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Infinitely Many Spacelike Periodic Trajectories on a Class of Lorentz Manifolds.

CARLO GRECO(*)

ABSTRACT - Let us consider $\mathbf{R}^4$ equipped with a Lorentzian tensor g with signature $(+, +, +, -)$. In this paper we prove, under suitable assumptions on g , the existence of infinitely many spacelike geodesics $z(s) = (x(s), t(s))$ with the periodicity conditions $x(s + 1) = x(s)$, $t(s + 1) = t(s) + T$ ($T > 0$) on the Lorentz manifold $(\mathbf{R}^4, g)$.

1. Introduction.

Let us consider the manifold $(\mathbf{R}^4, g)$, where $g(z) = g(x, t)$ is a Lorentz tensor on $\mathbf{R}^4$, with signature $(+, +, +, -)$. Let $z(s) = (x(s), t(s))$ be a geodesic on $(\mathbf{R}^4, g)$, and suppose that $t(0) = 0$, and there exist $\sigma, T > 0$ such that $x(s + \sigma) = x(s)$, $t(s + \sigma) = t(s) + T$ for every $s \in \mathbf{R}$. Then we shall say that z is a σ -periodic T -trajectory on $(\mathbf{R}^4, g)$. Moreover, if z is a geodesic, there exists $E_z \in \mathbf{R}$ such that $g(z(s))[\dot{z}(s), \dot{z}(s)] \equiv E_z$, and z called spacelike, null or timelike if $E_z > 0$ or, respectively, $E_z = 0$, $E_z < 0$ (see [14], p. 69).

Suitable Lorentz manifolds are used in Relativity theory in order to describe the physical space-time. Then, timelike (or, respectively, null) periodic trajectories corresponds to periodic orbits of a particle of positive mass (or, respectively, of a light ray). Spacelike geodesics are not trajectories of particles, but they are important in order to study geometrical properties of a semiriemannian manifold.

Some multiplicity results for timelike periodic trajectories on $(\mathbf{R}^4, g)$ are given, for instance, in [5] and [9] under the assumption that

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the gravitational field vanish at infinity, so that g tends to the Minkowski metric at infinity (see Remark 1.3 below for further informations).

In this paper we consider a completely different behavior at infinity for g , and we are able to prove that, for any $T > 0$, there are infinitely many spacelike 1-periodic T -trajectories on the semiriemannian manifold $(\mathbf{R}^4, g)$.

Let $\{g_{ij}\}_{i,j=1,\dots,4}$ be the components of g . We suppose that g not depend to the time, $g_{ij} = g_{ji} \in C^1(\mathbf{R}^3, \mathbf{R})$, and $g_{i4} = 0$ for $i = 1, 2, 3$. We set, for simplicity, $\alpha = \{\alpha_{ij}\}_{i,j=1,2,3} = \{g_{ij}\}_{i,j=1,2,3}$, and $\beta = -g_{44}$, so that we have, for every $x \in \mathbf{R}^3$ and every $\begin{pmatrix} \xi \\ \tau \end{pmatrix} \in \mathbf{R}^4$:

$$g(x) \left[\begin{pmatrix} \xi \\ \tau \end{pmatrix}, \begin{pmatrix} \xi \\ \tau \end{pmatrix} \right] = \alpha(x)[\xi, \xi] - \beta(x)\tau^2.$$

Moreover we assume that there exist α_0, α_1 , $R > 0$, $p > 2$ and $q \in]0, p-2[$ such that for every $x, \xi \in \mathbf{R}^3$:

$$(1.1) \quad \alpha(x)[\xi, \xi] \geq \alpha_0 |\xi|^2,$$

$$(1.2) \quad (q\alpha(x) - \alpha'(x)(x))[\xi, \xi] \geq \alpha_1 |\xi|^2,$$

$$(1.3) \quad p\beta(x) \leq (\beta'(x)|x) \quad \text{if } |x| \geq R,$$

$$(1.4) \quad 0 < \beta_0 \equiv \beta(0) = \min_{\mathbf{R}^3} \beta,$$

$$(1.5) \quad \lim_{|x| \rightarrow 0} \frac{\beta(x) - \beta_0}{|x|^2} = 0,$$

$$(1.6) \quad \alpha(x) = \alpha(-x), \quad \beta(x) = \beta(-x).$$

Then we have the following theorem.

