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Representable equivalences for closed categories of modules

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Representable Equivalences for Closed Categories of Modules.

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0. Introduction.

0.1. All rings considered in this paper have a nonzero identity and all modules are unital. For every ring R, Mod-R (R-Mod) denotes the category of all right (left) R-modules. The symbol M_R (${}_RM$) is used to emphasize that M is a right (left) R-module.

Categories and functors are understood to be additive. Any subcategory of a given category is full and closed under isomorphic objects. N denotes the set of positive integers.

0.2. Recall that a non empty subcategory $\mathcal{G}_R$ of Mod-R is *closed* if $\mathcal{G}_R$ is closed under taking submodels, homomorphic images and arbitrary direct sums. Clearly $\mathcal{G}_R$ is a Grothendieck category.

It is easy to show that a closed subcategory $\mathcal{G}_R$ of Mod-R has a generator and for every generator P_R of $\mathcal{G}_R$ we have:

$$\mathcal{G}_R = \mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$$

where $\mathrm{Gen}(P_R)$ is the subcategory of Mod-R generated by P_R and $\overline{\mathrm{Gen}}(P_R)$ is the smallest closed subcategory of Mod-R containing $\mathrm{Gen}(P_R)$.

0.3. Let $\mathcal{G}_R$ be a closed subcategory of Mod-R, P_R a generator of $\mathcal{G}_R$, $A = \mathrm{End}(P_R)$. In the search for subcategories of Mod-A which are equivalent to $\mathcal{G}_R$, the functors:

$$H = \mathrm{Hom}_R(P_R, -)\colon \text{Mod-}R \to \text{Mod-}A\,,$$

$$T = - \otimes_A P\colon \text{Mod-}A \to \text{Mod-}R\,,$$

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play a crucial role. Indeed we have the following representation theorem:

Let A and R be two rings, $\mathcal{D}_A$ a subcategory of Mod-A *such that $A_A \in \mathcal{D}_A$, $\mathcal{G}_R$ a closed subcategory of* Mod-R. *Assume that an equivalence (F, G) between $\mathcal{D}_A$ and $\mathcal{G}_R$ is given:*

$$\mathcal{D}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{G}_R .$$

Then there exists a bimodule ${}_AP_R$ such that

1) $P_R \in \mathcal{G}_R$, $A \cong \mathrm{End}(P_R)$ *canonically.*
2) *The functors F and G are naturally equivalent to the functors $T|_{\mathcal{D}_A}$ and $H|_{\mathcal{G}_R}$ respectively.*
3) $\mathcal{G}_R = \mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$, $\mathcal{D}_A = \mathrm{Im}(H)$.

On the other hand a remarkable result of Zimmermann-Huisgen [ZH] and Fuller [F] states that, if $P_R \in$ Mod-R and $A = \mathrm{End}(P_R)$, the following conditions are equivalent:

(*a*) $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

(*b*) The functor H: $\mathrm{Gen}(P_R) \to$ Mod-A is full and faithful and ${}_AP$ is flat.

Therefore H induces an equivalence between $\mathrm{Gen}(P_R)$ and $\mathrm{Im}(H)$. We say that a module P_R of Mod-R is a *W-module* if $\mathrm{Gen}(P_R)$ is a closed subcategory of Mod-R or, equivalently, if $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

0.4. Let P_R be a W-module, $A = \mathrm{End}(P_R)$. The main purpose of this paper is to find a satisfactory description of $\mathrm{Im}(H)$. Instead of using the Popescu-Gabriel Theorem (cf. [St] Theorem 4.1. Chap. X) we prefer to proceed in a more concrete manner using always the role of the functors H and T that lead to an interesting torsion theory on Mod-A.

Set

$$\mathrm{Ker}(T) = \{L \in \text{Mod-}A\colon\ L \otimes_A P = 0\}.$$

Since ${}_AP$ is flat, $\mathrm{Ker}(T)$ is a localizing subcategory of Mod-A, i.e. $\mathrm{Ker}(T)$ is the torsion class of a hereditary torsion theory in Mod-A. The corresponding torsion-free class is obtained in the following manner: let Q_R be a fixed, but arbitrary, injective cogenerator of Mod-R, $K_A = \mathrm{Hom}_R(P_R, Q_R)$, $\mathcal{D}(K_A)$ the subcategory of Mod-A cogenerated by

K_A. Then $\mathcal{D}(K_A)$ is the requested torsion-free class and K_A is injective in Mod-A.

The Gabriel filter Γ—consisting of right ideals of A—associated to the torsion theory $(\mathrm{Ker}(T), \mathcal{D}(K_A))$ is given by

$$\Gamma = \left\{ I \leqslant A_A \colon \frac{A}{I} \in \mathrm{Ker}(T) \right\}.$$

Equivalently

$$\Gamma = \{ I \leqslant A_A \colon IP = P \}.$$

For every $L \in$ Mod-A denote by L_Γ the module of quotients of L with respect to Γ. Set

$$\mathrm{Mod}-(A, \Gamma) = \{ L \in \text{Mod-}A \colon L = L_\Gamma \}$$

The main result on the torsion theory $(\mathrm{Ker}(T), \mathcal{D}(K_A))$ is the following: for every $L \in$ Mod-A

$$L_\Gamma = HT(L).$$

Then it is easy to show that $\mathrm{Im}(H) = \mathrm{Mod}-(A, \Gamma)$.

0.5. Various properties of W-modules are investigated, in particular their connection with Fuller's Theorem on Equivalences.

The work ends with an example concerning the closed subcategory of Mod-R consisting of semisimple modules.

0.6. Remark. The class $\mathrm{Ker}(T)$ was also investigated by [WW].

1. Representable equivalences.

1.1. Through this paper we use the following standing notations. Let A, R be two rings and ${}_AP_R$ a bimodule (left on A and right on R). Consider the adjoint functors:

$$T = - \otimes_A P \colon \text{Mod-}A \to \text{Mod-}R,$$

$$H = \mathrm{Hom}_R(P_R, -) \colon \text{Mod-}R \to \text{Mod-}A.$$

For every $L \in$ Mod-A and $M \in$ Mod-R there exist the natural morphisms:

$$\sigma_L \colon L \to HT(L) = \mathrm{Hom}_R(P_R, L \otimes_A P)$$

$$\sigma_L(l) \colon p \mapsto l \otimes p \qquad (p \in P, l \in L)$$

and

$$\rho_m\colon TH(M) = \mathrm{Hom}_R(P_R, M) \otimes_A P \to M,$$

$$\rho_M(f \otimes p) = f(p) \qquad (f \in \mathrm{Hom}_A(P, M),\ p \in P).$$

In the sequel the functors T and H will be suitably restricted and corestricted.

