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A. B. ANTONEVICH
J. APPELL
V. A. PROKHOROV
P. P. ZABREJKO

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# Quasi-Iteration Methods of Chebyshev Type for the Approximate Solution of Operator Equations.

A. B. ANTONEVICH - J. APPELL - V. A. PROKHOROV - P. P. ZABREJKO (*)

SUMMARY - We describe a general method for the approximate solution of the operator equation $Ax = f$, where $A$ is a bounded linear invertible operator in a Banach space $X$. The method builds on the construction of a sequence of polynomials of $A$ which approximates the operator $A^{-1}$ in the norm of $\mathscr{L}(X)$. We show that such a construction is possible if and only if $0 \in \Lambda_\infty(A)$, where $\Lambda_\infty(A)$ denotes the connected component of the resolvent set of $A$ containing $\infty$. Moreover, we estimate the rate of convergence in terms of the Green's function $g(z, \infty)$ of $\Lambda_\infty(A)$. Finally, we discuss an effective algorithm for constructing a sequence of polynomials of $A$ which tends to $A^{-1}$ with sharp Ljapunov exponent $\exp[-g(0, \infty)]$.

SUNTO - Descriviamo un metodo generale per la soluzione approssimata dell'equazione $Ax = f$, dove $A$ è un operatore lineare limitato invertibile in uno spazio di Banach $X$. Il metodo è basato sulla costruzione di una successione di polinomi in $A$ che tende all'operatore $A^{-1}$ nella norma di $\mathscr{L}(X)$. Dimostriamo che questa costruzione è possibile se e solo se $0 \in \Lambda_\infty(A)$, dove $\Lambda_\infty(A)$ è la componente connessa illimitata del risolvente di $A$. Inoltre, diamo delle stime per la velocità di convergenza che utilizzano la funzione di Green $g(z, \infty)$ di $\Lambda_\infty(A)$. Infine, discutiamo un algoritmo effettivo per costruire una successione di polinomi in $A$ che tende a $A^{-1}$ con esponente di Ljapunov preciso $\exp[-g(0, \infty)]$.

(*) Indirizzo degli AA.: J. APPELL: Math. Institut, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germania; A. B. ANTONEVICH, V. A. PROKHOROV, P. P. ZABREJKO: Belgosuniversitet, Mat. Fakultet, Pl. Nezavisimosti, BR-220050 Minsk, Belorussia.

Let $A$ be a bounded linear operator with bounded inverse in some Banach space $X$ and $f \in X$. One of the most commonly used methods for obtaining approximate solutions to the operator equation

$$Ax = f \tag{1}$$

is the iteration scheme

$$x_{j+1} = Bx_j + g \qquad (j = 0, 1, 2, \dots), \tag{2}$$

where $g \in X$ depends on $A$ and $f$, and $B$ is an operator constructed from $A$ in such a way that equation (1) and the equation

$$x = Bx + g \tag{3}$$

are equivalent, and the iterations (2) converge to the common solution of (1) and (3) as fast as possible.

Usually, a good choice for the operator $B$ is some polynomial $p(A)$ of $A$. It is not hard to see (see e.g. [1, 2, 10]) that in this case the equations (1) and (3) are equivalent if and only if $p(0) = 1$ and $p(\lambda) \neq 1$ on the spectrum $\operatorname{sp} A$ of $A$. Further, in the basic cases [6, 7] the iteration sequence (2) converges if and only if

$$\max_{\lambda \in \operatorname{sp} A} |p(\lambda)| < 1; \tag{4}$$

moreover, the smaller the left-hand side of (4), the faster the convergence of (2). Consequently, the natural problem arises to construct (precisely or at least approximately) polynomials $p$ whose *Chebyshev norm* on $\operatorname{sp} A$, i.e.

$$\|p\|_{\operatorname{sp} A} = \max_{\lambda \in \operatorname{sp} A} |p(\lambda)| \tag{5}$$

is minimal among all polynomials of degree $\leqslant n$. Iteration methods involving such polynomials are usually called *Chebyshev iteration methods* for the solution of the operator equation (1). Some results and problems on Chebyshev iteration methods may be found in [6, 8-13].

In the situation described above, the iterations (2) may always be written in the form

$$x_j = w_j(A) f \qquad (j = 0, 1, 2, \dots) \tag{6}$$

with some polynomial $w_j$. Therefore, the Chebyshev iteration method may be considered as a special case of the following more general problem: *given the operator $A \colon X \to X$ and the element $f \in X$ as above, find polynomials $w_j$ such that $w_j(A) f$ approximates the exact solution*

$x = A^{-1}f$ *of* (1), i.e. the operators $w_j(A)$ approximate (in a sense to be made precise) the operator $\psi(A) = A^{-1}$.