THEOREM 1.1. *If (1.1)-(1.6) are satisfied, then, for every $T > 0$, there exist infinitely many spacelike 1-periodic T -trajectories on $(\mathbf{R}^4, g)$.*

REMARK 1.1. If $x_0 \in \mathbf{R}^3$ and $\beta'(x_0) = 0$, it is easy to check that $z(s) = (x_0, Ts)$ is a trivial periodic trajectory. We shall see later that the trajectories given by Theorem 1.1 are not trivial, and are geometrically distinct.

REMARK 1.2. Condition (1.3) is a sort of superquadraticity condition at infinity. It has been introduced by P. H. Rabinowitz in the theory of Hamiltonian systems. (1.4) implies that there exists $c_1 > 0$ such that, for every $x \in \mathbf{R}^3$, with $|x| \geq R$:

$$(1.7) \quad \beta(x) \geq c_1 |x|^p.$$

Condition (1.3) means that $\sum_{i,j=1}^3 [q\alpha_{ij}(x) - (\alpha'_{ij}(x)|x)] \xi_i \xi_j \geq \alpha_1 |\xi|^2$; it is satisfied, for instance, if $\alpha(x) = \{\delta_{ij}\}_{i,j=1,2,3}$. Moreover, because of (1.3), there exists $c_2 > 0$ such that

$$(1.8) \quad \|\alpha(x)\| \leq c_2 |x|^q$$

for $|x| \geq 1$. Infact, let $x \in \mathbf{R}^3$ with $|x| \geq 1$. Since

$$d(t^{-q} \alpha(tx/|x|)[\xi, \xi])/dt \leq 0,$$

we have

$$|x|^{-q} \alpha(x)[\xi, \xi] \leq \alpha(x/|x|)[\xi, \xi] \leq c_2 |\xi|^2 \quad \text{where } c_2 = \max_{|y|=1} \|\alpha(y)\|.$$

REMARK 1.3. The problem of geodesics for a Lorentz manifold (M, g) has been recently studied by many authors (see [2]-[5], [7]-[12]). If particular, in the papers [5], [9], are given multiplicity results for timelike periodic trajectories on $(\mathbf{R}^4, g)$ under the assumption $\beta(x)$ bounded.

The main difficult in the variational approach of this kind of problems is that the action functional

$$\int g(z)[\dot{z}, \dot{z}] = \int \alpha(x)[\dot{x}, \dot{x}] - \int \beta(x) t^2$$

is strongly indefinite, i.e. it is not of the form identity + compact, even «modulo compact perturbations». In order to avoid this difficult, we use the convexity of the functional with respect to $\dot{t}$ and search for the critical points of a functional f depending only on x .

If $\beta(x)$ is bounded as in [9] (or it is subquadratic), the functional f is bounded from below, and satisfies easily the Palais-Smale compactness condition. In our case f is unbounded, so we need some linking argument; moreover more care is required in order to prove compactness conditions.

In Section 2 we expose the functional framework and we prove the compactness condition using assumptions (1.1)-(1.5). Then we prove Theorem 1.1 with a mountain pass argument by using (1.6).

2. Proof of the results.

In the following we assume that (1.1)-(1.5) hold. Let us consider a geodesic $z(s) = (x(s), t(s))$ on $(\mathbf{R}^4, g)$; then z satisfies the geodesic equations:

$$\frac{d}{ds}[\alpha(x)\dot{x}] = \frac{1}{2}(\alpha'(x)[\dot{x}, \dot{x}] - \beta'(x)t^2),$$

$$\frac{d}{ds}[\beta(x)t] = 0.$$

If z is a σ -periodic T -trajectory, we shall call the minimal period of x , the minimal period of z . Notice that, if $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$ are σ -periodic T -trajectories on $(\mathbf{R}^4, g)$, with $z_1 \neq z_2$, then z_1 and z_2 are geometrically distinct.

