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Wiener Tauberian Theorems for Ultradistributions.

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SUMMARY - The purpose of this paper is the extension of Wiener Tauberian theorems for distributions ([6]) on ultradistribution spaces. Because of that, we give the versions of Beurling's and Wiener's theorems for bounded ultradistributions. The corollary of our main theorem is the following one. Let f be an ultradistribution such that f/c is a bounded ultradistribution, where c is a smooth function which behaves as $L(e^x)e^{ax}$, $x \rightarrow \infty$, L is a slowly varying function at ∞ and $a \in \mathbf{R}$. If for an ultradifferentiable function ϕ with the property $\mathcal{F}[\phi](\xi - i\alpha) \neq 0$, $\xi \in \mathbf{R}$,

$$\lim_{x \rightarrow \infty} \frac{(f * \phi)(x)}{L(e^x)e^{ax}} = a \int \phi(t) e^{-at} dt, \quad a \in \mathbf{R},$$

then for every ultradifferentiable function ψ

$$\frac{(f * \psi)(x)}{L(e^x)e^{ax}} \rightarrow a \int \psi(t) e^{-at} dt, \quad x \rightarrow \infty.$$

1. Notation and notions.

With N and $\mathbf{R}$ are denoted the sets of natural and real numbers; $N_0 = N \cup \{0\}$. If f is a function on $\mathbf{R}$, then $\check{f}$ denotes the function defined by $\check{f}(x) = f(-x)$, $x \in \mathbf{R}$. $\mathcal{C}^\infty$ denotes the space of smooth functions on $\mathbf{R}$ and $\mathcal{L}^1$ is the space of Lebesgue integrable functions (classes) on $\mathbf{R}$ with the usual norm $\|\cdot\|_{\mathcal{L}^1}$. For an f from $\mathcal{L}^1$ the Fourier transform is denoted by $\mathcal{F}f$ or $\hat{f}$. $\mathcal{L}_{loc}^1$ and $\mathcal{L}^\infty$ are defined in a usual way.

We shall always denote by L a slowly varying function ([1]). Recall,

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L is measurable, positive and

$$\frac{L(xh)}{L(h)} \rightarrow 1, \quad h \rightarrow \infty, \quad x > 0.$$

For every $\delta > 0$ there is $C_\delta > 0$ such that

$$(1) \quad \frac{1}{C_\delta} \min \left\{ \left(\frac{x}{y} \right)^\delta, \left(\frac{y}{x} \right)^\delta \right\} \leq \frac{L(x)}{L(y)} \leq C_\delta \max \left\{ \left(\frac{x}{y} \right)^\delta, \left(\frac{y}{x} \right)^\delta \right\},$$

$$x > 0, \quad y > 0.$$

For the notation and properties of the spaces $\mathcal{O}^{(M_\alpha)}$, $\mathcal{O}^{\{M_\alpha\}}$ and their strong duals $\mathcal{O}'^{(M_\alpha)}$ (the space of Beurling ultradistributions) and $\mathcal{O}'^{\{M_\alpha\}}$ (the space of Roumieu ultradistributions) we refer to [3]. We shall assume that the sequence M_α satisfies conditions (M.1), (M.2) and (M.3)' ([3]).

As in [5] we put

$$\mathcal{O}_{\mathcal{E}^1}^{(M_\alpha)} = \text{proj}_{h \rightarrow \infty} \mathcal{O}_{\mathcal{E}^1, h}^{M_\alpha}, \quad \mathcal{O}_{\mathcal{E}^1}^{\{M_\alpha\}} = \text{ind}_{h \rightarrow 0} \mathcal{O}_{\mathcal{E}^1, h}^{M_\alpha},$$

where $\mathcal{O}_{\mathcal{E}^1, h}^{M_\alpha}$, $h > 0$, is a Banach space of smooth functions φ on $\mathbf{R}$ with the finite norm

$$\|\varphi\|_{\mathcal{E}^1, h} = \sup \left\{ \frac{h^\alpha}{M_\alpha} \|\phi^{(\alpha)}\|_{\mathcal{E}^1}; \alpha \in \mathbf{N}_0 \right\}.$$

The common notation for (M_α) and $\{M_\alpha\}$ will be $*$. The space $\mathcal{O}^*$ is dense in $\mathcal{O}_{\mathcal{E}^1}^*$ and the inclusion mapping is continuous. The strong dual of $\mathcal{O}_{\mathcal{E}^1}^*$ is denoted by $\mathcal{B}'^*$.