The simplest way is here, of course, to approximate $A^{-1}$ in the operator norm, i.e. in $\mathscr{L}(X)$. Recall (see e.g. [14]) that, for every function $\phi$, the estimate

$$\|\phi\|_{\mathrm{sp}A} \leqslant \|\phi(A)\| \tag{7}$$

holds, and one has equality in (7) in the special case when $A$ is a self-adjoint or, more generally, a normal operator in Hilbert space. Consequently, the problem of finding polynomial approximations to $A^{-1}$ reduces to (or is at least closely related to) the classical problem of approximating continuous functions by polynomials.

The iterations (6) are usually obtained by computing first the elements

$$f, Af, A^2f, \ldots, A^nf, \ldots \tag{8}$$

and then putting

$$x_n = \sum_{j=0}^{n} c_j^n A^j f \qquad (n = 1, 2, \ldots), \tag{9}$$

where the coefficients $c_0^n, c_1^n, \ldots, c_n^n$ $(n = 1, 2, \ldots)$ coincide with the corresponding coefficients of the polynomial $w_n$. In this way, our problem splits into two steps: an iteration procedure (calculate the elements (8)), and a non-iteration procedure (calculate the coefficients of the linear combinations (9)). The combination of these two procedures will be called *Chebyshev quasi-iteration method* in what follows.

The purpose of this paper is to give basic convergence results for the Chebyshev quasi-iteration method, as well as some error estimates. We point out that our quasi-iteration method is completely different from what is called *semi-iteration method* by some authors (see e.g. the papers [3,4] and the survey article [5]). In fact, in those papers the authors consider iteration methods of polynomial type to transform equation (1) into another equation. Such transforms may give interesting results already in the finite dimensional case $X = \mathbb{R}^N$. On the other hand, our results which are given below do not make sense in finite dimensional spaces $X$, since the sequences which characterize, say, the convergence rate are all eventually zero in $X = \mathbb{R}^N$.

## 1. – Convergence conditions.

The first question which comes into mind is, of course, the following: *under what condition is it possible to approximate the operator* $A^{-1}$ *by a polynomial* $p(A)$ *in the norm of* $\mathscr{L}(X)$*?* The answer is surprisingly simple:

THEOREM 1. *Let $A$ be a bounded invertible operator in a Banach space $X$. Then the operator $A^{-1}$ is the limit (in $\mathscr{L}(X)$) of polynomials of $A$ if and only if $0 \in \Lambda_\infty(A)$, where $\Lambda_\infty(A)$ denotes the connected component of the resolvent set of $A$ containing $\infty$.*

PROOF. By Runge's theorem (see e.g. [18]), the fact that $0 \in \Lambda_\infty(A)$ is equivalent to the fact that the function $\psi$ defined by

$$\psi(\lambda) = \frac{1}{\lambda} \tag{10}$$

may be approximated uniformly on $\operatorname{sp} A$ by polynomial functions.

Suppose first that the operator $A^{-1}$ may be approximated by polynomials of $A$. Since, by (7),

$$\|\psi - p\|_{\operatorname{sp} A} \leq \|A^{-1} - p(A)\|$$

for any polynomial $p$, we may then approximate the function (10) *a fortiori* by polynomial functions on $\operatorname{sp} A$.

The proof of the converse implication is somewhat less trivial. Suppose that $0 \in \Lambda_\infty(A)$. We denote by $\mathscr{C}(A)$ the closure of the subalgebra of $\mathscr{L}(X)$ consisting of all analytic functions of $A$, and by $\mathscr{P}(A)(\subset \mathscr{C}(A))$ the closure of the algebra of all polynomials of $A$. It is clear that $A^{-1} \in \mathscr{C}(A)$; we have to show that $A^{-1} \in \mathscr{P}(A)$ or, equivalently, that $0 \notin \operatorname{sp}_0 A$, where $\operatorname{sp}_0 A$ denotes the spectrum of $A$ in the algebra $\mathscr{P}(A)$.

As is well-known [14], the spectra $\operatorname{sp} A$ and $\operatorname{sp}_0 A$ are related by the relations

$$\operatorname{sp} A \subseteq \operatorname{sp}_0 A, \qquad \partial(\operatorname{sp}_0 A) \subseteq \partial(\operatorname{sp} A).$$

This means, in particular, that the spectrum $\operatorname{sp}_0 A$ may be constructed from the spectrum $\operatorname{sp} A$ by adding certain components of the resolvent set of $A$. But $\Lambda_\infty(A)$ cannot be among these components, because the spectrum of a bounded operator is always a bounded subset of the complex plane. This shows that $0 \notin \operatorname{sp}_0 A$ as claimed.