In fact, if $z_2(s) = z_1(\varphi(s))$ for some reparametrization $\varphi(s)$, from geodesic equations we have $\varphi(s) = as + b$ for some $a, b \in \mathbf{R}$ (see [14], p. 69), so that $t_2(s) = t_1(as + b)$. Since $\dot{t}_1(s) \neq 0$ for any $s \in \mathbf{R}$, from $t_1(0) = 0 = t_2(0) = t_1(b)$, we have $b = 0$, and from $t_1(as + a\sigma) = t_2(s + \sigma) = t_2(s) + T = t_1(as) + T = t_1(as + \sigma)$, we have $a\sigma = \sigma$ and $a = 1$, which is impossible.

In particular, if z_1 and z_2 have not the same minimal period, then they are geometrically distinct.

REMARK 2.1. We observe now that, if $z(s) = (x(s), t(s))$ is a k^{-1} -periodic Tk^{-1} -trajectory, x and $\dot{t}$ are also 1-periodic and $t(s+1) = t(s) + T$. Infact, it is easy to check that $t(s+1) = t(s + (k-h)/h) + Th/k$ for every $h = 1, \dots, k$; then z is a 1-periodic T -trajectory on $(\mathbf{R}^4, g)$, with minimal period less or equal to $1/k$. So, in order to prove Theorem 1.1, we can show that there exists $k_0 \in \mathbf{N}$ such that, for every $k \in \mathbf{N}$ with $k \geq k_0$, there exists a k^{-1} -periodic Tk^{-1} -trajectory $z(s) = (x(s), t(s))$, with $\dot{x} \neq 0$.

Let $k \in \mathbf{N}$ be free for the moment, and let us consider the functional

$$I(x, \eta) = \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] ds - \int_0^{1/k} \beta(x)(T/k + \eta)^2 ds,$$

defined on $H^{1,2}(S^{1/k}, \mathbf{R}^3) \times L_0(S^{1/k}, \mathbf{R})$, where $H^{1,2}(S^{1/k}, \mathbf{R}^3)$ is the Sobolev space of k^{-1} -periodic functions $x: \mathbf{R} \rightarrow \mathbf{R}^3$ with

$x, \dot{x} \in L^2([0, 1/k])$, and

$$L_0(S^{1/k}, \mathbf{R}) = \left\{ \eta \in L^2(S^{1/k}, \mathbf{R}) \mid \int_0^{1/k} \eta ds = 0 \right\}.$$

It is easy to check that, if (x, η) is a critical point of I , then $z(s) = (x(s), t(s))$, where $t(s) = Ts/k + \int_0^s \eta ds$ is a critical point of the action functional

$$\int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] ds - \int_0^{1/k} \beta(x) \dot{t}^2 ds;$$

so, it is a 1-periodic T -trajectory on $(\mathbf{R}^4, g)$, with minimal period less or equal to $1/k$ (see Remark 2.1).

Notice that, because of (1.4), for every $x \in H \equiv H^{1,2}(S^{1/k}, \mathbf{R}^3)$, the functional $\eta \mapsto \int_0^{1/k} \beta(x)(T/k + \eta)^2 ds$ is strictly convex, so it possess a unique minimum point $\eta_x \in L_0(S^{1/k}, \mathbf{R})$. Let $f: H \rightarrow \mathbf{R}$ be the functional

$$f(x) = \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] ds - \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 ds + \frac{\beta_0 T^2}{k^3}.$$

LEMMA 2.2. *The function $x \mapsto \eta_x$ is continuous from H to $L_0(S^{1/k}, \mathbf{R})$; moreover $f \in C^1(H, \mathbf{R})$ and*

$$\langle f'(x), y \rangle = \left\langle \frac{\partial I}{\partial x}(x, \eta_x), y \right\rangle,$$

so that, $x \in H$ is a critical point of f if and only if (x, η_x) is a critical point of I .

PROOF. The proof is contained in [9]. We recall it for the reader convenience. First of all we observe that $\int_0^{1/k} \beta(x)(T/k + \eta_x) \eta_x ds = 0$, because of η_x is a critical point of the functional $\eta \mapsto \int_0^{1/k} \beta(x)(T/k + \eta)^2 ds$.

