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On a Question of Deaconescu About Automorphisms. - II.

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This note is concerned with groups G satisfying the property that $\text{Aut } H \cong N_G(H)/C_G(H)$ for all subgroups H of G . Such groups were referred to in [3] as MD -groups. It was shown there that the only non-trivial finite MD -groups are \mathbb{Z}_2 and S_3 , and Theorem 2.1 of the same paper gave a complete classification of infinite, metabelian MD -groups. Our first result here shows that all (infinite) soluble MD -groups are in fact metabelian. Indeed, rather more than this is proved. Recall that a group G is a *radical* group if the iterated series of Hirsch-Plotkin radicals reaches G . (Thus, for instance, locally nilpotent groups and hyperabelian groups are radical.) The class of radical groups is certainly closed under forming sections—in particular, every finite section of such a group is soluble.

We shall prove the following:

THEOREM 1. *Let G be an infinite, radical MD -group. Then G is metabelian and is thus an extension of a torsion-free, locally cyclic subgroup A having finite type at every prime p by a cyclic group $\langle x \rangle$ of order 2, such that $a^x = a^{-1}$ for all $a \in A$.*

As a consequence, we have:

COROLLARY. Let G be an infinite locally soluble MD -group. Then G is metabelian.

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Our other result concerns periodic MD -groups, and shows that \mathbb{Z}_2 and S_3 are the only such groups which are nontrivial.

THEOREM. 2. *Let G be a periodic MD -group. Then G is finite.*

PROOF OF THEOREM 1. In view of Theorem 2.1 of [3] (which also shows that a group G having the structure described is an MD -group) all we need do is show that an infinite radical MD -group is metabelian. Let G be a such a group. The automorphism group of each of the following has a finite insoluble section: $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}_p \times \mathbb{Z}_p$ for $p > 3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. It follows that every abelian subgroup of G has finite rank and hence, by a theorem of Baer and Heineken (see p. 89 of [5]), that G has finite rank. Since $\text{Aut } C_{p^\infty}$ is uncountable, we see that every periodic abelian subgroup of G is reduced.

Now let H be the Hirsch-Plotkin radical of G and let P be a Sylow p -subgroup of H , for some prime p . Then H is hypercentral and P is reduced and hence finite. (For the relevant facts about locally nilpotent groups of finite rank, see Section 6.3 of [5].) If $p > 3$, then, by what was said in the previous paragraph, the centre of P is cyclic and, if nontrivial, contains a G -invariant cycle $\langle z \rangle$ of order p . Since $\text{Aut } \mathbb{Z}_p$ is cyclic and of order $p - 1$, while G/G' has exponent (at most) 2, by 1.3 of [3], we have a contradiction. Thus $P = 1$, and the torsion subgroup of H is a finite $(2, 3)$ -group. It follows easily (see [5]) that H is nilpotent. If H is finite then so is G , contrary to hypothesis. Thus the centre Z of H contains an element a of infinite order. Since the torsion subgroup T of Z is finite, we may assume that $\langle a \rangle \triangleleft G$.

Suppose next that b is some nontrivial element of T . Then the map $a \rightarrow ab$, $t \rightarrow t$, for all $t \in T$, determines an automorphism α of $\langle a \rangle \times T = B$, say. Now $\text{Aut } B$ is finite and so, by the MD -property, the natural embedding of $G/C_G(B)$ in $\text{Aut } B$ is an isomorphism. So α is induced by conjugation by some element g of G , contradicting the fact that $\langle a \rangle \triangleleft G$. Thus Z and hence H is torsion-free. Further, because $\mathbb{Z} \times \mathbb{Z}$ does not embed in G we see that H is locally cyclic. It follows that $[H, G'] = 1$ since $\text{Aut } H$ is abelian. But, in a radical group, the centraliser of the Hirsch-Plotkin radical H is contained in H (see [4, Lemma 2.32]). The result follows.

PROOF OF THE COROLLARY. Suppose that the MD -group G is locally soluble. We notice from the proof of Theorem 1 that every abelian subgroup of G has *bounded* rank (at most 3) and hence, by a result of Merzlyakov (see p. 89 of [5]), G has finite rank. But a locally soluble group with finite rank is hyperabelian ([5, Lemma 10.39]) and the result follows from Theorem 1.

PROOF OF THEOREM 2. Let G be a periodic MD-group and let p be any prime such that G contains an element of order p . Every infinite elementary abelian p -group has an automorphism of infinite order. The same is true of C_p and so every abelian p -subgroup of G is finite. It follows that every locally finite p -subgroup is finite, and thus we may choose a maximal finite p -subgroup P of G . Let N, C respectively denote the normaliser and centraliser of P in G and assume that P does not have prime order. Then $\text{Out } P$ contains an element of order p [1] and thus, if X/C is any Sylow p -subgroup of N/C , then $|X/C| > |P/Z(P)|$. Choosing X so that $P \leq X$, we see that there exists $x \in X$ such that $x^p \in PC$ but $x \notin PC$. Since G is periodic, we may assume that x is a p -element. But then $\langle x, P \rangle$ is a p -subgroup properly containing P , a contradiction. Thus all (locally) finite p -subgroups have prime order.

Next, suppose that G has a section K/L of order 4. Certainly K/L is not cyclic, since G has no elements of order 4, and so $K = \langle L, a, b \rangle$, where each of a and b has order 2. Now $\langle a, b \rangle$ is a periodic image of the infinite dihedral group and therefore it is finite and contains a 2-subgroup of order at least 4, a contradiction. We deduce that G has no subgroups H such that $\text{Aut } H$ contains a subgroup of order 4. In particular G contains no subgroups of type $\mathbb{Z}_p \times \mathbb{Z}_q$ where p and q are odd primes. Since G itself contains no subgroups of order 4, it also cannot contain a subgroup of type $\mathbb{Z}_p \times \mathbb{Z}_2$. It follows that every abelian subgroup of G is of prime order. We now claim that G contains no elements of order p for any prime $p > 3$. Suppose (for a contradiction) that G has a subgroup $\langle a \rangle$ of order p , where p is the least such prime greater than 3. Then $C_G \langle a \rangle = \langle a \rangle$ and so the normaliser N of $\langle a \rangle$ has order $p(p-1)$. By the minimality of p , we see that $p-1 = 2^k 3^l$ for some k, l and, since the Sylow subgroups of N are of prime order, we have $k = l = 1$ and $p = 7$. But $\text{Aut } \mathbb{Z}_7 = \mathbb{Z}_6$ and G has no elements of order 6. This contradiction establishes our claim. We now see that every finite subgroup of G has order at most 6 and (of course) that G has exponent dividing 6. By M. Hall's solution of the Burnside problem for exponent six [2], G is locally finite and therefore finite. The result follows.

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