Unfortunately, Theorem 1 gives no information about how to construct a sequence $(p_n)_n$ of polynomials such that $p_n(A)$ converges to

$A^{-1}$ in $\mathscr{L}(X)$. However, the Riesz formula

(11)
$$\phi(A) = \frac{1}{2\pi i} \int_{\Gamma} \phi(\lambda) R(\lambda, A)\, d\lambda$$

(with $R(\lambda, A) = (\lambda I - A)^{-1}$ denoting the resolvent of $A$ and $\Gamma$ denoting any closed positively oriented contour around $\operatorname{sp} A$) implies the important estimate

(12)
$$\|A^{-1} - p(A)\| \leqslant c\|\psi - p\|_{\Gamma},$$

where $c = c(\Gamma, A)$ is a constant which depends only on $\Gamma$ and $A$ and may be estimated by

(13)
$$c = c(\Gamma, A) \leqslant \frac{1}{2\pi} \int_{\Gamma} \|R(\lambda, A)\|\, |d\lambda|.$$

Combining the estimates (7) and (12), we arrive at the following very simple, though useful

THEOREM 2. *Let $A$ be a bounded invertible operator in a Banach space $X$. Suppose that $0 \in \Lambda_\infty(A)$, let $\psi$ be defined by* (10), *and let $(p_n)_n$ be a sequence of polynomial functions. Then the condition*

(14)
$$\lim_{n \to \infty} \|\psi - p_n\|_{\operatorname{sp} A} = 0$$

*is necessary, and the condition*

(15)
$$\lim_{n \to \infty} \|\psi - p_n\|_{\mathcal{O}(\operatorname{sp} A)} = 0$$

*is sufficient for $p_n(A) \to A^{-1}$, i.e.*

(16)
$$\lim_{n \to \infty} \|A^{-1} - p_n(A)\| = 0;$$

*here $\mathcal{O}(\operatorname{sp} A)$ denotes an arbitrary neighbourhood of* $\operatorname{sp} A$. *Moreover, if $A$ is a self-adjoint or, more generally, normal operator in Hilbert space, all three conditions* (14), (15) *and* (16) *are equivalent.*

Theorem 2 reduces the problem of finding approximate solutions to the operator equation (1) to that of approximating the rational function (10) on certain subsets $\mathfrak{M} \supseteq \operatorname{sp} A$ of the complex plane by polynomial functions. Such subsets are often called *localizations* of the spectrum of $A$; there are useful in many cases where it is difficult or even impossible to know the spectrum $\operatorname{sp} A$ explicitly. (Localizations of some typical spectra will be considered in the last section.) Theorem 2 shows that *any statement on the approximability of the function* (10) *leads to a statement on the approximate solvability of the operator equation* (1) *by iterations of the form* (6).

## 2. – The rate of convergence.

Theorem 2 shows that there is a close relation between the rate of convergence of $p_n(A)$ to $A^{-1}$ and the rate of convergence of $p_n(\lambda)$ to $\lambda^{-1}$. It is therefore natural to look for polynomials $p_n$ for which the convergence of $p_n(\lambda)$ to $\lambda^{-1}$ is as fast as possible. Moreover, one should try to choose $p_n$ as polynomial with the fastest convergence among all polynomials of degree $\leq n$ on $\mathfrak{M}$. We point out, however, that the explicit construction of such polynomials is extremely difficult even for quite simple subsets $\mathfrak{M} \subset \mathbb{C}$ (see e.g. [12, 13, 18]). In practice it is therefore necessary to construct such polynomials approximately by means of, for example, various interpolation formulas.

Let $\mathfrak{M}$ be a compact subset of the complex plane with the property that $0 \notin \mathfrak{M}$ but $0 \in \Lambda_\infty(\mathfrak{M})$, where $\Lambda_\infty(\mathfrak{M})$ denotes the unbounded connected component of $\mathbb{C} \backslash \mathfrak{M}$. In what follows, we shall make extensive use of the classical and generalized Green's function $g_{\mathfrak{M}}(., \infty)$ of $\Lambda_\infty(\mathfrak{M})$ with singularity $\log|z|$ at infinity. Recall (see e.g. [11, 16, 17]) that the *classical Green's function* $g_{\mathfrak{M}}(., \infty)$ is defined for a compact set $\mathfrak{M}$ whose boundary consists of a finite number of closed Jordan curves, is a harmonic function on $\Lambda_\infty(\mathfrak{M})$, may be continuously extended to 0 on $\partial\Lambda_\infty(\mathfrak{M})$, and has a logarithmic singularity at infinity (in the sense that $g_{\mathfrak{M}}(z, \infty) - \log|z|$ is bounded for $z$ near $\infty$). The *generalized Green's function* is defined for arbitrary compact sets $\mathfrak{M}$ by putting

$$g_{\mathfrak{M}}(z, \infty) = \lim_{n \to \infty} g_{\mathfrak{M}(n)}(z, \infty), \tag{17}$$

where $(\mathfrak{M}(n))_n$ is a sequence of compact sets of the type described above, such that

$$\mathfrak{M}(1) \supseteq \mathfrak{M}(2) \supseteq \ldots \supseteq \mathfrak{M}(n) \supseteq \ldots \supseteq \mathfrak{M}, \qquad \bigcap_{n=1}^{\infty} \mathfrak{M}(n) = \mathfrak{M}. \tag{18}$$

