x_k) \rangle dt \leq \\
 & \leq \int_{-n_k - l_k}^{n_k + l_k} \langle \dot{x}_k, D_t \text{grad } \varphi(x_k) \rangle dt + \int_{-n_k - l_k}^{n_k + l_k} \langle \text{grad}_x \psi(t, x_k), \text{grad } \varphi(x_k) \rangle dt + \\
 & + \frac{1}{\sqrt{k}} \|\text{grad } \varphi(x_k)\|_{n_k + l_k} = \\
 & = \int_{-n_k - l_k}^{n_k + l_k} H^\varphi(x_k)[\dot{x}_k, \dot{x}_k] dt + \int_{-n_k - l_k}^{n_k + l_k} \langle \text{grad}_x \psi(t, x_k), \text{grad } \varphi(x_k) \rangle dt + \\
 & + \frac{1}{\sqrt{k}} \|\text{grad } \varphi(x_k)\|_{n_k + l_k} .
 \end{aligned}$$

Since  $H^\varphi(x)[v, w] = \langle D_v \text{grad } \varphi(x), w \rangle$  for all  $v, w \in T_x M$  (e.g. Lemma 49 of [14, p.86]),

$$\begin{aligned}
 \|\text{grad } \varphi(x_k)\|_{n_k + l_k}^2 &= \int_{-n_k - l_k}^{n_k + l_k} \langle D_t \text{grad } \varphi(x_k), D_t \text{grad } \varphi(x_k) \rangle dt = \\
 &= \int_{-n_k - l_k}^{n_k + l_k} H^\varphi(x_k)[\dot{x}_k, D_t \text{grad } \varphi(x_k)] dt .
 \end{aligned}$$

Thus from (V4)-(φ2)

$$\begin{aligned}
 \|\text{grad } \varphi(x_k)\|_{n_k + l_k}^2 &\leq c'_0 \int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k| |D_t \text{grad } \varphi(x_k)| dt \leq \\
 &\leq c'_0 \left( \int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k|^2 dt \right)^{1/2} \|\text{grad } \varphi(x_k)\|_{n_k + l_k} ,
 \end{aligned}$$

that is,

$$(3.6) \quad \|\text{grad } \varphi(x_k)\|_{n_k + l_k} \leq \left( \int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k|^2 dt \right)^{1/2}.$$

Therefore, combining (3.4)-(3.6) and (V4)-(ψ2), we have

$$(3.7) \quad \int_{-n_k - l_k}^{n_k + l_k} \mu W_+(t, x_k) dt \leq \\ \leq \int_{-n_k - l_k}^{n_k + l_k} H^\varphi(x_k)[\dot{x}_k, \dot{x}_k] dt + (\mu - c_0'') \int_{-n_k - l_k}^{n_k + l_k} \psi(t, x_k) dt + \\ + \frac{1}{\sqrt{k}} c_0' \left( \int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k|^2 dt \right)^{1/2}.$$

By (3.1) and (3.7), we have for large  $k$

$$b_\infty + 1 \geq I_{n_k + l_k}(x_k) = \int_{-n_k - l_k}^{n_k + l_k} \left[ \frac{1}{2} |\dot{x}_k|^2 + \psi(t, x_k) - W(t, x_k) \right] dt \geq \\ \geq \int_{-n_k - l_k}^{n_k + l_k} \left( \frac{1}{2} |\dot{x}_k|^2 - \frac{1}{\mu} H^\varphi(x_k(t))[\dot{x}_k, \dot{x}_k] \right) dt + \frac{c_0''}{\mu} \int_{-n_k - l_k}^{n_k + l_k} \psi(t, x_k) dt + \\ + \int_{-n_k - l_k}^{n_k + l_k} W_-(t, x_k) dt - \frac{1}{\mu \sqrt{k}} c_0' \left( \int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k|^2 dt \right)^{1/2}.$$

Thus we by (V4)-(φ2)

$$b_\infty + 1 \geq c_0 \int_{-n_k - l_k}^{n_k + l_k} |\dot{x}_k|^2 dt - \frac{1}{\mu \sqrt{k}} c_0' \left( \int_{-n_k - l_k}^{n_k + l_k} dt + |\dot{x}_k|^2 dt \right)^{1/2} + \\ + \frac{c_0''}{\mu} \int_{-n_k - l_k}^{n_k + l_k} \psi(t, x_k) dt + \int_{-n_k - l_k}^{n_k + l_k} W_-(t, x_k) dt.$$