So $\int_0^{1/k} \beta(x) \eta_x^2 ds = -(T/k) \int_0^{1/k} \beta(x) \eta_x ds$, and then

$$(2.1) \quad \|\eta_x\| \leq \frac{T\|\beta(x)\|_\infty}{k\beta_0}.$$

Now, let $x, y \in H$. Clearly

$$(2.2) \quad I(x, \eta_y) - I(y, \eta_y) \leq f(x) - f(y) \leq I(x, \eta_x) - I(y, \eta_x),$$

and $I(x, \eta_x) - I(y, \eta_x) \rightarrow 0$ as $y \rightarrow x$. Moreover, since

$$\begin{aligned} I(x, \eta_y) - I(y, \eta_y) &= \\ &= \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] - \alpha(y)[\dot{y}, \dot{y}] ds - \int_0^{1/k} (\beta(x) - \beta(y))(T/k + \eta_y)^2 ds, \end{aligned}$$

using (2.1) we get $I(x, \eta_y) - I(y, \eta_y) \rightarrow 0$ as $y \rightarrow x$, so f is continuous.

We prove now that $x \mapsto \eta_x$ is continuous. Infact, arguing by contradiction, we suppose that there exist $x \in H$, $(x_n) \subset H$ and $\varepsilon > 0$ such that $x_n \rightarrow x$ and $\|\eta_x - \eta_{x_n}\| \geq \varepsilon$. Since $\int_0^{1/k} \beta(x)(T/k + \eta)^2 ds$ is strictly convex, we have

$$\sup \{I(x, \eta) \mid \eta \in L_0(S^{1/k}, \mathbf{R}), \|\eta - \eta_x\| = \varepsilon/2\} \leq I(x, \eta_x) - \delta$$

for some $\delta > 0$. Let $\mu_n \in \partial B(\eta_x, \varepsilon/2) \cap \{\eta_x + \lambda(\eta_{x_n} - \eta_x) \mid \lambda \in [0, 1]\}$; since $I(x_n, \cdot)$ is concave, we have $I(x_n, \mu_n) \geq I(x_n, \eta_x)$, so that

$$\begin{aligned} I(x, \eta_x) - \delta &\geq I(x, \mu_n) = I(x, \mu_n) - \\ &\quad - I(x_n, \mu_n) + I(x_n, \mu_n) \geq I(x, \mu_n) - I(x_n, \mu_n) + I(x_n, \eta_x). \end{aligned}$$

Since (μ_n) is bounded and $x_n \rightarrow x$, we get $I(x, \mu_n) - I(x_n, \mu_n) \rightarrow 0$, and $I(x_n, \eta_x) \rightarrow I(x, \eta_x)$, and then we have a contradiction.

Finally, fix $x, y \in H$, and let $\tau > 0$. From (2.2) we have

$$\begin{aligned} \frac{I(x + \tau y, \eta_x) - I(x, \eta_x)}{\tau} &\leq \\ &\leq \frac{f(x + \tau y) - f(x)}{\tau} \leq \frac{I(x + \tau y, \eta_{x + \tau y}) - I(x, \eta_{x + \tau y})}{\tau}. \end{aligned}$$

For $\tau \rightarrow 0$ we get $\langle f'(x), y \rangle = \langle \partial I(x, \eta_x) / \partial x, y \rangle$, so the lemma is proved. ■

REMARK 2.3. Notice that $\int_0^{1/k} \beta(x)(T/k + \eta_x) \eta ds = 0$ for every

$\eta \in L_0(S^{1/k}, \mathbf{R})$. In other words, there exists $c_x \in \mathbf{R}$ such that $\beta(x(s))(T/k + \eta_x(s)) = c_x$ for every $s \in \mathbf{R}$. Since $c_x \leq 0$ implies $T/k + \eta_x(s) \leq 0$, so $T/k^2 = \int_0^{1/k} (T/k + \eta_x) ds \leq 0$, we have $c_x > 0$, and then $T/k + \eta_x(s) > 0$ for every s . Moreover $\beta(x)(T/k + \eta_x)^2 = c_x(T/k + \eta_x)$, so $c_x = (k^2/T) \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 ds$.