The generalized Green's function is harmonic on the whole complex plane and has a logarithmic singularity at infinity if the logarithmic capacity $\operatorname{cap}\mathfrak{M}$ of $\mathfrak{M}$ (see e.g. [13, 16, 17]) is positive, but is identically infinite if $\operatorname{cap}\mathfrak{M} = 0$.

In some cases, the Green's function $g_{\mathfrak{M}}(., \infty)$ may be calculated explicitly. For instance, if the domain $\Lambda_\infty(\mathfrak{M})$ is simply connected and its boundary contains at least two points, the important formula [13, 20]

$$g_{\mathfrak{M}}(z, \infty) = \log |\varphi_{\mathfrak{M}}(z)| \tag{19}$$

holds, where $\varphi_{\mathfrak{M}}$ is the conformal *Riemann function* which maps $\Lambda_\infty(\mathfrak{M})$ onto the exterior of the unit disc with $\varphi_{\mathfrak{M}}(\infty) = \infty$.

In the sequel we shall use the abbreviations

$$e_n = \inf_{p \in \mathscr{P}_n} \|\psi - p\|_{\mathfrak{M}}, \qquad \tilde{e}_n = \inf_{p \in \mathscr{P}_n} \|A^{-1} - p(A)\|, \tag{20}$$

where $\mathscr{P}_n$ denotes the set of all polynomials of degree $\leqslant n$, and $\psi$ is defined by (10). Evidently,

$$e_n \leqslant \tilde{e}_n \qquad (n = 1, 2, \ldots). \tag{21}$$

From Theorem 1 it follows that the relations

$$\lim_{n \to \infty} e_n = 0, \qquad \lim_{n \to \infty} \tilde{e}_n = 0$$

are equivalent. However, one can make a much more precise statement. To this end, we first need an auxiliary

LEMMA. *Let $\mathfrak{M}$ be an arbitrary compact subset of the complex plane with $0 \in \Lambda_\infty(\mathfrak{M})$. Then the equality*

$$\lim_{n \to \infty} \sqrt[n]{e_n} = e^{-g_{\mathfrak{M}}(0, \infty)} \tag{22}$$

*holds.*

PROOF. The inequality

$$\limsup_{n \to \infty} \sqrt[n]{e_n} \leqslant e^{-g_{\mathfrak{M}}(0, \infty)}$$

follows from the Bernstein-Walsh theorem (see e.g. [18]) on the approximation of analytic functions by polynomials. We have therefore to show that

$$\liminf_{n \to \infty} \sqrt[n]{e_n} \geqslant e^{-g_{\mathfrak{M}}(0, \infty)}. \tag{23}$$

Now, observe that, for any $p \in \mathscr{P}_n$,

$$\max_{\lambda \in \mathfrak{M}} |\lambda^{-1} - p(\lambda)| \geqslant c \max_{\lambda \in \mathfrak{M}} |1 - \lambda p(\lambda)|, \tag{24}$$

where $c = \min \{ |\lambda^{-1}| : \lambda \in \mathfrak{M}\} > 0$. By the Bernstein-Walsh lemma on the growth of polynomials $r \in \mathscr{P}_n$ we have

$$|r(\lambda)| \leqslant \|r\|_{\mathfrak{M}} e^{n g_{\mathfrak{M}}(\lambda, \infty)} \qquad (\lambda \in \Lambda_\infty(\mathfrak{M}),\ r \in \mathscr{P}_n).$$

Applying this to the polynomial $r(\lambda) = 1 - \lambda p(\lambda)\ (\in \mathscr{P}_{n+1}!)$ we obtain

$$\|\psi - p\|_{\mathfrak{M}} \geqslant c\, \|r\|_{\mathfrak{M}} \geqslant c e^{-(n+1) g_{\mathfrak{M}}(0, \infty)}.$$

Since $p \in \mathscr{P}_n$ is arbitrary, we conclude that

$$e_n \geqslant c e^{-(n+1) g_{\mathfrak{M}}(0, \infty)},$$

and hence (23) follows by taking the $n$-th root.