The spaces of tempered ultradistributions are defined as the strong duals of the following testing function spaces ([4])

$$\mathcal{S}^{(M_\alpha)} = \text{proj}_{h \rightarrow \infty} \mathcal{S}_h^{M_\alpha}, \quad \mathcal{S}^{\{M_\alpha\}} = \text{ind}_{h \rightarrow 0} \mathcal{S}_h^{M_\alpha},$$

where $\mathcal{S}_h^{M_\alpha}$, $h > 0$, is the Banach space of smooth functions φ on $\mathbf{R}$ with the finite norm:

$$\gamma_h(\varphi) = \sup \left\{ \frac{h^{\alpha+\beta}}{M_\alpha M_\beta} \|(1+x^2)^{\alpha/2} \varphi^{(\beta)}\|_{\mathcal{E}^\infty}, \alpha, \beta \in \mathbf{N}_0 \right\}.$$

The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto $\mathcal{S}^*$ and $\mathcal{O}^*$ is dense in $\mathcal{S}^*$. Clearly, $\mathcal{S}^*$ is dense in $\mathcal{O}_{\mathcal{E}^1}^*$ and the inclusion mapping is continuous.

Recall from [2] that a sequence of continuous and bounded functions

f_n on $\mathbf{R}$ converges narrowly to a continuous bounded function f_0 if and only if f_n converges to f_0 uniformly on bounded sets in $\mathbf{R}$ and

$$\|f_n\|_{\mathcal{E}^\infty} \rightarrow \|f_0\|_{\mathcal{E}^\infty}, \quad n \rightarrow \infty.$$

We shall always assume that $\alpha, \beta \in \mathbf{R}$ and $\alpha > \beta$. Put

$$(2) \quad c_0(x) = \begin{cases} L(e^x) e^{\alpha x}, & x \geq 0, \\ e^{\beta x}, & x < 0. \end{cases}$$

The following regularization of this function will be used.

$$(3) \quad c(x) = (c_0 * \omega)(x), \quad x \in \mathbf{R},$$

where $\omega \in \mathcal{O}^*$, $\text{supp } \omega \subset [-1, 1]$, $\omega \geq 0$ and $\int \omega(t) dt = 1$.

We shall denote by η a function from $\mathcal{E}^{\mathbf{R}}_*$ with the properties

$$(4) \quad \eta(x) = 1, \quad x > x_0 > 0, \quad \eta(x) = 0, \quad x < -x_0.$$

2. Assertions.

THEOREM 1. *Let $f \in \mathcal{B}'^*$ and $K \in \mathcal{O}^*_{\mathcal{E}^1}$ such that $\mathcal{F}[K](\xi) \neq 0$, $\xi \in \mathbf{R}$. If*

$$\lim_{x \rightarrow \infty} (f * K)(x) = a \int_{\mathbf{R}} K(t) dt, \quad a \in \mathbf{R},$$

*then for every $\psi \in \mathcal{O}^*_{\mathcal{E}^1}$,*

$$\lim_{x \rightarrow \infty} (f * \psi)(x) = a \int_{\mathbf{R}} \psi(t) dt.$$

THEOREM 2. *Let $f \in \mathcal{O}'^*$ and $K \in C^\infty$. Assume:*

(i) $f/c \in \mathcal{B}'^*$.

(ii) *There exists $\delta > 0$ such that*

$$\eta \check{K} e^{(\alpha + \delta) \cdot}, (1 - \eta) \check{K} e^{\beta \cdot} \in \mathcal{O}^*_{\mathcal{E}^1}.$$

(iii) $\mathcal{F}[K](\xi - i\alpha) \neq 0$, $\xi \in \mathbf{R}$.