Thus we get from (V4)-(ψ1)

$$\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2 dt, \quad \int_{-n_k-l_k}^{n_k+l_k} \psi(t, x_k) dt, \quad \int_{-n_k-l_k}^{n_k+l_k} W_-(t, x_k) dt \leq C_1,$$

where  $C_1 > 0$  is independent of  $k$ . Thus we also get from (3.7)

$$\int_{-n_k-l_k}^{n_k+l_k} W_+(t, x_k) dt \leq C'_1,$$

for some  $C'_1 > 0$  independent of  $k$ . Therefore

$$\int_{-n_k-l_k}^{n_k+l_k} |V(t, x_k)| dt \leq C_2. \quad \blacksquare$$

LEMMA 3.4. *For large  $k$ ,*

$$x_k([-n_k-l_k, n_k+l_k]) \not\subset B(x_0, 2\varepsilon_0).$$

PROOF. Recall that  $\delta_1 = 2\varepsilon_0$ . If  $x_k([-n_k-l_k, n_k+l_k]) \subset B(x_0, 2\varepsilon_0) = B(x_0, \delta_1)$ , we get from Lemma 2.3 and (3.2) that

$$I_{n_k+l_k}(x_k) \leq \frac{C_0}{k}.$$

But by Lemma 2.2 this contradicts (3.1) for large  $k$ .  $\blacksquare$

In the next section, we take a limit as  $t \rightarrow \infty$  to get a homoclinic solution.

#### 4. Limit processes as $k \rightarrow \infty$ .

By Lemmas 3.2 and 3.4, for large  $k$  we can find a  $s_k \in \{-n_k, \dots, 0, \dots, n_k\}$  such that

$$(4.1) \quad x_k([s_k, s_k+1]) \cap \partial B(0, 2\varepsilon_0) \neq \emptyset.$$

We define  $y_k(t): \mathbf{R} \rightarrow M$  by

$$y_k(t) = \begin{cases} x_k(t + s_k), & \text{for } t \in [-n_k-l_k-s_k, n_k+l_k-s_k], \\ x_0, & \text{for } t \notin [-n_k-l_k-s_k, n_k+l_k-s_k]. \end{cases}$$

We remark

$$(4.2) \quad -n_k - l_k - s_k \leq -l_k \rightarrow -\infty ,$$

$$(4.3) \quad n_k + l_k - s_k \geq l_k \rightarrow \infty .$$

By Proposition 3.3, we have

$$(4.4) \quad \int_{\mathbf{R}} |\dot{y}_k|^2 dt \leq C_1 ,$$

$$(4.5) \quad \int_{\mathbf{R}} |V(t, y_k(t))| dt \leq C_2 ,$$

where  $C_1, C_2 > 0$  are independent of  $k \in \mathbf{N}$ .

By (4.1) and (4.4),  $y_k(t)$  is bounded in  $W_{\text{loc}}^{1,2}(\mathbf{R}, \mathbf{R}^N)$  and we can extract a subsequence—still denote by  $k$ —such that

$$(4.6) \quad y_k(t) \rightarrow y(t)$$

weakly in  $W_{\text{loc}}^{1,2}(\mathbf{R}, \mathbf{R}^N)$  and strongly in  $C_{\text{loc}}(\mathbf{R}, M)$ .

From (4.4) and (4.5), we deduce

$$(4.7) \quad \int_{\mathbf{R}} |\dot{y}|^2 dt \leq \liminf_{k \rightarrow \infty} \int_{\mathbf{R}} |\dot{y}_k|^2 dt \leq C_1 ,$$

$$(4.8) \quad \int_{\mathbf{R}} |V(t, y(t))| dt \leq \liminf_{k \rightarrow \infty} \int_{\mathbf{R}} |V(t, y_k(t))| dt \leq C_2 .$$

Moreover

LEMMA 4.1.

$$y_k(t) \rightarrow y(t) \quad \text{strongly in } W_{\text{loc}}^{1,2}(\mathbf{R}, \mathbf{R}^N) .$$

PROOF. It suffices to prove

$$(4.9) \quad \int_{\mathbf{R}} |\dot{y}_k(t) - \dot{y}(t)|^2 \phi(t) dt \rightarrow 0$$

for any  $\phi(t) \in C_0^\infty(\mathbf{R}, \mathbf{R})$ .