THEOREM 3. *Let $A$ be a bounded invertible operator in a Banach space $X$, and let $\mathfrak{M}$ be a compact subset of the complex plane with $0 \in \Lambda_\infty(\mathfrak{M})$. Suppose that* $\operatorname{sp} A \subseteq \mathfrak{M}$ *(resp.,* $\operatorname{sp} A = \mathfrak{M}$*). Then the estimate*

$$\limsup_{n \to \infty} \sqrt[n]{\tilde{e}_n} \leqslant e^{-g_{\mathfrak{M}}(0, \infty)} \tag{25}$$

*(resp., the equality*

$$\lim_{n \to \infty} \sqrt[n]{\tilde{e}_n} = e^{-g_{\mathfrak{M}}(0, \infty)}) \tag{26}$$

*holds.*

PROOF. It suffices to prove the second assertion; thus, let $\mathfrak{M} = \operatorname{sp} A$, and suppose first that $\partial\mathfrak{M}$ consists of a finite number of closed Jordan curves. First of all, it follows from (21) and (22) that

$$\liminf_{n \to \infty} \sqrt[n]{\tilde{e}_n} \geqslant e^{-g_{\mathfrak{M}}(0, \infty)}. \tag{27}$$

We claim that

$$\limsup_{n \to \infty} \sqrt[n]{\tilde{e}_n} \leqslant e^{-g_{\mathfrak{M}}(0, \infty)},$$

or, equivalently, that

$$\limsup_{n \to \infty} \sqrt[n]{\|A^{-1} - w_n(A)\|} \leqslant e^{-g_{\mathfrak{M}}(0, \infty)} \tag{28}$$

for an appropriate sequence of polynomials $w_n \in \mathscr{P}_n$. Now, choose any sequence of polynomials $w_n \in \mathscr{P}_n$ satisfying

$$\lim_{n\to\infty} \sqrt[n]{\|\psi - w_n\|_{\mathfrak{M}}} = e^{-g_{\mathfrak{M}}(0,\infty)},$$

where $\psi$ is given by (10). For every $p \in \mathscr{P}_n$ we have

$$\max_{\lambda\in\mathfrak{M}} |\lambda^{-1} - p(\lambda)| \leqslant C \max_{\lambda\in\mathfrak{M}} |1 - \lambda p(\lambda)|, \tag{29}$$

where $C = \max\{|\lambda|^{-1}\colon \lambda \in \mathfrak{M}\}$. Again by the Bernstein-Walsh lemma on the growth of polynomials, we see that

$$\limsup_{n\to\infty} \sqrt[n]{\|\psi - w_n\|_{L_r(\mathfrak{M})}} \leqslant re^{-g_{\mathfrak{M}}(0,\infty)}$$

for $1 < r < e^{g_{\mathfrak{M}}(0,\infty)}$, where

$$L_r(\mathfrak{M}) = \{\lambda\colon \lambda \in \mathbb{C},\ g_{\mathfrak{M}}(\lambda,\infty) = \log r\}.$$

Applying (12) we get

$$\limsup_{n\to\infty} \sqrt[n]{\|A^{-1} - w_n(A)\|} \leqslant re^{-g_{\mathfrak{M}}(0,\infty)};$$

letting $r$ tend to 1 we conclude that (28) holds. This proves the assertion in the case when $\partial\mathfrak{M}$ consists of a finite number of closed Jordan curves.

Now let $\mathfrak{M}$ be an arbitrary compact subset of the complex plane. Choose a sequence $(\Lambda(k))_k$ of domains which contain $\infty$, exhaust $\Lambda_\infty(\mathfrak{M})$ (i.e. $\Lambda_\infty(\mathfrak{M}) = \Lambda(1) \cup \Lambda(2) \cup \ldots \cup \Lambda(k) \cup \ldots$), and satisfy $\overline{\Lambda}(k) \subseteq \Lambda(k+1)$. Since the sets $\mathfrak{M}(k) = \mathbb{C}\setminus\Lambda(k)$ satisfy then the conditions (18), from what we have proved before we deduce that

$$\limsup_{n\to\infty} \sqrt[n]{\|A^{-1} - w_n(A)\|} \leqslant e^{-g_{\mathfrak{M}(k)}(0,\infty)}.$$

Passing in this estimate to the limit $k \to \infty$, we obtain

$$\limsup_{n\to\infty} \sqrt[n]{\|A^{-1} - w_n(A)\|} \leqslant e^{-g_{\mathfrak{M}}(0,\infty)},$$

and it remains to use (21) and (22). This proves the assertion in the general case.