LEMMA 2.4. Fix $\rho > 0$ and $x \in H$, and set $I = \{s \in [0, 1/k] \mid |x(s)| \leq \rho\}$. Then, if $|I| > 0$,

$$\int_0^{1/k} \beta(x)(T/k + \eta_x)^2 ds \leq \frac{T^2 M}{k^4 |I|},$$

where $M = \max\{\beta(x) \mid |x| \leq \rho\}$, and $|I|$ is the Lebesgue measure of I .

PROOF. Let c_x be as in Remark 2.3, so that $T/k + \eta_x(s) = c_x/\beta(x(s))$ for every $s \in \mathbf{R}$. If $s \in I$, we have $c_x/M \leq T/k + \eta_x(s)$, and then $c_x^2/M \leq \beta(x(s))(T/k + \eta_x(s))^2$.

Integrating on I , we have:

$$\frac{c_x^2}{M} |I| \leq \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 ds = \frac{Tc_x}{k^2}.$$

Then $c_x \leq TM/k^2 |I|$, so that the lemma is proved. ■

LEMMA 2.5. Let $0 < r < \rho$ and $(x_n) \subset H$ be such that $\text{dist}(\text{Im}(x_n), 0) \leq r$ and $\|x_n\|_\infty \geq \rho$. Then

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_{x_n})^2 ds \leq \frac{T^2 M}{k^4 (\rho - r)^2} \|\dot{x}_n\|_2^2,$$

where $M = \max\{\beta(x) \mid |x| \leq \rho\}$.

PROOF. Let $I_n = \{s \in [0, 1/k] \mid |x_n(s)| \leq \rho\}$; since $\|x_n\|_\infty \geq \rho$, $|I_n| > 0$, so that

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_{x_n})^2 ds \leq T^2 M/k^4 |I_n|$$

because of Lemma 2.4. Moreover, since $\text{dist}(\text{Im}(x_n), 0) \leq r$, we have

$$\rho - r \leq \int_{I_n} |\dot{x}_n| ds \leq \|\dot{x}_n\|_2 |I_n|^{1/2}, \text{ and the lemma follows. } \blacksquare$$

We say that a functional $f: H \rightarrow \mathbf{R}$ verifies the Palais-Smale-Cerami (PSC) condition (see [6]) if every sequence $(x_n) \subset H$ such that $f(x_n) \rightarrow c \in \mathbf{R}$ and $\langle f'(x_n), x_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, possesses a convergent subsequence.

We have the following lemma.

LEMMA 2.6. *There exists $k_0 \in N$ such that, for every $k \geq k_0$, the functional f satisfies the PSC-condition.*

PROOF. Let $M = \max \{ \beta(x) \mid |x| \leq R + 1 \}$ (R is defined in (1.3)), and let $k_0 \in N$ be such that $\alpha_0 - T^2 M / k_0^4 > 0$. Fix $k \in N$ with $k \geq k_0$, and let us consider a sequence $(x_n) \subset H$ such that $f(x_n) \rightarrow c \in \mathbf{R}$ and $\langle f'(x_n), x_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. First of all, we prove that $(\|\dot{x}_n\|_2)$ is bounded modulo subsequences. Infact, we distinguish two cases:

1) case: for every $n \in N$, $\text{dist}(\text{Im}(x_n), 0) > R$ (modulo subsequences). Then $p\beta(x_n(s)) \leq (\beta'(x_n(s)) |x_n(s))$ for every s (see (1.3)), so, from $f(x_n) \rightarrow c$ we get (setting $\eta_n \equiv \eta_{x_n}$):

$$p \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds \leq pc + \int_0^{1/k} (\beta'(x_n) |x_n)(T/k + \eta_n)^2 ds + o(1).$$

Since $\langle f'(x_n), x_n \rangle \rightarrow 0$, we have

$$\begin{aligned} \int_0^{1/k} \alpha'(x_n)(x_n)[\dot{x}_n, \dot{x}_n] ds + 2 \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds - \\ - \int_0^{1/k} (\beta'(x_n) |x_n) \left(\frac{T}{k} + \eta_n \right)^2 ds = o(1), \end{aligned}$$

so that

$$\int_0^{1/k} (q\alpha(x_n) - \alpha'(x_n)(x_n))[\dot{x}_n, \dot{x}_n] ds \leq pc + o(1),$$

then $(\|\dot{x}_n\|_2)$ is bounded because of (1.2).