1.2. Let A, R be two rings, $\mathcal{D}_A$ and $\mathcal{G}_R$ subcategories of Mod-A and Mod-R respectively. Assume that a category equivalence (F, G) between $\mathcal{D}_A$ and $\mathcal{G}_R$ is given:

$$\mathcal{D}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{G}_R, \qquad G \circ F \approx 1_{\mathcal{D}_A}, \qquad F \circ G \approx 1_{\mathcal{G}_R}.$$

In this situation we always assume that $A_A \in \mathcal{D}_A$.

Set $P_R = F(A)$. Then we have the bimodule ${}_AP_R$, with $A = \mathrm{End}(P_R)$ canonically.

1.3. LEMMA. *in the situation* (1.2) *the functor G is naturally equivalent to the functor* $\mathrm{Hom}_R(P_R, -)|_{\mathcal{G}_R}$.

PROOF. Let $M \in \mathcal{G}_R$ and consider the following natural isomorphisms:

$$G(M) \cong \mathrm{Hom}_A(A, G(M)) \cong \mathrm{Hom}_R(F(A), FG(M)) \cong \mathrm{Hom}_R(P_R, M).$$

Thus $G \approx H|_{\mathcal{G}_R}$.

1.4. DEFINITION. We say that the equivalence (F, G) is *representable* by the bimodule ${}_AP_R(P_R = F(A))$ if $F \approx T|_{\mathcal{D}_A}$ and $G \approx H|_{\mathcal{G}_R}$. In this case we say that the bimodule ${}_AP_R$ *represents* the equivalence (F, G).

1.5. Let $P_R \in$ Mod-R and let $\mathrm{Gen}(P_R)$ be the subcategory of Mod-R generated by P_R. Recall that a module $M \in$ Mod-R is in $\mathrm{Gen}(P_R)$ if there exists an exact sequence $P_R^{(X)} \to M \to 0$ where X is a suitable set. $\mathrm{Gen}(P_R)$ is closed under taking epimorphic images and arbitrary direct sums. Denote by $\overline{\mathrm{Gen}}(P_R)$ the smallest closed subcategory of Mod-R containing $\mathrm{Gen}(P_R)$. $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$ if and only if $\mathrm{Gen}(P_R)$ is closed under taking submodules. Let ${}_AP_R$ be a bimodule and let Q_R be a fixed, but arbitrary, cogenerator of Mod-R. Set $K_A = \mathrm{Hom}_R(P, Q)$ and denote by $\mathcal{D}(K_A)$ the subcategory of Mod-A cogenerated by K_A.

1.6. LEMMA. *Let ${}_AP_R$ be a bimodule. Then* $\mathrm{Im}(T) \subseteq \mathrm{Gen}(P_R)$ *and* $\mathrm{Im}(H) \subseteq \mathcal{D}(K_A)$.

PROOF. See [MO$_2$] Prop. 2.2.

For every $M \in \text{Mod-}R$ set $t_P(M) = \sum\{\mathrm{Im}(f)\colon f \in \mathrm{Hom}_R(P, M)\}$. Then $t_p(M) \in \mathrm{Gen}(P_R)$ and $\mathrm{Hom}_R(P_R, M) \cong \mathrm{Hom}_R(P_R, t_p(M))$ in a natural way.

1.7. LEMMA. *Let ${}_AP_R$ be a bimodule. Then*

a) $\mathrm{Im}(H) = H(\mathrm{Gen}(P_R))$;

b) $M \in \mathrm{Gen}(P_R)$ *if and only if* ρ_M *is surjective*;

c) $L \in \mathcal{D}(K_A)$ *if and only if* σ_L *is injective.*

PROOF. See [MO$_2$] page 207.

1.8. PROPOSITION. *The equivalence* (F, G) *is representable by the bimodule* ${}_AP_R$ $(P_R = F(A))$ *if and only if for every* $L \in \mathcal{D}_A$ *and for every* $M \in \mathcal{G}_R$ *the canonical morphisms* σ_L *and* ρ_M *are both isomorphisms.*

2. W-modules.

2.1. Let $\mathcal{G}_R$ be a closed subcategory of Mod-R. Then $\mathcal{G}_R$ has a generator P_R and

$$\mathcal{G}_R = \mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R). \tag{1}$$

Indeed let $\wp$ the filter of all right ideals I of R such that $R/I \in \mathcal{G}_R$. Then $P_R = \bigoplus_{I \in \wp} R/I$ is a generator of $\mathcal{G}_R$ and it is easy to check that (1) holds.

2.2. DEFINITION. Let $P_R \in \text{Mod-}R$, $A = \mathrm{End}(P_R)$. Consider the functors $H = \mathrm{Hom}_R(P_R, -)$ and $T = - \otimes_A P$. We say that P_R is a *W_0-module* if

(*) *the functor* $H\colon \mathrm{Gen}(P_R) \to \text{Mod-}A$ *subordinates an equivalence between* $\mathrm{Gen}(P_R)$ *and* $\mathrm{Im}(H)$

(whose inverse is given by $T|_{\mathrm{Im}(H)}$).

2.3. REPRESENTATION THEOREM. *Let* $\mathcal{D}_A$ *and* $\mathcal{G}_R$ *be subcategories of* Mod-A *and* Mod-R *respectively. Assume that* $A_A \in \mathcal{D}_A$ *and that* $\mathcal{G}_R$ *is closed under taking arbitrary direct sums and homomorphic images.*

Suppose that a category equivalence (F, G) between $\mathcal{D}_A$ and $\mathcal{G}_R$ is given:

$$\mathcal{D}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{G}_R .$$

Then (F, G) is representable by the bimodule ${}_AP_R$ $(P_R = F(A)$, $A = \mathrm{End}(P_R))$ and $\mathcal{G}_R = \mathrm{Gen}(P_R)$, $\mathcal{D}_A = \mathrm{Im}(H)$. Therefore P_R is a W_0-module.

PROOF. By Lemma (1.2), $G \approx H|_{\mathcal{G}_R}$. Since $\mathrm{Gen}(P_R) \subseteq \mathcal{G}_R$ and by Lemma (1.6), the functor $T|_{\mathcal{D}_A}$ is a left adjoint of the functor G. Since (F, G) is an equivalence F is a left adjoint of G. Therefore $F \approx T|_{\mathcal{D}_A}$. Thus by Lemma (1.6) $\mathcal{G}_R = \mathrm{Gen}(P_R)$. Finally, by Lemma (1.7), $\mathcal{D}_A = = \mathrm{Im}(H)$.

2.4. Under the assumptions of Theorem (2.3) suppose that $\mathcal{G}_R$ is a closed subcategory of Mod-R. Then P_R is a W_0-module such that $\mathcal{G}_R = \mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

2.5. Let $P_R \in$ Mod-R and assume that $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$. Then the condition $(*)$ of 2.2. holds by the following important

2.6. THEOREM. *Let $P_R \in$ Mod-R, $A = \mathrm{End}(P_R)$. The following conditions are equivalent*:

(a) *For every positive integer n, P_R generates all submodules of P_R^n.*

(b) $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

(c) *${}_AP$ is flat and the functor H: $\mathrm{Gen}(P_R) \to$ Mod-A is full an faithful.*

Moreover if the above conditions are fulfilled, then

1) *H subordinates an equivalence between* $\mathrm{Gen}(P_R)$ *and* $\mathrm{Im}(H)$.