Observe that the assertion of Theorem 3 is void in the finite dimensional case $X = \mathbb{R}^N$. Indeed, by the classical Cayley-Hamilton theorem the inverse matrix $A^{-1}$ of a non-singular matrix $A \in \mathbb{R}^{N\times N}$

may be expressed as a polynomial $p(A)$ of $A$ with $p \in \mathscr{P}_{N-1}$. Consequently, all characteristics $\tilde{e}_n$ are then zero for $n \geq N$.

## 3. - Extremal sequences of polynomials.

Theorem 3 makes it possible to calculate (or estimate) the rate of convergence of the quasi-iteration methods introduced above. Moreover, its proof provides a rather effective method for obtaining the corresponding approximations.

In fact, in the proof of Theorem 3 we have shown that the sequence of polynomials $w_n$ satisfying (29) gives the estimate (28) for the corresponding operators. This means that, if we construct the quasi-iterations of the original operator equation (1) by means of these polynomials, we get the best possible convergence rate.

In what follows, we say that the sequence $(w_n)_n$ is an *exact approximation* of the function (10) on a compact set $\mathfrak{M}$ if the polynomial $w_n$ gives, for each $n$, the best possible approximation to (10) among all polynomials in $\mathscr{P}_n$. Similarly, we say that $(w_n)_n$ is a *maximal approximation* of the function (10) on $\mathfrak{M}$ if

$$\lim_{n \to \infty} \sqrt[n]{\|\psi - w_n\|_{\mathfrak{M}}} = e^{-g_{\mathfrak{M}}(0,\, \infty)}. \tag{30}$$

As already observed, the explicit construction of an exact approximation $(w_n)_n$ on a given compact set $\mathfrak{M}$ is possible only in very special cases. For maximal approximations the situation is nicer, as we shall show now.

From the trivial relation

$$|\lambda^{-1} - w(\lambda)| = \left| \frac{1 - \lambda w(\lambda)}{\lambda} \right|$$

it follows that a sequence $(w_n)_n$ is a maximal approximation if and only if the polynomials $p_n(\lambda) = 1 - \lambda w_n(\lambda)$ satisfy

$$\lim_{n \to \infty} \sqrt[n]{\|p_n\|_{\mathfrak{M}}} = e^{-g_{\mathfrak{M}}(0,\, \infty)}. \tag{31}$$

The equality (31) is in turn equivalent to the existence of a sequence $(\pi_n)_n$ of polynomials $\pi_n$ satisfying

$$\lim_{n \to \infty} \left( \frac{|\pi_n(0)|}{\|\pi_n\|_{\mathfrak{M}}} \right)^{1/n} = e^{g_{\mathfrak{M}}(0,\, \infty)}. \tag{32}$$

Indeed, since $p_n(0) = 1$, (30) implies (31) simply by putting $\pi_n(\lambda) = p_n(\lambda)$. Conversely, if $(\pi_n)_n$ is a sequence of polynomials satisfying (32), we may put $p_n(\lambda) = \pi(\lambda)/\pi_n(0)$. Polynomials $\pi_n \in \mathscr{P}_n$ which satisfy

$$\lim_{n \to \infty} \left( \frac{|\pi_n(\lambda)|}{\|\pi_n\|} \right)^{1/n} = e^{g_{\mathfrak{M}}(\lambda, \infty)} \tag{33}$$

for all $\lambda \in \Lambda_\infty(\mathfrak{M})$ are of great importance in the approximation theory for analytic functions. Usually, such sequences are called *extremal sequences for the set* $\mathfrak{M}$. A detailed account of the properties of extremal sequences of polynomials, as well as a large list of references may be found, for example, in [18].

The simplest way of constructing extremal sequences of polynomials for a given compact set $\mathfrak{M}$ is as follows. Define polynomials $\pi_n$ by

$$\pi_n(\lambda) = (\lambda - \xi_{0,n})(\lambda - \xi_{1,n})\dots(\lambda - \xi_{n,n}) \qquad (n = 1, 2, \dots), \tag{34}$$

where

$$\xi_{0,n}, \xi_{1,n}, \dots, \xi_{n,n} \in \mathfrak{M}. \tag{35}$$

A system of points (35) for which the condition (33) is fulfilled is called *uniformly distributed*. For example, if the domain $\Lambda_\infty(\mathfrak{M})$ is simply connected, its boundary contains at least two points, and the conformal Riemann function $\varphi_{\mathfrak{M}}$ of $\mathfrak{M}$ may be continuously extended to the boundary, as a uniformly distributed system one may choose the points $\xi_{k,n} = \varphi_{\mathfrak{M}}^{-1}(e^{2\pi k/n})$ $(k = 0, 1, \dots, n)$. (This is the so-called *system of Fejér points.*) There are also other uniformly distributed systems, such as the *system of Fekete points* or the *system of Leja points*. In the case when $\Lambda_\infty(\mathfrak{M})$ is simply connected and its boundary contains at least two points, another prominent example of an extremal sequence is that of the *Faber polynomials* (see [15]).