2) case: for every $n \in N$, $\text{dist}(\text{Im}(x_n), 0) \leq R$ (modulo subsequences). Then, if $(\|x_n\|_\infty)$ is bounded, we have $\beta(x_n(s)) \leq M_1$ for $n \in N$, $s \in \mathbf{R}$, so $\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds \leq M_1 T^2/k^3$, and the claim follows from the fact that $f(x_n) \rightarrow c$ as $n \rightarrow \infty$. So, we can assume $\|x_n\|_\infty \rightarrow \infty$. Let $I_n = \{s \in [0, 1/k] \mid |x_n(s)| \leq R + 1\}$; from Lemma 2.5 (with $r = R$ and $\rho = R + 1$), we have

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds \leq \frac{T^2 M}{k^4} \|\dot{x}_n\|_2^2.$$

Then, since $f(x_n) \rightarrow c$,

$$\int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds \leq \frac{T^2 M}{k^4} \|\dot{x}_n\|_2^2 + c + o(1),$$

so that (see (1.1)): $(\alpha_0 - T^2 M/k^4) \|\dot{x}_n\|_2^2 \leq c + o(1)$. Since $k \geq k_0$, the claim follows.

We set now $x_n = \xi_n + y_n$, where $\xi_n \in \mathbf{R}^3$, and $\int_0^{1/k} y_n(s) ds = 0$; we shall prove that (ξ_n) is bounded. In fact, we can assume that $y_n \rightarrow y$ weakly in $H^{1,2}$ and strongly in L^∞ ; then

$$|\xi_n| - (\|y\|_\infty + 1) \leq |x_n(s)| \leq |\xi_n| + (\|y\|_\infty + 1)$$

for n large enough, so that, since

$$\alpha(x_n(s))[\dot{x}_n(s), \dot{x}_n(s)] \leq c_2 |x_n(s)|^q |\dot{x}_n(s)|^2$$

(see (1.8)), we have $\int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds \leq c_3 |\xi_n|^q + c_4$ for some $c_3, c_4 > 0$.

On the other hand, $\beta(x_n(s)) \geq c_1 |x_n(s)|^p$, then $\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds \geq c_5 |\xi_n|^p + c_6$. Since $f(x_n) \rightarrow c$, we have

$$\begin{aligned} c_5 |\xi_n|^p + c_6 &\leq \int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds = \\ &= \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds - c + o(1) \leq c_3 |\xi_n|^q + c_4 + c + o(1), \end{aligned}$$

so (ξ_n) is bounded. Let us suppose $x_n \rightarrow x$ weakly in $H^{1,2}$ and strongly L^∞ . Then

$$\begin{aligned} \langle f'(x_n), x - x_n \rangle &= \int_0^{1/k} \alpha'(x_n)(x - x_n)[\dot{x}_n, \dot{x}_n] ds + \\ &+ 2 \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x} - \dot{x}_n] ds - \int_0^{1/k} (\beta'(x_n)|x - x_n|(T/k + \eta_n)^2 ds; \end{aligned}$$

because of (2.1) we have that (η_n) is bounded, so, the fact that

$$\langle f'(x_n), x - x_n \rangle = o(1) \quad \text{implies} \quad \int_0^{1/k} \alpha(x_n)[\dot{x}_n \dot{x} - \dot{x}_n^2] ds = o(1). \quad \text{Then}$$

$$\int_0^{1/k} |\dot{x} - \dot{x}_n|^2 ds \leq \alpha_0^{-1} \int_0^{1/k} \alpha(x_n)[\dot{x} - \dot{x}_n, \dot{x} - \dot{x}_n] ds = o(1), \quad \text{so that } x_n \rightarrow x$$