2) *The canonical image of R into* $\mathrm{End}({}_AP)$ *is dense if* $\mathrm{End}({}_AP)$ *is endowed with its finite topology.*

PROOF. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ are due to Zimmermann-Huisgen (cf. [ZH], Lemma 2.2). The statement (2) is due to Fuller ([F], Lemma 1.3).

2.7. DEFINITION. Let $P_R \in$ Mod-R. We say that P_R is a *W-module* if $\mathrm{Gen}(P_R)$ is a closed subcategory of Mod-R or equivalently $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

3. Some properties of W-modules.

3.1. PROPOSITION. *Let* P_R *be a* W*-module,* $A = \mathrm{End}(P_R)$, $B = \mathrm{End}({}_AP)$. *Then the bimodule* ${}_AP_B$ *is faithfully balanced and* $\mathrm{Gen}(P_B)$ *is naturally equivalent to* $\mathrm{Gen}(P_R)$.

PROOF. By Proposition (4.12) of [AF], ${}_AP_B$ is faithfully balanced. Endow R with the P-topology τ. τ is a right linear topology on R and has as a basis of neighbourhoods of 0 the right ideals of the form $\mathrm{Ann}_R(F)$ where F is a finite subset of P. Let $\mathcal{F}_\tau$ be the filter of all right ideals of R which are open in (R, τ). Set $\mathcal{T}_\tau = \{M \in \text{Mod-}R: \forall x \in M, \mathrm{Ann}_R(x) \in \mathcal{F}_\tau\}$. Then $\mathcal{T}_\tau = \mathrm{Gen}(P_R)$. Indeed it is obvious that $\mathrm{Gen}(P_R) \subseteq \mathcal{T}_\tau$. On the other hand let $M \in \mathcal{T}_\tau$ and $x \in M$. Then $\mathrm{Ann}_R(x) \geqslant \mathrm{Ann}_R(p_1, \ldots, p_n)$ where $\{p_1, \ldots, p_n\}$ is a finite subset of P. We have

$$\frac{R}{\bigcap_{i=1}^{n} \mathrm{Ann}_R(p_i)} \hookrightarrow \bigoplus_{i=1}^{n} \frac{R}{\mathrm{Ann}_R(p_i)} \cong \bigoplus_{i=1}^{n} p_i R \in \mathrm{Gen}(P_R).$$

Since $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$ it is $R/\bigcap_{i=1}^{n} \mathrm{Ann}_R(p_i) \in \mathrm{Gen}(P_R)$. It follows that $xR \in \mathrm{Gen}(P_R)$ since xR is an homomorphic image of $R/\bigcap_{i=1}^{n} \mathrm{Ann}_R(p_i)$. By Theorem 2.2, B is the Hausdorff completion of (R, τ), since τ is the relative topology on $R/\mathrm{Ann}_R(P)$ of the finite topology of $\mathrm{End}({}_AP)$. Let $\hat{\tau}$ the topology of B. It is clear that $\hat{\tau}$ is the P-topology of B. For every $I \in \mathcal{F}_\tau$ let $\bar{I}$ be the closure of $I/\mathrm{Ann}_R(P)$ in B. Then $\overline{\mathcal{F}_\tau} = \{\bar{I}: I \in \mathcal{F}_\tau\}$ is a basis of neighbourhoods of 0 in $(B, \hat{\tau})$ and $R/I \cong B/\bar{I}$ both in Mod-R and in Mod-B. Therefore $\mathrm{Gen}(P_R) = \mathrm{Gen}(P_B)$.

3.2. REMARK. Let P_R be a W-module, $A = \mathrm{End}(P_R)$, $H = - \oplus_A P$. Then, in general, $\mathrm{Im}(H) \neq \mathcal{D}(K_A)$ (cf. Lemma 1.6), as the following example shows.

EXAMPLE. Let P_R a generator of Mod-R, $A = \mathrm{End}(P_R)$. Clearly P_R is a W-module. Assume that $\mathrm{Im}(H) = \mathcal{D}(K_A)$. Then by Proposition 3.2 of [MO$_2$], $\mathrm{Im}(H) = \text{Mod-}A$. (This is a generalization of Fuller's Theorem on Equivalences [F]). It follows that the functors T and H give an equivalence between Mod-A and Mod-R. By a well known result of Morita [M], P_R is a progenerator of Mod-R. If P_R is a generator non progenerator in Mod-R then $\mathrm{Im}(H) \neq \mathcal{D}(K_A)$

3.3. The Remark 3.2 shows that the theory of W-modules is not trivial even if P_R is a generator of Mod-R so that $\mathrm{Gen}(P_R) = $ Mod-R. (See [WW]).

3.4. We conclude this section giving another generalization of Fuller's Theorem on Equivalences. Namely, if P_R is a W-module and if $\mathrm{Im}(H)$ is closed under taking homomorphic images, then $\mathrm{Im}(H) =$ $=$ Mod-A. For this purpose we need some preliminar results.

3.5. Let $P_R \in$ Mod-R, $A = \mathrm{End}(P_R)$, $M \in \mathrm{Gen}(P_R)$. Consider an epimorphism $h\colon P_R^{(X)} \to M \to 0$ where X is a suitable set. Clearly $h =$ $= (h_x)_{x \in X}$ where $h_x \in \mathrm{Hom}_R(P_R, M)$. Therefore there exists a natural injection

$$i\colon \sum_{x \in X} h_x A \to \mathrm{Hom}_R(P_R, M).$$

An Azumaya's Lemma (cf. [A], Lemma 1) guarantees that, if ρ_M is injective, then the canonical morphism

$$T(i)\colon \Big(\sum_{x \in X} h_x A\Big) \otimes_A P \to \mathrm{Hom}_R(P_R, M) \otimes_A P$$

is surjective.