We summarize our discussion above with the following Theorem 4. For its formulation, we recall that the Ljapunov exponent of a sequence $(c_n)_n$ is defined by $\limsup_{n \to \infty} \sqrt[n]{|c_n|}$.

THEOREM 4. *Let $A$ be a bounded invertible operator in a complex Banach space $X$. Let $\mathfrak{M} \supseteq \operatorname{sp} A$ be some compact set with $0 \in \Lambda_\infty(\mathfrak{M})$, and let $(w_n)_n$ be a sequence which is a maximal approximation of the function* (10) *on $\mathfrak{M}$. Then the sequence of iterations* (6) *converges, for any $f \in X$, to the solution of the operator equation* (1). *Moreover, the Ljapunov exponent of the convergence is* $\leqslant \exp[-g_{\mathfrak{M}}(0, \infty)]$.

## 4. Iterative and quasi-iterative methods.

We return to the Chebyshev quasi-iteration method. Let $\mathfrak{M}$ be a compact subset of the complex plane. Recall [6] that the numbers

$$\chi_n(\mathfrak{M}) = \inf\{\|p\|_{\mathfrak{M}}:\ p \in \mathscr{P}_n,\ p(0) = 1\} \qquad (n = 1, 2, \dots), \tag{36}$$

and

$$\chi(\mathfrak{M}) = \lim_{n\to\infty} \sqrt[n]{\chi_n(\mathfrak{M})} \tag{37}$$

are called *Chebyshev characteristic of order n* and *Chebyshev limit characteristic*, respectively, of the set $\mathfrak{M}$. As was shown in [6], the Chebyshev iteration method of order $n$ (resp., of arbitrary order) for equation (1) works if and only if $\chi_n(\mathrm{sp}A) < 1$ (resp., $\chi(\mathrm{sp}A) < 1$). Moreover, in this case $\chi_n(\mathrm{sp}A)$ (resp., $\chi(\mathrm{sp}A)$) coincides with the minimal Ljapunov exponent of the convergence.

As was observed in [8], the inequality $\chi_1(\mathfrak{M}) < 1$ holds if the set $\mathfrak{M}$ is situated on one side of a straight line passing through the origin. Analogous conditions for the higher order characteristics $\chi_n(\mathfrak{M})$ are not known. Nevertheless, our results obtained above allow us to give a condition for the estimate $\chi(\mathfrak{M}) < 1$ which is both necessary and sufficient. In fact, we shall show now that $\chi(\mathfrak{M}) < 1$ if and only if $0 \in \Lambda_\infty(\mathfrak{M})$. Moreover, we shall obtain an explicit formula for $\chi(\mathfrak{M})$.

So, let $\mathfrak{M}$ be an arbitrary compact subset of the complex plane, and denote by $\Lambda_\infty(\mathfrak{M})$ the connected component of $\mathbb{C}\backslash\mathfrak{M}$ containing $\infty$.

THEOREM 5. *Let $\mathfrak{M} \subset \mathbb{C}$ be compact. Then the estimate $\chi(\mathfrak{M}) < 1$ holds if and only if $0 \in \Lambda_\infty(\mathfrak{M})$. Moreover, in this case the equality*

$$\chi(\mathfrak{M}) = e^{-g_{\mathfrak{M}}(0,\,\infty)} \tag{38}$$

*holds.*

PROOF. Suppose that $0 \notin \Lambda_\infty(\mathfrak{M})$. For any $p \in \mathscr{P}_n$ with $p(0) = 1$ we have then $\|p\|_{\mathfrak{M}} \geq 1$. Consequently, $\chi_n(\mathfrak{M}) \geq 1$, and thus $\chi(\mathfrak{M}) \geq 1$ as well.

Conversely, suppose that $0 \in \Lambda_\infty(\mathfrak{M})$. Combining (24) and (29), we see that

$$c\chi_n(\mathfrak{M}) \leq e_n \leq C\chi_n(\mathfrak{M}) \qquad (n = 1, 2, \dots).$$

Taking the $n$-th root in this inequality and passing to the limit $n \to \infty$, we conclude that (38) holds. The estimate $\chi(\mathfrak{M}) < 1$ follows from the fact that $g_{\mathfrak{M}}(0, \infty)$ is positive.

We remark that the one-sided estimate

$$\chi(\mathfrak{M}) \geq e^{-g_{\mathfrak{M}}(0,\,\infty)} \tag{39}$$

is always true: In fact, from the Bernstein-Walsh lemma on the growth of polynomial functions it follows that

$$\chi_n(\mathfrak{M}) \geq e^{-ng_{\mathfrak{M}}(0,\,\infty)} \qquad (n = 1, 2, \ldots)$$

which implies (39).