For a given  $\phi(t) \in C_0^\infty(\mathbf{R}, \mathbf{R})$ , we define

$$v_k(t) = P(y_k(t))(y_k(t) - y(t))\phi(t) \in T_{y_k(t)}M$$

and

$$\bar{v}_k(t) = v_k(t - s_k).$$

Since  $\text{supp } \phi(t) \subset (-l_k, l_k)$  for large  $k$ ,  $\text{supp } \bar{v}_k(t) \subset (-n_k - l_k, n_k + l_k)$  and  $\bar{v}_k \in T_{x_k} \Omega_{n_k + l_k}^1$  for large  $k$ . Now using the fact that

$$D_t v_k = P(y_k(t)) \frac{d}{dt} (P(y_k(t))(y_k(t) - y(t)) \phi(t))$$

and  $y_k(t)$  is bounded in  $W_{\text{loc}}^{1,2}(\mathbf{R}, \mathbf{R}^N)$ , we have

$$\|D_t \bar{v}_k\|_{n_k + l_k} = \int_{\mathbf{R}} |D_t v_k(t)|^2 dt \quad \text{is bounded as } k \rightarrow \infty.$$

Therefore by (3.2)

$$\begin{aligned} (4.10) \quad & \int_{\mathbf{R}} [\langle \dot{y}_k, D_t v_k \rangle - \langle \text{grad}_x V(t, y_k), v_k \rangle] dt = \\ & = \int_{-n_k - l_k}^{n_k + l_k} [\langle \dot{x}_k, D_t \bar{v}_k \rangle - \langle \text{grad}_x V(t, x_k), \bar{v}_k \rangle] dt \quad \text{as } k \rightarrow \infty. \end{aligned}$$

By (4.6), we also have

$$\begin{aligned} & \int_{\mathbf{R}} \langle \text{grad}_x V(t, y_k), v_k \rangle dt = \\ & = \int_{\mathbf{R}} \langle \text{grad}_x V(t, y_k), P(y_k), P(y_k)(y_k(t) - y(t)) \phi(t) \rangle dt \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Thus it follows from (4.10)

$$(4.11) \quad \int_{\mathbf{R}} \langle \dot{y}_k, D_t v_k \rangle dt \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

On the other hand,

$$\begin{aligned} D_t v_k &= D_t((1 - Q(y_k))(y_k - y) \phi) = P(y_k) \frac{d}{dt} ((1 - Q(y_k))(y_k - y) \phi) = \\ &= P(y_k)(\dot{y}_k - \dot{y} - Q(y_k)(\dot{y}_k - \dot{y}) - dQ(y_k)[\dot{y}_k](y_k - y)) \phi + \dot{\phi} P(y_k)(y_k - y) = \\ &= (\dot{y}_k - \dot{y} + (1 - P(y_k)) \dot{y} - P(y_k) dQ(y_k)[\dot{y}_k](y_k - y)) \phi + \dot{\phi} P(y_k)(y_k - y). \end{aligned}$$

Here we used the fact  $\dot{y}_k \in T_{y_k}M$ . From (4.6), (4.11) we can deduce

$$\int_R \langle \dot{y}_k, \dot{y}_k - \dot{y} \rangle \phi(t) dt \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Thus we get (4.9). ■

Using the above Lemma, we can show the following

PROPOSITION 4.2.  *$y(t)$  satisfies*

$$D_t \dot{y} + \text{grad}_x V(t, y(t)) = 0 \quad \text{in } \mathbf{R}.$$

PROOF. It suffices to prove

$$\int_R [\langle \dot{y}, D_t v \rangle - \langle \text{grad}_x V(t, y), v \rangle] dt = 0$$

for all  $v \in C_0^\infty(\mathbf{R}, \mathbf{R}^N)$  with  $v(t) \in T_{y(t)}M$ . We set

$$v_k(t) = P(y_k(t))v(t), \quad \bar{v}_k(t) = v_k(t - s_k).$$

By (4.2)-(4.3),  $\bar{v}_k(t) \in T_{x_k} \Omega_{n_k + l_k}^1$  for large  $k$ . We can see  $\int |D_t v_k|^2$  is bounded as  $k \rightarrow \infty$  and as in the proof of Lemma 4.1 we have

$$(4.12) \quad \int_R [\langle \dot{y}_k, D_t v_k \rangle - \langle \text{grad}_x V(t, y_k), v_k \rangle] dt \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Using Lemma 4.1, we have

$$(4.13) \quad \int_R \langle \dot{y}_k, D_t v_k \rangle dt = \int_R \langle \dot{y}_k, \dot{v}_k \rangle dt \rightarrow \int_R \langle \dot{y}, \dot{v} \rangle = \int_R \langle \dot{y}, D_t v \rangle dt,$$

$$(4.14) \quad \int_R \langle \text{grad}_x V(t, y_k), v_k \rangle dt \rightarrow \int_R \langle \text{grad}_x V(t, y), v \rangle dt,$$

as  $k \rightarrow \infty$ . Combining (4.12)-(4.14), we get the desired result. ■