(iv) $\lim_{x \rightarrow \infty} (f * K)(x) / (L(e^x) e^{\alpha x}) = a \int_{\mathbf{R}} K(t) e^{-\alpha t} dt, \quad a \in \mathbf{R}.$

Then for every $\psi \in C^\infty$ for which

$$(*) \quad \eta \check{\psi} e^{(\alpha + \delta) \cdot}, \quad (1 - \eta) \check{\psi} e^{\beta \cdot} \in \mathcal{O}_{\mathbb{R}^1}^*,$$

there holds

$$\lim_{x \rightarrow \infty} \frac{(f * \psi)(x)}{L(e^x) e^{\alpha x}} = a \int \psi(t) e^{-\alpha t} dt.$$

REMARK. It is an open problem whether the assumption that the set $\{(f(\cdot + h))/c(h); h \in \mathbf{R}\}$ is bounded in $\mathcal{O}'^*$ implies that $f/c \in \mathcal{B}'^*$. Note that for distributions the corresponding assertion holds (see [7]).

COROLLARY 1. Let $f \in \mathcal{O}'^*$ such that $f/c \in \mathcal{B}'^*$ and let $\phi \in \mathcal{O}^*$ such that $\mathcal{F}[\phi](\xi - i\alpha) \neq 0$, $\xi \in \mathbf{R}$. If

$$\lim_{x \rightarrow \infty} \frac{(f * \phi)(x)}{L(e^x) e^{\alpha x}} = a \int \phi(t) e^{-\alpha t} dt, \quad a \in \mathbf{R},$$

then for every $\psi \in \mathcal{O}^*$

$$\lim_{x \rightarrow \infty} \frac{(f * \psi)(x)}{L(e^x) e^{\alpha x}} = a \int \psi(t) e^{-\alpha t} dt.$$

3. Proofs.

PROOF OF THEOREM 1. First we need the following version of Beurling's theorem ([2]) for bounded ultradistributions.

«Let $f \in \mathcal{B}'^*$. A point ξ_0 belongs to $\text{supp } \widehat{f}$ if and only if there is a sequence of functions $\{\varphi_n\}$ from $\mathcal{S}^*$ such that

$$f_n(x) = (f * \varphi_n)(x), \quad x \in \mathbf{R}, \quad n \in \mathbf{N},$$

converges narrowly to $f_0(x) = e^{ix\xi_0}$, $x \in \mathbf{R}$, $n \rightarrow \infty$ ».

The proof of this assertion is the same as for bounded distributions since all the properties of Schwartz's test functions which were used in [2, pp. 230-231], have been proved in [3] and [5] for ultradifferentiable functions. The same holds for the next assertion, based on the previous one, which is analogous to the Theorem on p. 232 in [2].

«Let $f \in \mathcal{B}'^*$ and $K \in \mathcal{O}_{\varepsilon^1}^*$. If $K * f \equiv 0$ on $\mathbf{R}$, then $\widehat{K}(\xi) = 0$ for $\xi \in \text{supp } \widehat{f}$ ».

First, we shall prove that the set $\mathcal{M}$ which consists of finite linear combinations of translations of $K \in \mathcal{O}_{\varepsilon^1}^*$ is dense in $\mathcal{O}_{\varepsilon^1}^*$. By the property of dual pairing, $\mathcal{M}$ is dense in $\mathcal{O}_{\varepsilon^1}^*$ if and only if for every $S \in \mathcal{B}'^*$, $S * \check{K} = 0 \Leftrightarrow S = 0$. For, if $\mathcal{M}$ is not dense, there exists an $S_0 \in \mathcal{B}'^*$, $S_0 \neq 0$ such that $S_0 * \check{K} = 0$. Thus $\mathcal{F}[\check{K}](\xi) = 0$, $\xi \in \text{supp } \mathcal{F}[S_0]$. Since we assume $\mathcal{F}[\check{K}](\xi) = \mathcal{F}[K](-\xi)$ is never zero, we conclude that $\mathcal{M}$ is dense in $\mathcal{O}_{\varepsilon^1}^*$.