3.6. Lemma. *Let P_R be a W-module, $A = \mathrm{End}(P_R)$ and assume that $\mathrm{Im}(H)$ is closed under taking homomorphic images. Let $M \in \mathrm{Gen}(P_R)$ and let $h = (h_x)_{x \in X}$ an epimorphism of $P_R^{(X)}$ onto M. Then*

$$\sum_{x \in X} h_x A = \mathrm{Hom}_R(P, M).$$

Proof. We have in Mod-A the exact sequence

$$0 \to \sum_{x \in X} h_x A \xrightarrow{i} \mathrm{Hom}_R(P, M) \to V \to 0.$$

By assumption $V \in \mathrm{Im}(H)$. Applying the exact functor $- \otimes_A P$ we get the exact sequence:

$$0 \to \Big(\sum_{x \in X} h_x A\Big) \otimes_A P \xrightarrow{T(i)} \mathrm{Hom}_R(P, M) \otimes_A P \to V \otimes_A P \to 0.$$

Since P_R is a W-module, ρ_M is an isomorphism (cf. Theorem 2.3 and Proposition 1.8). Therefore, by Azumaya's Lemma, $\mathrm{T}(i)$ is surjective so that $V \otimes_A P = 0$. It follows $V = 0$ since the bimodule ${}_AP_R$ represents a category equivalence between $\mathrm{Im}(H)$ and $\mathrm{Gen}(P_R)$.

3.7. DEFINITION. Recall that a module $P_R \in$ Mod-R is *Σ-quasi-projective* if for every diagram with exact row

$$\begin{tikzcd} & P_R \arrow[d, "f"] & \\ P_R^{(X)} \arrow[r, "h"'] & M \arrow[r] & 0 \end{tikzcd}$$

there exists $\alpha \in \mathrm{Hom}_R(P_R, P_R^{(X)})$ such that $f = h \circ \alpha$.

3.8. DEFINITION. Let $P_R \in$ Mod-R, $A = \mathrm{End}(P_R)$. Recall that P_R is *self-small* if for every set $X \neq \emptyset$ we have

$$\mathrm{Hom}_R(P_R, P_R^{(X)}) \cong \mathrm{Hom}_R(P_R, P_R)^{(X)} = A^{(X)}$$

canonically.

3.9. PROPOSITION. *Let* $P_R \in$ Mod-R, $A = \mathrm{End}(P_R)$. *The following conditions are equivalent*:

(*a*) *For every* $M \in \mathrm{Gen}(P_R)$ *and for every epimorphism* $h = (h_x)_{x \in X}: P_R^{(X)} \to M \to 0$ *we have*:

$$\sum_{x \in X} h_x A = \mathrm{Hom}_R(P, M)$$

(*b*) P_R *is Σ-quasi-projective and self-small.*

PROOF. $(a) \Rightarrow (b)$. Consider the diagram (1) of 3.7. By assumption we have $f = \sum_{x \in X} h_x a_x$, with $a_x \in A$ and almost all a_x's vanish. Consider the morphism $g: P \to P_R^{(X)}$ given by $g = (a_x)_{x \in X}$. Then $f = h \circ g$. Therefore P_R is Σ-quasi-projective. Let us show that P_R is self-small. Let $i_x: P_R \to P_R^{(X)}$ the x-th inclusion and consider the diagram with exact row

$$\begin{tikzcd} & & P_R \arrow[d, "f"] & \\ 0 \arrow[r] & P_R^{(X)} \arrow[r, "i = (i_x)_{x \in X}"'] & P_R^{(X)} \arrow[r] & 0 \end{tikzcd}$$

We have $f = \sum_{x \in X} i_x a_x$ with $a_x \in A$ and almost all a_x's vanish. Let $g = (a_x)_{x \in X}$. Then $g \in A^{(X)}$ and $f = i \circ g$, hence $\mathrm{Hom}_R(P_R, P_R^{(X)}) \cong A^{(X)}$.

$(b) \Rightarrow (a)$. Let $f \in \mathrm{Hom}_R(P, M)$ and let $h\colon P_R^{(X)} \to M \to 0$ be an epimorphism. Then there exists a morphism $g\colon P_R \to P_R^{(X)}$ such that $f = h \circ g$. On the other hand $g = (a_x)_{x \in X}$ with $a_x \in A$ and almost all a_x's vanish, hence $f \in \sum_{z \in X} h_x A$.

3.10. THEOREM. *Let P_R be a W-module, $A = \mathrm{End}(P_R)$ and assume that* $\mathrm{Im}(H)$ *is closed under taking homomorphic images. Then* $\mathrm{Im}(H) = \text{Mod-}A$.

PROOF. We have $T(A^{(X)}) = A^{(X)} \otimes_A P \cong P_R^{(X)}$ in a natural way. By Lemma 3.6 and Proposition 3.9 P_R is self-small, hence $H(P_R^{(X)}) = \mathrm{Hom}_R(P_R, P_R^{(X)}) \cong A^{(X)}$. Thus $A^{(X)} \in \mathrm{Im}(H)$. Let $L \in \text{Mod-}A$. There exists an exact sequence $A^{(X)} \to L \to 0$, so that $L \in \mathrm{Im}(H)$.

4. The torsion theory $(\mathrm{Im}(T), \mathcal{D}(K_A))$.

From now on we assume the reader familiar with some elementary facts on torsion theories. See [St] or [N].

4.1. In all this section P_R is a W-module with $A = \mathrm{End}(P_R)$. Set, as usual, $T = - \otimes_A P$ and $H = \mathrm{Hom}_R(P_R, -)$. The bimodule ${}_AP_R$ represents an equivalence between $\mathrm{Im}(H)$ and $\mathrm{Gen}(P_R) = \overline{\mathrm{Gen}}(P_R)$.

4.2. Consider the following subcategory of Mod-A

$$\mathrm{Ker}(T) = \{L \in \text{Mod-}A\colon\ L \otimes_A P = 0\}.$$

Clearly $\mathrm{Im}(H) \cap \mathrm{Ker}(T) = 0$.

4.3. LEMMA. $\mathrm{Ker}(T)$ *is a localizing subcategory of* Mod-A, *i.e.* $\mathrm{Ker}(T)$ *is the torsion class for a hereditary torsion theory on* Mod-A.

PROOF. It is obvious that Ker (T) is closed under taking homomorphic images, direct sums and extensions. On the other hand, since ${}_AP$ is flat, Ker(T) is closed under taking submodules.

The Gabriel filter Γ canonically associated to the localizing subcategory $\operatorname{Ker}(T)$ is given by setting

$$\Gamma = \left\{ I \leq A_A \colon \frac{A}{I} \in \operatorname{Ker}(T) \right\}.$$

Clearly

$$\Gamma = \{ I \leq A_A \colon IP = P \}.$$

Let $L \in$ Mod-A. The torsion submodule $t_\Gamma(L)$ of L is defined by setting

$$t_\Gamma(L) = \{ x \in L \colon \operatorname{Ann}_A(x) \in \Gamma \}.$$

Then the category of torsion-free modules is

$$\mathcal{F}_\Gamma = \{ L \in \text{Mod-}A \colon t_\Gamma(L) = 0 \}.$$

For every $L \in$ Mod-A, $L/t_\Gamma(L)$ is torsion free. If no confusion arises, we write $t(L)$ instead of $t_{\Gamma(L)}$

Let Q_R be a fixed, but arbitrary, injective cogenerator of Mod-R, $K_A = \operatorname{Hom}_R(P_R, Q_R)$, $\mathcal{D}(K_A)$ the subcategory of Mod-A, cogenerated by K_A. Since ${}_AP$ is flat, K_A is injective in Mod-A.