From Theorem 5 it follows, in particular, that $\chi(\mathfrak{M}) = 0$ if and only if $\operatorname{cap}\mathfrak{M} = 0$. In the finite dimensional case $X = \mathbb{R}^N$, a similar formula to (38) has been proved in [3].

As already observed, the Chebyshev iteration method in [6] is a special case of our Chebyshev quasi-iteration method. Indeed, if $\chi_n(\mathfrak{M}) < 1$ choose $p \in \mathscr{P}_n$ with $p(0) = 1$ and $\|p\|_{\mathfrak{M}} < 1$ and let $q(\lambda) = (1 - p(\lambda))/\lambda$. We have then

$$\frac{1}{\lambda} = \frac{q(\lambda)}{1 - p(\lambda)} = \sum_{k=0}^{\infty} p^k(\lambda)\,q(\lambda) \qquad (\lambda \in \mathfrak{M}). \tag{40}$$

Since the series in (40) converges uniformly on $\mathfrak{M}$, as approximating polynomials for the function (10) on $\mathfrak{M}$ we may take the partial sums

$$w_n(\lambda) = \sum_{k=0}^{n} p^k(\lambda)\,q(\lambda) \qquad (n = 1, 2, \ldots).$$

As a straightforward calculation shows, the rate of convergence of the sequence $(w_n)_n$ is then $\sqrt[n]{\chi_n(\mathfrak{M})}$, and hence is arbitrarily close to $\chi(\mathfrak{M})$ for large $n$.

## 5. - Concluding examples and remarks.

In this final section we give some examples and additional remarks. First, let us consider two examples of how the Chebyshev limit characteristic $\chi(\mathfrak{M})$, the Riemann function $\varphi_{\mathfrak{M}}$, and the Faber polynomials $\Phi_{n,\mathfrak{M}}$ may look like for specific subsets $\mathfrak{M}$ of the complex plane.

Consider first the disc $\mathfrak{M} = \{z\colon z \in \mathbb{C},\ |z - a| \leq R\}$, where $|a| > R$. Here we have $\chi(\mathfrak{M}) = R/|a|$, $\varphi_{\mathfrak{M}}(z) = (z - a)/R$, and $\Phi_{n,\mathfrak{M}}(z) = (z - a)^n$. In particular, the dependence of these characteristics on $a$ and $R$ is very simple.

On the other hand, consider a segment $\mathfrak{M} = [z_1, z_2] = \{(1 - \tau)z_1 + \tau z_2\colon 0 \leq \tau \leq 1\}$ which does not contain 0. Put $|z_1 - z_2| = 2L$ and

$\arg(z_1 - z_2) = \alpha$. In this case we have

$$\chi(\mathfrak{M}) = \frac{1}{|\zeta + \sqrt{|\zeta^2 - 1|}|},$$

where $\zeta = -(z_1 + z_2)e^{-i\alpha}/2L$, and

$$\varphi_{\mathfrak{M}}(z) = \xi + \sqrt{\xi^2 - 1},$$

where $\xi = \xi(z) = e^{-i\alpha}[z - (z_1 + z_2)/2]/L$, and the branch of the root is chosen in such a way that $\varphi_{\mathfrak{M}}(\infty) = \infty$. The Faber polynomials in this case are

$$\Phi_{n,\mathfrak{M}}(z) = 2^{-n}[(\xi + \sqrt{\xi^2 - 1})^n + (\xi - \sqrt{\xi^2 - 1})^n].$$

In contrast to the preceding example, the characteristics are here very sensitive with respect to any change of the position of $\mathfrak{M}$ in the complex plane.

Theorem 3 and Theorem 4 show that, for a suitable choice of the polynomials $w_n$ in the quasi-iteration method, or of the polynomial $p$ in the iteration method, the rate of convergence of the approximation is better than of an arbitrary geometric progression with ratio $q \in (e^{-g(0,\infty)}, 1)$. In the case of a compact operator $A$ (which frequently occurs in applications) the corresponding iterations may converge faster than any geometric progression. However, when trying to construct the polynomials $w_n$ and $p$ explicitly, one usually encounters a lot of both technical and principal difficulties. For example, we do not have more precise information on the relation between the sequences of norms $\|A^{-1} - w_n(A)\|$ and $\|\psi - w_n\|_{\mathrm{sp}A}$ than that given in the above theorems. In particular, our results allow us only to state that these two sequences of norms are weakly equivalent, and hence exhibit the same rate of convergence as $n \to \infty$. Effective error estimates may be obtained so far, however, only in some special cases (for instance, if $A$ is a normal operator in Hilbert space).

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