From that and previous statement we obtain the proof of the quoted Wiener theorem.

Note ([5]), $f \in \mathcal{B}'^*$ if and only if it is of the form

$$f = \sum_{\alpha=0}^{\infty} D^{\alpha} F_{\alpha}, \quad F_{\alpha} \in \mathcal{L}^{\infty}, \quad \alpha \in N_0,$$

where D is derivative in $\mathcal{B}'^*$ and F_{α} , $\alpha \in N_0$, are such that for some $h > 0$ (in the (M_{α}) -case), respectively, for every $h > 0$ (in the $\{M_{\alpha}\}$ -case)

$$(5) \quad \sum_{\alpha=0}^{\infty} \frac{M_{\alpha}}{h^{\alpha}} \|F_{\alpha}\|_{\mathcal{L}^{\infty}} = K_h < \infty.$$

Let $\psi \in \mathcal{O}_{\varepsilon^1}^*$. Since $\mathcal{M}$ is dense in $\mathcal{O}_{\varepsilon^1}^*$, then: In the (M_p) case, for every $\varepsilon > 0$ and every $h > 0$, there is $H_h \in \mathcal{M}$ such that

$$(6) \quad \|H_h - \psi\|_{\mathcal{L}^1, h} < \varepsilon.$$

In the $\{M_p\}$ case we have that for every $\varepsilon > 0$ there is $h > 0$ and H such that (6) holds.

In the $\{M_p\}$ case, the assumption of the theorem and Lebesgue's theorem give that for $x > x_0(\varepsilon)$, where $x_0(\varepsilon)$ is large enough,

$$\begin{aligned} & \left| (f * \psi)(x) - a \int_{\mathbf{R}} \psi(\xi) dt \right| \leq \left| ((\psi - H) * f)(x) - a \int_{\mathbf{R}} (\psi(t) - H(t)) dt \right| + \\ & + \left| (H * f)(x) - a \int_{\mathbf{R}} H(t) dt \right| \leq \\ & \leq \left| \sum_{i=0}^{\infty} \int_{\mathbf{R}} (\psi - H)^{(i)}(t) F_{\alpha}(x - t) dt \right| + a \int_{\mathbf{R}} |\psi(t) - H(t)| dt + \end{aligned}$$

$$\begin{aligned}
& + \left| (H * f)(x) - a \int_{\mathbf{R}} H(t) dt \right| \leq \\
& \leq \sup_{\alpha} \frac{h^{\alpha}}{M_{\alpha}} \|(\psi - H)^{(\alpha)}\|_{x^1} \sum_{\alpha=0}^{\infty} \frac{M_{\alpha}}{h^{\alpha}} \|F_{\alpha}\|_{x^{\infty}} + a\varepsilon + \\
& + \left| (H * f)(x) - a \int_{\mathbf{R}} H(t) dt \right| \leq \varepsilon K_h + a\varepsilon + \varepsilon.
\end{aligned}$$

The (M_p) - case can be proved similarly. The proof is completed.

PROOF OF THEOREM 2. We shall only prove the (M_{α}) -case since this proof can be simply transferred to the $\{M_{\alpha}\}$ -case.

The proof is organized as follows. In Part I we shall prove estimations (7), (7') and (8) which will be used in Part II for the proof that $\mathcal{F}(Ke^{-\alpha \cdot})(\xi)$, $\xi \in \mathbf{R}^n$, and the convolution $f * K$ exist. In Part III we will prove the assertion of Theorem 2.