4.4. LEMMA.

$$\operatorname{Ker}(T) = \{ L \in \text{Mod-}A \colon \operatorname{Hom}_A(L, K_A) = 0 \}.$$

PROOF. For every $L \in$ Mod-A we have the canonical isomorphisms:

$$\operatorname{Hom}_A(L, K_A) = \operatorname{Hom}_A(L, \operatorname{Hom}_R(P, Q)) \cong \operatorname{Hom}_R(L \otimes_A P, Q_R).$$

Since Q_R is a cogenerator in Mod-R we have

$$\operatorname{Hom}_A(L, K_A) = 0 \Leftrightarrow L \otimes_A P = 0 \Leftrightarrow L \in \operatorname{Ker}(T).$$

4.5. PROPOSITION.

$$\mathcal{F}_\Gamma = \mathcal{D}(K_A).$$

PROOF. Let $L \in \mathcal{F}_\Gamma$. Then $t_\Gamma(L) = 0$. Let $l \in L$, $l \neq 0$. Then $lA \notin \operatorname{Ker}(T)$ hence, by Lemma 4.4, $\operatorname{Hom}_A(lA, K_A) \neq 0$. Let $f \colon lA \to K_A$ a non zero morphism. Since K_A is injective in Mod-A, f extends to a morphism $\bar{f} \colon L \to K_A$ and $\bar{f}(l) \neq 0$. It follows $L \in \mathcal{D}(K_A)$.

Conversely let $L \in \mathcal{D}(K_A)$ and let $L' \leq L$ such that $L' \in \operatorname{Ker}(T)$. By

Lemma 4.4 we have $\mathrm{Hom}_A(L', K_A) = 0$. On the other hand there exists an exact sequence $0 \to L \to K_A^X$ where X is a suitable set. Then $L' = 0$, so that $L \in \mathcal{F}_\Gamma$.

4.6. COROLLARY. *(a) The torsion theory* $(\mathrm{Ker}(T), \mathcal{C}(K_A))$ *is cogenerated by the injective module* K_A.

(b) Since $\mathrm{Im}(H) \subseteq \mathcal{C}(K_A)$, *the modules in* $\mathrm{Im}(H)$ *are torsion-free.*

4.7. PROPOSITION. *For every* $L \in \text{Mod-}A$ *consider the canonical morphism* $\sigma_L \colon L \to \mathrm{Hom}_R(P_R, L \otimes_A P)$. *Then:*

$$t_\Gamma(L) = \mathrm{Ker}(\sigma_L).$$

PROOF. We have:

$$\mathrm{Ker}(\sigma_L) = \{l \in L \colon l \otimes p = 0, \forall p \in P\} = \{l \in L \colon lA \otimes_A P = 0\} =$$

$$= \{l \in L \colon lA \in \mathrm{Ker}(T)\} = t_\Gamma(L).$$

4.8. Let $L \in \text{Mod-}A$, I, $J \in \Gamma$, $I \geq J$. Consider the natural morphism

$$\mathrm{Hom}_A(I, L) \to \mathrm{Hom}_A(J, L)$$

given by restrictions. For every $L \in \text{Mod-}A$ set:

$$L_\Gamma = \varinjlim_{I \in \Gamma} \mathrm{Hom}_A\left(I, \frac{L}{t_\Gamma(L)}\right)$$

and, since A is torsion-free

$$A_\Gamma = \varinjlim_{I \in \Gamma} \mathrm{Hom}_A(I, A).$$

It is well known that L_Γ is a right A-module, A_Γ is a ring and moreover L_Γ is a right A_Γ-module.

A_Γ is called the *ring of quotients* of A and L_Γ the *module of quotients* of L with respect to the Gabriel filter Γ. For every $L \in \text{Mod-}A$, L_Γ is also called the *localization* of L at Γ.

For every $L \in \text{Mod-}A$ there exists a canonical morphism $\varphi_L \colon L \to L_\Gamma$

such that

$$\mathrm{Ker}(\varphi_L) = t_\Gamma(L), \quad \frac{L_\Gamma}{\varphi_L(L)} \in \mathrm{Ker}(T), \qquad \varphi_L(L) \in \mathcal{D}(K_A)$$

$\varphi_A: A \to A_\Gamma$ is a ring morphism.

4.9. Lemma. *Let* $I \in \Gamma$.
Then $\mathrm{Hom}_A(A/I, A) = 0$ *and* $\mathrm{Ext}^1_A(A/I, A) = 0$.

Proof. See [WW] Proposition 1.2.

4.10. Corollary. *The canonical morphism* $\varphi_A: A \to A_\Gamma$ *is a ring isomorphism.*

Proof. By Corollary 4.6 A is torsion-free. Let $I \in \Gamma$ and let $\alpha_I: I \to A$ be the canonical inclusion. By Lemma 4.9 the exact sequence

$$0 \to I \xrightarrow{\alpha_I} A \to A/I \to 0$$

gives rise to the exact sequence:

$$0 = \mathrm{Hom}_A(A/I, A) \to \mathrm{Hom}_A(A, A) \xrightarrow{\alpha^*_I} \mathrm{Hom}_A(I, A) \to \mathrm{Ext}^1_A(A/I, A) = 0.$$

Therefore $\alpha^*_I: \mathrm{Hom}_A(A, A) \to \mathrm{Hom}_A(I, A)$ is an isomorphism i.e. any morphism $I \to A$ extends uniquely to an element of A. Then, if $I, J \in \Gamma$ and $I \geqslant J$, the restriction map $\mathrm{Hom}_A(I, A) \to \mathrm{Hom}_A(J, A)$ is an isomorphism.

4.11. Definitions. Recall that a module $L \in$ Mod-A is *Γ-injective* if for every $I \in \Gamma$ the restriction morphism

$$\mathrm{Hom}_A(A, L) \to \mathrm{Hom}_A(I, L) \tag{1}$$

is surjective.

L is Γ-injective if and only if $\mathrm{Ext}^1_A(N, L) = 0$ for every $N \in \mathrm{Ker}(T)$.

A module $L \in$ Mod-A is called *Γ-closed* if for every $I \in \Gamma$ the above morphism (1) is an isomorphism.

The following results are classical in torsion theories.