END OF THE PROOF OF THEOREM 0.1. By (4.1), (4.6), we observe

$$y([0, 1]) \cap \partial B(x_0, 2\varepsilon_0) \neq \emptyset.$$

In particular,  $y(t) \neq x_0$ .

To complete the proof of Theorem 0.1, we need to show

$$y(t) \rightarrow x_0 = 0, \quad \dot{y}(t) \rightarrow 0,$$

as  $t \rightarrow \pm \infty$ . We deal only with the case «+». (The case «-» can be treated in a similar way).

First, we remark  $y(t)$  is bounded on  $\mathbf{R}$ . Indeed, by (4.7),  $y(t)$  is uniformly continuous on  $\mathbf{R}$  and using (V3) and (4.8), our claim follows.

From the equation (0.1), we have for any  $h > 0$

$$\begin{aligned} \max_{s \in [0, h]} |\dot{y}(t+s) - \dot{y}(t)| &\leq \int_t^{t+h} |D_\tau \dot{y}(\tau)| d\tau \leq \\ &\leq \int_t^{t+h} |\text{grad}_x V(\tau, y(\tau))| d\tau \leq mh, \end{aligned}$$

where  $m = \max_{\tau \in \mathbf{R}} |\text{grad}_x V(\tau, y(\tau))|$ .

Thus, if there is a sequence  $t_j \rightarrow \infty$  such that  $|\dot{y}(t_j)| \not\rightarrow 0$ . Setting

$$\delta = \min \left\{ \limsup_{j \rightarrow \infty} |\dot{y}(t_j)|, 1 \right\} > 0, \quad h = \frac{\delta}{2m},$$

we see

$$|\dot{y}(t_j + s)| \geq \frac{\delta}{3} > 0 \quad \text{for all } s \in [0, h]$$

for large  $j \in \mathbf{N}$ . Therefore we have

$$\int_{\mathbf{R}} |\dot{y}(t)|^2 dt \geq \sum_{j=1}^{\infty} \int_{t_j}^{t_j+h} |\dot{y}(t)|^2 dt = \infty.$$

This contradicts (4.7). Thus

$$\dot{y}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Next we set

$Z = \{(t, x) \in [0, 1] \times \partial; \text{ there exists some } (t_j)_{j=0}^\infty \text{ such that } t_j \rightarrow \infty,$

$$\min_{i \in \mathbb{Z}} |t - t_j - i| \rightarrow 0, y(t_j) \rightarrow x \text{ as } j \rightarrow \infty \}.$$

Using the uniform continuity of  $y(t)$  on  $\mathbf{R}$  again, we can see that

$$(4.15) \quad Z \subset \{(t, x); V(t, x) = 0\}.$$

Let us show  $Z \subset [0, 1] \times \{x_0\}$ . For any  $(s, x) \in Z$  there exists  $(t_j)_{j=1}^\infty$  such that

$$t_j \rightarrow \infty, \quad \min_{i \in \mathbb{Z}} |s - t_j - i| \rightarrow 0, \quad y(t_j) \rightarrow x \quad \text{as } j \rightarrow \infty.$$

Then by the continuous dependence of solutions on data, we have

$$y(t + t_j) \rightarrow z(t) \quad \text{in } C_{\text{loc}}^2(\mathbf{R}, \mathbf{R}^N),$$

where  $z(t)$  solves

$$D_t \dot{z} + \text{grad}_x V(t, z) = 0, \quad z(s) = x, \quad \dot{z}(s) = 0.$$

If  $x \neq x_0$ , using (4.15) and (V2), we can see  $z(t) \neq x$  and we have

$$\int_{\mathbf{R}} |\dot{y}|^2 dt \geq \sum_{j=1}^{\infty} \int_{t_j}^{t_j + \delta} |\dot{y}(t)|^2 dt = \infty,$$

since

$$\int_{t_j}^{t_j + \delta} |\dot{y}(t)|^2 dt \rightarrow \int_s^{s + \delta} |\dot{z}|^2 dt > 0 \quad \text{as } j \rightarrow \infty.$$