Part I. Note, from the assumption that $(1 - \eta)\check{K}e^{\beta \cdot}$ and $\eta\check{K}e^{(\alpha + \delta) \cdot}$ belong to $\mathcal{O}_{x^1}^{(M_{\alpha})}$ and (M.2) it follows that for every $r > 0$

$$\begin{aligned}
(7) \quad \sup \left\{ \frac{r^m}{M_m} [\|e^{\beta x} ((1 - \eta(x))\check{K}(x))^{(m)}\|_{x^1} + \right. \\
\left. + \|e^{(\alpha + \delta)x} (\eta(x)\check{K}(x))^{(m)}\|_{x^1}], m \in N_0 \right\} < \infty.
\end{aligned}$$

Since $e^{(\alpha + \delta)x} \leq e^{\beta x}$, for $x < 0$, we also have that for every $r > 0$

$$(7') \quad \sup \left\{ \frac{r^m}{M_m} \|e^{(\alpha + \delta)x} ((1 - \eta(x))\check{K}(x))^{(m)}\|_{x^1(-\infty, 0)}, m \in N \right\} < \infty.$$

We need the following estimate:

For every $r > 0$ there is $C > 0$ such that

$$(8) \quad \sup_{k \in N_0} \left\{ \frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h) e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \right\} \leq \begin{cases} Ce^{\alpha x + \delta |x|}, & x+h > 0, \\ Ce^{\beta x}, & x+h < 0, \end{cases}$$

where we choose δ such that $0 < \delta < \alpha - \beta$.

Let $r > 0$, $k \in N_0$ and $x+h > 1$. By using (3) and (1) we have (with

suitable constants)

$$\begin{aligned} \frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| &\leq \\ &\leq \frac{r^k}{M_k} \int_{-1}^1 \frac{L(e^{x+h-t})}{L(e^h)} e^{\alpha(x-t)} |\omega^{(k)}(t)| dt + \frac{|r\alpha|^k}{M_k} e^{\alpha x} \leq \\ &\leq C_1 e^{\alpha x + \delta|x|} \|\omega\|_{[-1,1],r} + C_2 e^{\alpha x} \leq C e^{\alpha x + \delta|x|}. \end{aligned}$$

Similarly, for $x+h < -1$ we get that for given $r > 0$ there is $C > 0$ such that

$$\sup_{k \in N_0} \left\{ \frac{r^k}{M_k} \left| \left(\frac{c_0(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \right\} < C e^{\beta x}.$$

Let $u = x+h \in [-1, 1]$, $r > 0$ and $k \in N_0$. From (1) we have that

$$\frac{1}{L(e^{u-x})} \leq C e^{\delta|x|}, \quad x \in \mathbf{R},$$

where $C > 0$ and $\delta > 0$. Thus,

$$\begin{aligned} \frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| &\leq \\ &\leq \|\omega\|_{[-1,1],r} \sup_{t \in [-1,1]} \{c_0(x+h-t)\} \frac{1}{L(e^{u-x})e^{\alpha(u-x)}} + \\ &\quad + C_2 e^{\alpha x} \leq C_3 e^{\alpha x + \delta|x|} + C_2 e^{\alpha x}. \end{aligned}$$

These inequalities and the assumption $\delta \in (0, \alpha - \beta)$ imply (8).

Part II. Let $\psi \in \mathcal{S}^{(M_x)}$ be such that $\psi(x) = 1$ on $(-\infty, -1)$ and $\psi(x) = 0$ on $[0, \infty)$. We have

$$\begin{aligned} (9) \quad e^{\alpha x} \check{K}(x) &= e^{\alpha x} \check{K}(x)(1 - \eta(x))\psi(x) + \\ &\quad + e^{\alpha x} \check{K}(x)(1 - \eta(x))(1 - \psi(x)) + e^{\alpha x} \check{K}(x)\eta(x), \quad x \in \mathbf{R}. \end{aligned}$$

Since the multiplication in $\mathcal{O}_{\mathcal{L}_1}^{(M_x)}$ is the inner operation, one can easily

prove that all the members on the right side of (9) are from $\mathcal{O}_{\varepsilon_1}^{(M_x)}$ and so the same holds for $e^{\alpha \cdot} \check{K}$. This implies that $\mathcal{F}(Ke^{-\alpha \cdot})(\xi) = \mathcal{F}(\check{K})(\xi - i\alpha)$, $\xi \in \mathbf{R}^n$, exists.