4.12. Theorem. *Let* $L \in$ Mod-A. *The following conditions are equivalent:*

(*a*) L *is Γ-closed;*

(*b*) $L \in \mathcal{D}(K_A)$ *and* L *is* Γ*-injective;*

(*c*) *for every morphism* $\alpha\colon U \to V$ *in* Mod-A *such that* $\mathrm{Ker}(\alpha) \in \mathrm{Ker}(T)$ *and* $\mathrm{Coker}(\alpha) \in \mathrm{Ker}(T)$, *the transposed morphism* $\mathrm{Hom}_A(V, L) \to \mathrm{Hom}_A(U, L)$ *is an isomorphism;*

(*d*) *the canonical morphism* $\varphi_L\colon L \to L_\Gamma$ *is an isomorphism.*

4.13. COROLLARY. *For every* $L \in$ Mod-A, L_Γ *is* Γ*-closed.*

5. A characterization of Im (H).

5.1. In all this section we work in situation 4.1.

Denote by Mod-(A,Γ) the subcategory of Mod-A whose objects are all the Γ-closed modules in Mod-A. By Theorem 4.12 we can write

$$\text{Mod-}(A, \Gamma) = \{L \in \text{Mod-}A\colon\ L = L_\Gamma\}.$$

Our main result is the following theorem which, together with Theorem 2.6, gives easily the Popescu-Gabriel Theorem in our setting.

5.2. THEOREM. *Let* $P_R \in$ Mod-R *be a W-module,* $A = \mathrm{End}(P_R)$, $H = \mathrm{Hom}_R(P_R, -)$, $T = - \otimes_A P$. *Then for every* $L \in$ Mod-A *we have*

$$L_\Gamma = HT(L).$$

PROOF. For every $L \in$ Mod-A, we have

$$T(L_\Gamma) = T(L). \tag{1}$$

Indeed, consider the exact sequence

$$0 \to t_\Gamma(L) \to L \xrightarrow{\varphi_L} L_\Gamma \to L_\Gamma/\varphi_L(L) \to 0.$$

Tensoring by ${}_AP$ and since $t_\Gamma(L)$ and $L_\Gamma/\varphi_L(L)$ are in $\mathrm{Ker}(T)$, we get (1).

We now prove that, for every $L \in$ Mod-A, $L_\Gamma \in \mathrm{Im}(H)$ from which it will follow

$$L_\Gamma \cong HT(L)$$

by Theorem 2.3.

Indeed assume $L = L_\Gamma$. Since $L \in \mathcal{D}(K_A)$ (cf. Theorem 4.12), σ_L is injective. Consider the exact sequence

$$0 \to L \xrightarrow{\sigma_L} HT(L) \to \mathrm{Coker}(\sigma_L) \to 0. \tag{2}$$

Since T and H are adjoint functors there exists the commutative diagram

$$\begin{tikzcd} T(L) \arrow[r, "T(\sigma_L)"] \arrow[dr, "1_{T(L)}"'] & THT(L) \arrow[d, "\rho_{T(L)}"] \\ & T(L) \end{tikzcd}$$

Since $T(L) \in \mathrm{Gen}(P_R)$, $\rho_{T(L)}$ is an isomorphism hence $T(\sigma_L)$ is an isomorphism too. Applying T in (2) we get the exact sequence

$$0 \to T(L) \xrightarrow{T(\sigma_L)} THT(L) \to T(\mathrm{Coker}(\sigma_L)) = 0\,.$$

It follows $\mathrm{Coker}(\sigma_L) \in \mathrm{Ker}(T)$. Since L is Γ-injective, the exact sequence (2) splits hence:

$$HT(L) \cong L \oplus \mathrm{Coker}(\sigma_L);$$

therefore $\mathrm{Coker}(\sigma_L) = 0$ because HT(L) is torsion-free. Thus σ_L is an isomorphism and $L \in \mathrm{Im}(H)$.

5.3. Corollary. *Under the assumptions of Theorem* 5.2

$$\mathrm{Im}(H) = \mathrm{Mod}-(A, \Gamma)\,.$$

Proof. Let $L \in \mathrm{Im}(H)$. Then $L \cong HT(L)$, hence $L = L_\Gamma$. If $L = L_\Gamma$ then $L = HT(L)$, hence $L \in \mathrm{Im}(H)$.

6. The trace ideal of ${}_AP$ in A.

6.1. Let P_R be a W-module, $A = \mathrm{End}(P_R)$. Define the trace ideal τ of ${}_AP$ in A by setting

$$\tau = \sum \{\mathrm{Im}(f)\colon\ f \in \mathrm{Hom}_A(P, A)\}\,;$$

τ is a two-sided ideal of A.

6.2 Lemma ([WW], Proposition 1.5 and Theorem 1.6). *Let P_R be a W-module, $A = \mathrm{End}(P_R)$. Then* $\tau \subseteq \bigcap_{I \in \Gamma} I$.

If moreover P_R is a generator of Mod-R, *then*:

a) $\tau P = P$ *so that* $I \in \Gamma$ *if and only if* $I \supseteq \tau$;

b) $\tau^2 = \tau$;

c) the left annihilator of τ *is* 0;

d) τ *is finitely generated as a two-sided ideal;*

e) τ *is essential as a right ideal.*

6.3 COROLLARY. *Let* P_R *be a generator of* Mod-R, $A = \mathrm{End}(P_R)$. *Then for every* $L \in$ Mod-A

$$L_\Gamma = \mathrm{Hom}\left(\tau, \frac{L}{t_\Gamma(L)}\right).$$

7. An example: closed spectral subcategories of Mod-R.

7.1. Let $\mathcal{G}_R$ be a closed subcategory of Mod-R, P_R a generator of $\mathcal{G}_R$, $A = \mathrm{End}(P_R)$. Set, as usual, $T = - \otimes_A P$, $H = \mathrm{Hom}_R(P_R, -)$. Let Γ be the Gabriel filter associated to the hereditary torsion theory $(\mathrm{Ker}(T), \mathcal{D}(K_A))$. Then $\mathcal{G}_R$ is naturally equivalent to the subcategory $\mathrm{Im}(H) = \mathrm{Mod} - (A, \Gamma)$ of Mod-A.

Recall that the subcategory $\mathrm{Mod} - (A, \Gamma)$ is closed under taking injective envelopes and direct products in Mod-A.

7.2. We are interested in finding conditions in order that every module $L \in \mathrm{Mod} - (A, \Gamma)$ is injective in Mod-(A, Γ) or, equivalently, in Mod-A.

7.3 LEMMA. *The sequence in* Mod-(A, Γ)

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{1}$$

is exact in Mod-(A, Γ) *if and only if*

1) f *is injective;*

2) $\mathrm{Im}(f) = \mathrm{Ker}(g)$;

3) $N/\mathrm{Im}(g) \in \mathrm{Ker}(T)$.

