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Rendiconti del Seminario Matematico della Università di Padova, tome 93 (1995), p. 103-107

<http://www.numdam.org/item?id=RSMUP_1995__93__103_0>
An Algebraic Summation Over the Set of Partitions and Some Strange Evaluations.

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ABSTRACT - By means of formal power series operation, a general algebraic summation formula over the set of partitions is established. Several combinatorial identities are demonstrated as special cases.

Let \( \Omega \) be a subset of non-negative integers and \( x_0, x_1, \ldots, x_n \) the indeterminates. Evaluate

\[
E_{m \in \Omega} (x | \Omega) := \sum_{k \in \tau(m, n | \Omega)} \{x_{k_1} (x_{k_1} + x_{k_2}) \ldots (x_{k_1} + x_{k_2} + \ldots + x_{k_n})\}^{-1}
\]

where \( \tau(m, n | \Omega) \) is the set of all \( n \)-tuples \( \kappa = (k_1, k_2, \ldots, k_n) \in \Omega^n \) with sum \( m \).

For each solution \( \kappa = (k_1, k_2, \ldots, k_n) \in \tau(m, n | \Omega) \) of the equation \( k_1 + k_2 + \ldots + k_n = m (k_i \in \Omega) \), define its type by the partition \( \rho = \{0^{p_0} 1^{p_1} \ldots m^{p_m}\} \), if number \( k \) appears \( p_k \) times in this solution \( (k_1, k_2, \ldots, k_n) \) for \( 0 \leq k \leq m \) (it is obvious that \( p_k = 0 \) if \( k \notin \Omega \)). Then the solutions with the same type \( \rho \) are generated by the different permutations \( S(\rho) \) of multi-set \( p = \{0^{p_0}, 1^{p_1}, \ldots, m^{p_m}\} \). Thus, we can classify the solution-set \( \tau(m, n | \Omega) \) of that equation according to the partitions \( \sigma(m, n | \Omega) = \{ \rho = \{0^{p_0} 1^{p_1} \ldots m^{p_m}\} : \sum kp_k = m, \sum p_k = n \text{ and } p_k = 0 \text{ for } k \notin \Omega \} \), of number \( m \) into \( n \)-parts with each part restricted in \( \Omega \). Based on this observation, the summation defined by (1) can be decom-

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(**) Partially supported by NSF (Chinese) Youth Grant # 19901033.
posed as

\( E_{m \cdots n}(x | \Omega) = \)

\[ \sum_{\rho \in \sigma(m, n | \Omega)} \sum_{\pi \in S(\rho)} \left\{ x_{\pi(1)} (x_{\pi(1)} + x_{\pi(2)}) \cdots (x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(n)}) \right\}^{-1}. \]

As a crucial lemma, it is an easy exercise (Chu [2], 1989) to show, by the induction principle on \( n = \sum p_k \), that.

**LEMMA.**

\[ \sum_{\pi \in S(\rho)} \left\{ x_{\pi(1)} (x_{\pi(1)} + x_{\pi(2)}) \cdots (x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(n)}) \right\}^{-1} = \]

\[ = \left\{ \prod_{k=0}^{m} p_k ! x_k^{p_k} \right\}^{-1}. \]

It follows from substituting (3) into (2), that

\( E_{m \cdots n}(x | \Omega) = \)

\[ \sum_{\rho \in \sigma(m, n | \Omega)} \left\{ \prod_{k=0}^{m} p_k ! x_k^{p_k} \right\}^{-1} \]

which could be expressed equivalently as

\( E_{m \cdots n}(x | \Omega) = (n!)^{-1} \sum_{\kappa \in \tau(m, n | \Omega)} \left\{ \prod_{i=1}^{n} x_k \right\}^{-1} \]

in view of the fact that the number of different permutations \( S(\rho) \) of multi-set \( p = \{ 0^{p_0}, 1^{p_1}, \ldots, m^{p_m} \} \) is equal to \( n! / \prod p_i ! \). Both (4) and (5) lead to a general algebraic summation formula if we denote by \([t^m] P(t)\) the coefficients of \( t^m \) in the power series expansion of function \( P(t) \).

**THEOREM.**

\( E_{m \cdots n}(x | \Omega) := [t^m] \left\{ \sum_{k=0}^{n} t^k / x_k \right\} \right)^n / n! \).

Let \( \Gamma \) be the Gamma function. Denote the lower and upper factorials, respectively, by

\[ [z]_n = z(z - 1) \cdots (z - n + 1) = \Gamma(1 + z) / \Gamma(1 + z - n) \]

and

\[ (z)_n = z(z + 1) \cdots (z + n - 1) = \Gamma(z + n) / \Gamma(z). \]
Now we are ready to exhibit some special evaluations. For convenience, we denote by \( \Phi(t, x | \Omega) \) the power series inside the bracket in equation (6).