Since

$$(f * K)(h) = \left\langle \frac{f(x+h)}{c(x+h)}, c(x+h) \check{K}(x) \right\rangle, \quad h \in \mathbf{R},$$

the existence of the convolution $f * K$ will be proved if we prove that for every $h \in \mathbf{R}$,

$$c(\cdot + h) \check{K} \in \mathcal{O}_{\varepsilon_1}^{(M_x)},$$

because by (i) $f(\cdot + h)/c(\cdot + h) \in \mathcal{B}'^{(M_x)}$.

For a fixed $h \in \mathbf{R}$ and a ψ as in (9) we have

$$(10) \quad c(x+h) \check{K}(x) = c(x+h) \check{K}(x)(1 - \eta(x))\psi(x) + \\ + c(x+h) \check{K}(x)(1 - \eta(x))(1 - \psi(x)) + c(x+h) \check{K}(x)\eta(x), \quad x \in \mathbf{R}.$$

By using (8), (7), (7') and that $e^{\alpha \cdot} \check{K} \in \mathcal{O}_{\varepsilon_1}^{(M_x)}$ we shall prove that $c(\cdot + h) \check{K} \in \mathcal{O}_{\varepsilon_1}^{(M_x)}$. We shall only prove that $c(\cdot + h) \check{K}(1 - \eta)\psi$ is from $\mathcal{O}_{\varepsilon_1}^{(M_x)}$ for every $h \in \mathbf{R}$, because the proof that $c(\cdot + h) \check{K}\eta$ belongs to $\mathcal{O}_{\varepsilon_1}^{(M_x)}$ is similar and then one can easily see that

$$c(\cdot + h) \check{K}(1 - \eta)(1 - \psi) \in \mathcal{O}^{(M_x)}.$$

Since

$$c(x+h) \check{K}(x)(1 - \eta(x))\psi(x) = \left[e^{\alpha h} L(e^h) \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \check{K}(x) + \right. \\ \left. + e^{\alpha(x+h)} L(e^h) \check{K}(x) \right] (1 - \eta(x))\psi(x), \quad x \in \mathbf{R},$$

we have to prove that

$$(1 - \eta)\psi \left(\frac{c(\cdot + h)}{e^{\alpha h} L(e^h)} - e^{\alpha \cdot} \right) \check{K} \in \mathcal{O}_{\varepsilon_1}^{(M_x)}.$$

For every $r > 0$, $x \in \mathbf{R}$, $k \in N_0$, by using (M.2) and (8) we have

$$\begin{aligned} \frac{r^k}{M_k} \left\| \sum_{j=0}^k \binom{k}{j} (\check{K}(x)(1 - \eta(x))\psi(x))^{(k-j)} \left(\frac{c(x+h)}{e^{xh}L(e^h)} - e^{x\alpha} \right)^{(j)} \right\|_{\mathcal{E}^1} &\leq \\ &\leq \frac{AC}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{(2rH)^{k-j}}{M_{k-j}} \{ \|\check{K}(x)(1 - \eta(x))\psi(x)\|_{\mathcal{E}^1}^{(k-j)} e^{\beta x} \|_{\mathcal{E}^1} \} \leq \\ &\leq C_1 \sup_{\alpha \in N_0} \left\{ \frac{(2rH)^\alpha}{M_\alpha} \|\check{K}(x)(1 - \eta(x))\psi(x)\|_{\mathcal{E}^1}^{(\alpha)} e^{\beta x} \|_{\mathcal{E}^1} \right\}, \end{aligned}$$

where C and C_1 are suitable constants.