PROOF. Assume that (1) is exact in Mod-(A, Γ). Then we have the exact sequence

$$0 \to T(L) \xrightarrow{T(f)} T(M) \xrightarrow{T(g)} T(N) \to 0 \quad \text{in } \mathcal{G}_R. \tag{2}$$

Since $\mathcal{G}_R$ is closed, (2) is exact in Mod-R. Therefore the sequence

$$0 \to HT(L) \xrightarrow{HT(f)} HT(M) \xrightarrow{HT(g)} HT(N)$$

is exact in Mod-A; thus f is injective and $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Assume that $N/\mathrm{Im}(g) \notin \mathrm{Ker}(T)$. Then we have the exact sequence in Mod-A:

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to N/\mathrm{Im}(g) \to 0 .$$

Applying T we get $T(N/\mathrm{Im}(g)) \neq 0$, in contrast with (2).

Conversely, if conditions 1), 2) and 3) hold for the sequence (1), then the sequence (2) is exact in $\mathcal{G}_R$ and (1) is exact in Mod-(A, Γ).

7.4. Assume that every module in Mod-(A, Γ) is injective. Let

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be an exact sequence in Mod-(A,Γ). Since L is injective, we have

$$M = L \oplus L' \qquad \text{in Mod-}A\,,$$

where $L' \cong \mathrm{Im}(g) \leqslant N$. Let us show that

$$L' \cong N \text{ cononically.}$$

Observe that $L' \in \mathrm{Mod}-(A, \Gamma)$. In fact L' is torsion free and, being injective, it is Γ-injective. We have $N \cong L' \oplus L''$, with $L'' \cong N/\mathrm{Im}(g)$. Since $L' \in \mathcal{D}(K_A)$ and $L'' \in \mathrm{Ker}(T)$ we get $N \cong L'$.

7.5 Proposition. *Assume that every module in* Mod-(A, Γ) *is injective. The sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

with L, M, $N \in$ $\mathrm{Mod}-(A, \Gamma)$ *is exact in* Mod-A *if and only if it is exact in* Mod-A.

In this case (1) *splits.*

7.6 Lemma. *Let $M \in \mathcal{G}_R =$* $\mathrm{Gen}(P_R)$, *$N \in$* Mod-R *and let $f\colon M \to N$ be a morphism. Then* $\mathrm{Im}(f) \leqslant t_P(N)$.

Proof. Assume that $M = P_R^{(X)}$, where $X \neq \emptyset$ is a set. Let $h\colon P^{(X)} \to$ $\to N$ be a morphism. Then $h = (h_x)_{x \in X}$, with $h_x \in \mathrm{Hom}_R(P, N)$. Let $p \in P^{(X)}$. Then $p = (p_x)_{x \in X}$, with $p_x \in P$ and $p_x = 0$ for almost all $x \in X$.

We have

$$h(p) = \sum_{x \in X} h_x(p_x) \in t_p(N).$$

Let $M \in \mathrm{Gen}(P_R)$, $f \in \mathrm{Hom}_R(M, N)$. There exists a diagram

$$P^{(X)} \xrightarrow{h} M \xrightarrow{f} N$$

with h a surjective morphism. It is $f \circ h \in \mathrm{Hom}_R(P^{(X)}, N)$, hence $\mathrm{Im}(f \circ h) \leq t_P(N)$ and $\mathrm{Im}(f \circ h) = \mathrm{Im}(f)$.

7.7 Proposition. *Let $\mathcal{G}_R$ be a closed subcategory of* Mod-R, *P_R a generator of $\mathcal{G}_R$, $A = \mathrm{End}(P_R)$. The following conditions are equivalent:*

(a) *every module in* Mod-(A, Γ) *is injective*;

(b) *$\mathcal{G}_R$ is a spectral category.*

In this case every module in $\mathcal{G}_R$ is semisimple.

Proof. $(a) \Rightarrow (b)$ By Proposition 7.5 every short exact sequence in $\mathcal{G}_R$ splits. Therefore such a sequence splits in Mod-R. Then every module in $\mathcal{G}_R$ is semisimple so that $\mathcal{G}_R$ is spectral.

$(b) \Rightarrow (a)$ Let $L \in \mathrm{Mod} - (A, \Gamma) = \mathrm{Im}(H)$. Then $L = H(M)$, with $M \in \mathcal{G}_R$. Since Q_R is a cogenerator in Mod-R, there exists an exact sequence in Mod-R

$$0 \to M \to Q^X$$

where X is a suitable set. By Lemma 7.6, $\mathrm{Im}(f) \leq t_P(Q_R^X) \in \mathrm{Gen}(P_R) = \mathcal{G}_R$. Since $\mathcal{G}_R$ is spectral, M is a direct summand of $t_P(Q_R^X)$. Therefore $L = H(M)$ is a direct summand of $H(t_P(Q_R^X))$. On the other hand, $H(t_P(Q_R^X)) \cong H(Q_R^X) = K_A^X$ which is injective.

7.8 Proposition *Let $\mathcal{G}_R$ be a closed spectral subcategory of* Mod-R, *P_R a generator of $\mathcal{G}_R$ and $A = \mathrm{End}(P_R)$. Then:*

a) *for every $L \in$* Mod-A *the following conditions are equivalent:*

(i) $L \in \mathrm{Mod} - (A, \Gamma)$;

(ii) *L is a direct summand of a module of the form A^X, where X is a non empty set*;

b) *the ring A is von Neumann regular and right self-injective.*

PROOF. *a*) (*i*)$\Rightarrow$(*ii*) Let X be a non empty set. We show that $H(P_R^{(X)})$ is a direct summand of A^X. In fact:

$$H(P_R^{(X)}) \leqslant H(P_R^X) \cong H(t_P(P_R^X)) \cong A^X \in \mathrm{Mod}-(A, \Gamma).$$

Since $H(P_R^{(X)})$ is injective, $H(P_R^{(X)})$ is a direct summand of A^X. Let $L \in \mathrm{Mod}-(A, \Gamma)$ be an injective module. Then $L = H(M)$, for some $M \in \mathcal{G}_R$. Then H(M) is a direct summand of a module of the form $H(P_R^{(X)})$, hence L is a direct summand of A^X.

(ii)$\Rightarrow$(i) If L is a direct summand of A^X, L is torsion free and it is Γ-injective, being injective. Therefore $L \in \mathrm{Mod}-(A, \Gamma)$.

b) Since P_R is semisimple, A is von Neumann regular (cf. [St], Chap. I, Prop. 12.4). Clearly A is right self-injective.

7.9. Let $\mathcal{G}_R$ be a closed spectral subcategory of Mod-R, P_R a generator of $\mathcal{G}_R$ and $A = \mathrm{End}(P_R)$. In this case the filter Γ has a nice description using the trace ideal of ${}_AP$ in ${}_AA$.

7.10. Fix a simple module $S \in$ Mod-R and denote by $\Sigma(S)$ the spectral subcategory of Mod-R consisting of all semisimple modules which are a direct sum of copies of S.