(A) Let \( \Omega = \mathbb{N}_0 \), the set of non-negative integers, and \( x_k = y^{-k} / \binom{x}{k} \). Then the corresponding \( \Phi \)-function is \( \Phi(t, x | \Omega) = (1 + yt)^x \) which results in

\[
E_{m-n} \left( x_k = y^{-k} / \binom{x}{k} \mid \mathbb{N}_0 \right) = y^m \left( \frac{nx}{m} \right) / n!.
\]

Taking \( x = -1 \) and \( y = -1/a \) in (6), we get

\[
E_{m-n} \left( x_k = a^k \mid \mathbb{N}_0 \right) = a^{-m} \left( \frac{m+n-1}{m} \right) / n!
\]

whose special case corresponding to \( a = 2 \) is due to Knuth and Pittel[4].

\[
\sum_{k_1 + k_2 + \ldots + k_n = m} \left\{ 2^{k_1} (2^{k_1} + 2^{k_2}) \ldots (2^{k_1} + 2^{k_2} + \ldots + 2^{k_n}) \right\}^{-1} = 2^{-m} (n)_m / m! n!.
\]

Alternatively, the limiting case of (6) for \( 1/x \) and \( y \) tending to zero under condition \( xy = 1/c \) yields another formula

\[
E_{m-n} \left( x_k = k! c^k \mid \mathbb{N}_0 \right) = n^m c^{-m} / m! n!
\]

which gives, for \( c = 1 \), another evaluation of Knuth and Pittel[4].

\[
\sum_{k_1 + k_2 + \ldots + k_n = m} \left\{ k_1! (k_1! + k_2!) \ldots (k_1! + k_2! + \ldots + k_n!) \right\}^{-1} = n^m / m! n!.
\]

(B) If we take \( \Omega = \mathbb{N} \), the positive integers. Then for \( x_k = y / \binom{x}{k-1} \), it holds that \( \Phi(t, x | \Omega) = yt(1 + yt)^x \). From this, the shifted version of (7) follows

\[
E_{m-n} \left( x_k = y^{-k} / \binom{x}{k-1} \mid \mathbb{N} \right) = y^m \left( \frac{nx}{m-n} \right) / n!, \quad (m \geq n)
\]

which reduces when \( x = -1 \), \( y = -1/a \) and \( x = -2 \), \( y = -1/c \), re-
spectively, to the formulae:

\begin{align}
(11) \quad E_{m-n}(x_k = a^k \mid N) &= a^{-m} \binom{m-1}{n-1} / n!, \quad (m \geq n), \\
(12) \quad E_{m-n}(x_k = c^k / k \mid N) &= c^{-m} \binom{m+n-1}{m-n} / n!, \quad (m \geq n).
\end{align}

Similarly, one can compute the following summations:

\begin{align}
(13) \quad E_{m-n}(x_k = (-1)^{k-1} k x^k \mid N) &= x^{-m} S_1(m, n)/m!, \quad (m \geq n), \\
(14) \quad E_{m-n}(x_k = k x^k \mid N) &= x^{-m} S_1^*(m, n)/m!, \quad (m \geq n), \\
(15) \quad E_{m-n}(x_k = k! x^k \mid N) &= x^{-m} S_2(m, n)/m!, \quad (m \geq n),
\end{align}

where $S_1(m, n)$, $S_1^*(m, n)$ and $S_2(m, n)$ are the Stirling numbers defined by

\begin{align}
[z]_m = \sum_{n \leq m} S_1(m, n) z^n, \\
(z)_m = \sum_{n \leq m} S_1^*(m, n) z^n,
\end{align}

and

\begin{align}
z^m = \sum_{n \leq m} S_2(m, n)[z]_n.
\end{align}

(C) Recall the generating function of Hagen-Rothe coefficients (Gould [3], 1956)

\[ \sum_{k \geq 0} \frac{a}{a + bk} \binom{a + bk}{k} u^k = v^a, \quad u = (v - 1)v^{-b}. \]

We can extend the results displayed in (A) as an identity on binomial coefficients

\begin{align}
(16) \quad E_{m-n}(x_k = c^{-k} \left[ \frac{a}{a + bk} \binom{a + bk}{k} \right]^{-1} \mid N_0) =
&= \frac{an}{an + bm} \left( \frac{an + bm}{m} \right) c^m / n!,
\end{align}

as well as its Abel-analogue

\begin{align}
(17) \quad E_{m-n}(x_k = ac^{-k}(a + bk)^{-1}/k! \mid N_0) = anc^m (an + bm)^{m-1} / n!.
\end{align}
where the latter is the limiting version of the former under replacements $a \to aM$, $b \to bM$ and $c \to cM$ when $M \to \infty$.

More generally, for any Sheffer-sequences generated by

$$\sum_{k \in \Omega} t^k / X_k(\lambda) = t^\delta \exp[\lambda \psi(t)], \quad (\delta \in \Omega)$$

all the formulas exhibited above could be formally unified as

$$E_{m-n}(x_k = y^{-k} X_k(\lambda) | \Omega) = y^m X_{m-n}(n\lambda) / n! .$$

For particular settings of $\psi(t)$, this identity could be used to create numerous other evaluations. But the resulting relations are too messy to be stating unless when necessary.

**Remark.** The evaluations (7)-(17) demonstrated in this note may also be reformulated in the summations of (4)-(5). Some of them can be found in Chu [2] (1989).

**REFERENCES**


Manoscritto pervenuto in redazione l’8 giugno 1993 e, in forma revisionata, il 12 agosto 1993.