For the proof that the last supremum is bounded we have to use the following estimates

$$\begin{aligned} \frac{(4rH)^\alpha}{M_\alpha} \frac{1}{2^\alpha} \sum_{j=0}^\alpha \binom{\alpha}{j} \|\check{K}(x)(1 - \eta(x))^{(j)} e^{\beta x} \psi^{(\alpha-j)}\|_{\mathcal{E}^1} &\leq \\ &\leq C_2 \sup_{j \in N_0} \left\{ \frac{(4rH^2)^j}{M_j} \|\check{K}(x)(1 - \eta(x))^{(j)} e^{\beta x}\|_{\mathcal{E}^1} \right\} \cdot \\ &\quad \cdot \sup_{\substack{\alpha \in N_0 \\ j \leq \alpha}} \left\{ \frac{(4rH^2)^{k-j}}{M_{\alpha-j}} \|\psi^{(\alpha-j)}\|_{\mathcal{E}^\infty(-1, 0)} \right\} \leq C_3, \end{aligned}$$

where C_2 and C_3 are suitable constants. Thus, we have proved that the convolution $f * K$ exists.

Part III. We are going to prove that the assumptions of the theorem imply

$$(11) \quad \left(\frac{f}{c} * Ke^{-\alpha \cdot} \right)(h) = \left\langle \frac{f(x+h)}{c(x+h)}, \check{K}(x) e^{x\alpha} \right\rangle \rightarrow a \int_{\mathbf{R}} K(t) e^{-\alpha t} dt, \\ h \rightarrow \infty.$$

It is enough to prove that

$$\left\langle \frac{f(x+h)}{c(x+h)}, \left(\frac{c(x+h)}{e^{xh}L(e^h)} - e^{x\alpha} \right) \check{K}(x) \right\rangle \rightarrow 0, \quad \text{as } h \rightarrow 0.$$

Since $\mathcal{B}'^* \ni f/c = \sum_{i=0}^{\infty} D^i F_i$ such that (5) holds, we have to prove that

$$S_h = \sum_{i=0}^{\infty} (-1)^i \int_{\mathbf{R}} F_i(x+h) \left(\left(\frac{c(x+h)}{L(e^h)e^{ah}} - e^{ax} \right) \check{K}(x) \right)^{(i)} dx \rightarrow 0, \\ h \rightarrow \infty.$$

We have

$$S_h = \sum_{i=0}^N (-1)^i \int_{\mathbf{R}} F_i(x+h) \left(\left(\frac{c(x+h)}{L(e^h)e^{ah}} - e^{ax} \right) \check{K}(x) \right)^{(i)} dx + \\ + \sum_{i=N+1}^{\infty} (-1)^i \int_{\mathbf{R}} F_i(x+h) \left(\left(\frac{c(x+h)}{L(e^h)e^{ah}} - e^{ax} \right) \check{K}(x) \right)^{(i)} dx = S_{h,N} + S_{h,\infty}.$$

Because the sum in $S_{h,N}$ is finite, the proof that $S_{h,N} \rightarrow 0$, $h \rightarrow \infty$, is the same as in the main assertion of [6]. By using (5) we obtain

$$S_{h,\infty} \leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{M_i \|F_i\|_{\mathcal{L}^\infty}}{r^i} \frac{(2r)^i}{M_i} \left\| \left[\left(\frac{c(x+h)}{L(e^h)e^{ah}} - e^{ax} \right) \check{K}(x) \right]^{(i)} \right\|_{\mathcal{L}^1} \leq \\ \leq \frac{C}{2^{N+1}} \sum_{i=0}^{\infty} \frac{(2r)^i}{M_i} \left\| \left[\left(\frac{c(x+h)}{L(e^h)e^{ah}} - e^{ax} \right) \check{K}(x) \right]^{(i)} \right\|_{\mathcal{L}^1}.$$

So, if we prove that the last series is bounded with respect to h for $h \geq h_0$, the proof that $S_{h,\infty} \rightarrow 0$, $h \rightarrow \infty$, simply follows. Put

$$I_{m,h} = \left\| \left(\left(\frac{c(\cdot+h)}{e^{ah}L(e^h)} - e^{a\cdot} \right) \check{K} \right)^{(m)} \right\|_{\mathcal{L}^1}, \quad m \in \mathbf{N}_0, \quad h \geq h_0.$$

We are going to prove that for every $r > 0$, there is $C > 0$ such that

$$(12) \quad \sup_{m \in \mathbf{N}_0} \left\{ \frac{r^m}{M_m} I_{m,h} \right\} < C, \quad h > h_0.$$

This implies that the quoted series is bounded.