Fix a positive cardinal number α. Then

$$P_R = S^{(\alpha)}$$

is a projective generator and an injective cogenerator of $\Sigma(S)$. Let $D = \mathrm{End}(S_R)$, $A = \mathrm{End}(P_R)$. Then D is a division ring and A is the ring of all $\alpha \times \alpha$ matrices, with entries in D, whose columns have only a finite number of non zero elements. It follows that $A \cong \mathrm{End}(D^{(\alpha)})$, where $D^{(\alpha)}$ is considered as a right vector space over the division ring D.

Let Γ be the usual Gabriel filter on A. Let τ be the trace ideal of ${}_AP$ in ${}_AA$:

$$\tau = \sum \{\mathrm{Im}(g)\colon\ g \in \mathrm{Hom}_A({}_AP, A)\}.$$

a) ${}_AP$ *is a semisimple module in* A-Mod.

PROOF. Since P_R is an injective cogenerator of $\Sigma(S)$, then P_R is strongly quasi-injective in the sense of [MO_1]. Applying Proposition 6.10 of [MO_1] we have

$$\mathrm{Soc}({}_AP) = \mathrm{Soc}(P_R) = P$$

and thus ${}_AP$ is semisimple.

Let L_ω be the minimal two-sided non zero ideal of A. As it is well known, L_ω consists of all the endomorphism of $D^{(\alpha)}$ whose image is finite dimensional. L_ω has the following properties:

i) $L_\omega = \mathrm{Soc}({}_AA) = \mathrm{Soc}(A_A)$;

ii) the right ideals of A containing L_ω are exactly the essential right ideals of A.

Therefore we have, by i),

$$\tau = \sum\{\mathrm{Im}(g)\colon\ g \in \mathrm{Hom}_A({}_AP, L_\omega)\}.$$

Thus $\tau \leqslant L_\omega$.

b) The trace ideal $\tau = L_\omega$.

PROOF. Let us show that $\tau \neq 0$; it will follow that $\tau = L_\omega$, since τ is two-sided and L_ω is the minimal two-sided non zero ideal of A.

Let J be a maximal right ideal of R such that $R/J \cong S_R$. The exact sequence

$$0 \to J \to R \to R/J \to 0$$

gives rise, by applying $\mathrm{Hom}_R(-, P_R)$, to the exact sequence

$$0 \to \mathrm{Hom}_R(R/J, P_R) = \mathrm{Ann}_P(J) \to {}_AP.$$

Thus

$$\mathrm{Ann}_P(J) \cong \mathrm{Hom}_R(S, P_R) \cong D^{(\alpha)} \hookrightarrow {}_AP,$$

since S_R is finitely generated. Therefore ${}_AP$ contains a direct summand of the form $\mathrm{Ann}_P(J) \cong \mathrm{Hom}_R(S, P_R) \cong D^{(\alpha)}$ and it is well known that $\mathrm{Hom}_A(D^{(\alpha)}, A) \neq 0$.

c) Let f be an endomorphism of P_R such that $\mathrm{Im}(f)$ *is finitely generated. Then $f \in L_\omega$.*

PROOF. In fact f may be represented by an $\alpha \times \alpha$ matrix having only a finite number of non zero rows. Then this matrix represents an endomorphism of $D^{(\alpha)}$ whose image is finite dimensional. Therefore $f \in L_\omega$.

d) $L_\omega P = P$; hence $L_\omega \in \Gamma$ and thus

$$\Gamma = \{I \leqslant A_A\colon\ I \geqslant \tau\}.$$

PROOF. Let $x \in P_R$, $x \neq 0$, and let f be the projection of P_R onto the submodule F generated by x, such that $f(x) = x$. Since F is finitely generated, $f \in L_\omega$. Thus $L_\omega P = P$.

The last statement follows from Lemma 6.2.

We now consider closed spectral subcategories of Mod-R in the general case.

7.11 PROPOSITION. *Let $\mathcal{G}_R$ be a closed spectral subcategory of* Mod-R, *P_R a generator of $\mathcal{G}_R$ and $A = \mathrm{End}(P_R)$. Let Γ be the usual Gabriel filter on A and τ be the trace of ${}_AP$ in ${}_AA$. Then:*

a) $\mathrm{Soc}(A_A) = \mathrm{Soc}({}_AA) = \tau \neq 0$;

b) $\tau \in \Gamma$ *and* $\Gamma = \{I \leq A_A : I \geq \tau\}$.

Consequently Γ consists of all essential right ideals of A.

PROOF. Let $(S_\delta)_{\delta \in \Delta}$ be a system of representatives of all non isomorphic simple modules in $\mathcal{G}_R$. Set $D_\delta = \mathrm{End}_R(S_\delta)$. We have

$$P_R = \bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)},$$

where the α_δ's are non zero cardinal numbers. P_R is a projective generator and an injective cogenerator of $\mathcal{G}_R$. Next we have:

$$A = \mathrm{Hom}_R(P_R, P_R) \cong \mathrm{Hom}_R\left(\bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)}, \bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)}\right) \cong$$

$$\cong \prod_{\delta \in \Delta} \mathrm{Hom}_R(S_\delta^{(\alpha_\delta)}, S_\delta^{(\alpha_\delta)}) \cong \prod_{\delta \in \Delta} A_\delta,$$

where $A_\delta = \mathrm{End}_R(S_\delta^{(\alpha_\delta)})$.

Let τ be the trace ideal of ${}_AP$ in A; note that $\bigoplus_{\delta \in \Delta} A_\delta$ is essential in $A = \prod_{\delta \in \Delta} A_\delta$. Therefore:

$$\mathrm{Soc}(A_A) = \mathrm{Soc}\left(\bigoplus_{\delta \in \Delta} A_\delta\right) = \bigoplus_{\delta \in \Delta} \mathrm{Soc}(A_\delta) = \bigoplus_{\delta \in \Delta} L_\omega(\delta),$$

where $L_\omega(\delta)$ is the smallest two-sided ideal of the ring A_δ. Hence

$$\mathrm{Soc}(A_A) = \mathrm{Soc}({}_AA).$$

Then

$$\tau = \sum\left\{\mathrm{Im}(g) \colon g \in \mathrm{Hom}_A\left({}_AP, \bigoplus_{\delta \in \Delta} L_\omega(\delta)\right)\right\}.$$

Hence

$$\tau = \bigoplus_{\delta \in \Delta} L_\omega(\delta) = \mathrm{Soc}(A_A).$$

As we know, $\tau \subseteq \bigcap\{I \colon I \in \Gamma\}$. Let us show that $\tau \in \Gamma$. In fact

$$\Big(\bigoplus_{\delta' \in \Delta} L_\omega(\delta')\Big)\Big(\bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)}\Big) = \bigoplus_{\delta \in \Delta} L_\omega(\delta) S_\delta^{(\alpha_\delta)} = \bigoplus_{\delta \in \Delta} S_\delta^{(\alpha_\delta)} .$$

7.12 REMARK. We think that a number of more interesting examples may be constructed from the recent paper of Albu and Wisbauer [AW].

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