This contradicts (4.7). Therefore we get  $y(t) \rightarrow x_0$  as  $t \rightarrow \infty$ . ■

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## REFERENCES

- [1] A. AMBROSETTI - M. L. BERTOTTI, *Homoclinics for second order conservative systems*, in *Partial Differential Equations and Related Subjects* (ed. M. MIRANDA), Pitman Research Note in Math. Ser. (1992).
- [2] A. AMBROSETTI - V. COTI ZELATI, *Multiple homoclinic orbits for a class of conservative systems*, *Rend. Sem. Mat. Univ. Padova*, **89** (1993), pp. 177-194. See also *Multiplicité des orbites homoclines pour des Systèmes conservatifs*, *C. R. Acad. Sci. Paris*, **314** (1992), pp. 601-604.
- [3] V. BENCI - F. GIANNONI, *Homoclinic orbits on compact manifolds*, *J. Math. Anal. Appl.*, **157** (1991), pp. 568-576.
- [4] M. L. BERTOTTI, *Homoclinics for Lagrangian systems on Riemannian manifolds*, *Dyn. Sys. Appl.*, **1** (1992), pp. 341-368.
- [5] P. CALDIROLI, *Existence and multiplicity of homoclinic orbits for singular potentials on unbounded domains*, *Proc. Roy. Soc. Edinburgh* (to appear).
- [6] V. COTI ZELATI - I. EKELAND - E. SÉRÉ, *A variational approach to homoclinic orbits in Hamiltonian systems*, *Math. Ann.*, **288** (1990), pp. 133-160.
- [7] V. COTI ZELATI - P. H. RABINOWITZ, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, *J. Amer. Math. Soc.*, **4** (1991), pp. 693-727.
- [8] F. GIANNONI, *On the existence of homoclinic orbits on Riemannian manifolds*, *Ergodic Theo. Dyn. Sys.*, **14** (1994), pp. 103-127.
- [9] F. GIANNONI - P. H. RABINOWITZ, *On the multiplicity of homoclinic orbits on Riemannian manifolds for a class of second order Hamiltonian system*, *Nonlinear Diff. Eq. Appl.*, **1** (1994), pp. 1-46.
- [10] H. HOFER - K. WYSOCKI, *First order elliptic systems and the existence of homoclinic orbits in Hamiltonian system*, *Math. Ann.*, **288** (1990), pp. 483-503.
- [11] L. JEANJEAN, *Existence of connecting orbits in a potential well*, *Dyn. Sys. Appl.* (to appear).
- [12] V. KOZLOV, *Calculus of variations in the large and classical mechanics*, *Russ. Math. Surv.*, **40** (1985), pp. 37-71.
- [13] J. NASH, *The embedding problem for Riemannian manifolds*, *Ann. Math.*, **63** (1956), pp. 20-63.
- [14] B. O'NEILL, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press. (1983).
- [15] R. S. PALAIS, *Morse theory on Hilbert manifolds*, *Topology*, **2** (1963), pp. 299-340.
- [16] P. H. RABINOWITZ, *Periodic and heteroclinic orbits for a periodic Hamiltonian system*, *Ann. Inst. H. Poincaré: Analyse Non Linéaire*, **6** (1989), pp. 331-346.
- [17] P. H. RABINOWITZ, *Homoclinic orbits for a class of Hamiltonian systems*, *Proc. Roy. Soc. Edinburgh*, **114** (1990), pp. 33-38.
- [18] P. H. RABINOWITZ - K. TANAKA, *Some results on connecting orbits for a class of Hamiltonian system*, *Math. Zeit.*, **206** (1991), pp. 473-499.

- [19] E. SÉRÉ, *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Zeit., **209** (1992), pp. 27-42.
- [20] E. SÉRÉ, *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré: Analyse Non Linéaire (to appear).
- [21] E. SÉRÉ, *Homoclinic orbits in compact hypersurface in  $\mathbf{R}^{2N}$  of restricted contact type*, preprint.
- [22] K. TANAKA, *Homoclinic orbits for a singular second order Hamiltonian system*, Ann. Inst. H. Poincaré: Analyse Non Linéaire, **7** (1990), pp. 427-438.
- [23] K. TANAKA, *Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonic orbits*, J. Diff. Eq., **94** (1991), pp. 315-339.
- [24] K. TANAKA, *A note on the existence of multiple homoclinic orbits for a perturbed radial potential*, Nonlinear Diff. Eq. Appl. (to appear).

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