Let $x_0 > 0$ be as in (4). We have

$$\begin{aligned} I_{m,h} = & \int_{-\infty}^{-x_0} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) (1 - \eta(x)) \check{K}(x) \right]^{(m)} \right| dx + \\ & + \int_{-x_0}^{x_0} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \check{K}(x) \right]^{(m)} \right| dx + \\ & + \int_{x_0}^{\infty} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \eta(x) \check{K}(x) \right]^{(m)} \right| dx = I_1 + I_2 + I_3. \end{aligned}$$

By the Leibniz formula, (8), (7), (7') and (1) there are constants C_1 and C which do not depend on m and p (but depend on r) such that for $\delta \in (0, \alpha - \beta)$

$$\begin{aligned} I_1 \leq & \sum_{p=0}^m \binom{m}{p} \left[\int_{-\infty}^{-h} \left| \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)^{(p)} e^{-\beta x} \right| \cdot \right. \\ & \cdot |e^{\beta x} ((1 - \eta(x)) \check{K}(x))^{(m-p)}| dx + \\ & + \left. \int_{-h}^{-x_0} \left| \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)^{(p)} e^{-(\alpha x - \delta x)} \right| |e^{\alpha x - \delta x} ((1 - \eta(x)) \check{K}(x))^{(m-p)}| dx \right] \leq \\ \leq & \sum_{p=0}^m \binom{m}{p} C \left[\frac{M_p}{r^p} \frac{M_{m-p}}{r^{m-p}} \sup_{\substack{m,p \\ p \leq m}} \left\{ \frac{r^{m-p}}{M_{m-p}} \|e^{\beta x} (1 - \eta(x)) (\check{K}(x))^{(m-p)}\|_{\mathcal{L}^1} \right\} + \right. \\ & + \left. \frac{M_p}{r^p} \frac{M_{m-p}}{r^{m-p}} \sup_{\substack{m,p \\ p \leq m}} \left\{ \frac{r^{m-p}}{M_{m-p}} \|e^{\alpha x - \delta x} (1 - \eta(x)) \check{K}(x)\|_{\mathcal{L}^1(-\infty, 0)} \right\} \right] \leq \\ \leq & C_1 \sum_{p=0}^m \binom{m}{p} \frac{M_m}{r^m} = C_1 \frac{M_m}{(r/2)^m}. \end{aligned}$$

This gives that $\sup_{m \in \mathbb{N}_0} \{(r/2)^m / (M_m) I_1\} < C_1$.

In a similar way one can prove the corresponding estimates for I_2 and I_3 and the proof of (12) is completed.

Thus, we have proved (11).

If $\psi \in \mathcal{G}^{(M_\alpha)}$ satisfies the assumption given in (*) then $\psi e^\alpha \in \mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$ and

we have

$$\left(\frac{f}{c} * \psi e^{-\alpha \cdot} \right)(h) \rightarrow a \int_R \psi(x) e^{-\alpha x} dx, \quad h \rightarrow \infty.$$

As above we can prove that $(f * \psi)(h)$, $h \in \mathbf{R}$, exists.

For the proof that

$$\left\langle \frac{f(x+h)}{L(e^h)e^{\alpha h}}, \psi(x) \right\rangle \rightarrow a \int \psi(x) e^{-\alpha x} dx, \quad h \rightarrow \infty,$$

we have to prove that

$$\left\langle \frac{f(x+h)}{c(x+h)}, \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \psi(x) \right\rangle \rightarrow 0, \quad h \rightarrow \infty$$

but this is already done (with K instead of ψ) and the proof of the Theorem is completed.

The proof of Corollary 1 simply follows from the given Theorem 2 since the ϕ in the Corollary satisfies conditions assumed for K and functions from $\mathcal{O}^*$ satisfy condition (c) of Theorem 2.

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