strongly in H , and the lemma is proved. ■

Let $H = H^{1,2}(S^{1/k}, \mathbf{R}^3) = \mathbf{R}^3 \times Y$, where

$$Y = \left\{ x \in H \mid \int_0^{1/k} x(s) ds = 0 \right\}.$$

As well-known (see e.g. [13], p. 9), for every $y \in Y$ we have $\|\dot{y}\|_2 \geq a\|y\|$, and $\|y\|_\infty \leq b\|\dot{y}\|_2$, where $a = 2k\pi(1 + 4k^2\pi^2)^{-1/2}$, and $b = (1/12k)^{1/2}$.

We have now the following lemma.

LEMMA 2.7. *There exist $\delta, \rho > 0$ such that $f(y) \geq \delta$ for every $y \in Y$ with $\|y\| = \rho$. Moreover δ is independent of k .*

PROOF. Fix $\varepsilon > 0$ such that $\alpha_0 - \varepsilon T^2/\sqrt{12} > 0$. (1.5) implies that there exists $\rho_1 > 0$ such that $\beta(x) \leq \beta_0 + \varepsilon|x|^2$ for $|x| \leq \rho_1$. Set $\rho = \rho_1/b$ and

$$\delta = \frac{4\pi^2}{1 + 4\pi^2} \left(\alpha_0 - \frac{\varepsilon T^2}{12} \right) \rho_1^2 12.$$

For $y \in Y$ with $\|y\| = \rho$, we have $\|y\|_\infty \leq b\|\dot{y}\|_2 \leq b\|y\| = b\rho = \rho_1$, so that $\beta(y(s)) \leq \beta_0 + \varepsilon|y(s)|^2 \leq \beta_0 + \varepsilon b^2\|\dot{y}\|_2^2$. Then

$$\int_0^{1/k} \beta(y)(T/k + \eta_y)^2 ds \leq (\beta_0 + \varepsilon b^2\|\dot{y}\|_2^2) T^2/k^3,$$

so

$$\begin{aligned} f(y) &\geq \alpha_0 \|\dot{y}\|_2^2 - (\beta_0 + \varepsilon b^2 \|\dot{y}\|_2^2) T^2 / k^3 + \beta_0 T^2 / k^3 = (\alpha_0 - \varepsilon T^2 b^2 / k^3) \|\dot{y}\|_2^2 \geq \\ &\geq (\alpha_0 - \varepsilon T^2 b^2 / k^3) a^2 \|y\|^2 = a^2 (\alpha_0 - \varepsilon T^2 b^2 / k^3) \rho_1^2 / b^2. \end{aligned}$$

Since $a^2 (\alpha_0 - \varepsilon T^2 b^2 / k^3) \rho_1^2 / b^2 > \delta$ for every $k \in \mathbb{N}$, the lemma is proved. ■

REMARK 2.8. Lemma 2.5 implies that the functional

$$x \mapsto \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 ds,$$

under assumption (1.3) is not superquadratic at infinity on finite-dimensional subspaces of H . This fact make not possible to apply the standard linking theorem of f . In order to avoid this difficult, we consider the subspace $E = \{x \in H | x(s + 1/2k) = -x(s)\}$. Clearly $E \subset Y$; moreover we have the following lemma.

LEMMA 2.9. *Let us suppose that (1.6) holds. Then, every critical point $x \in E$ of the functional $f|_E$ is a critical point of f .*

PROOF. Let $x \in E$ be a critical point of $f|_E$, and $z \in H$; we shall prove that $\langle f'(x), z \rangle = 0$. In fact, set $z_1(s) = z(s) - z(s + 1/2k)$, and $z_2(s) = z(s) - z_1(s)$, so that $z_1 \in E$, and $z = z_1 + z_2$. Since $\langle f'(x), z_1 \rangle = 0$, we have $\langle f'(x), z \rangle = \langle f'(x), z_2 \rangle$. From Remark 2.3, there exists $c_x > 0$ such that $\beta(x(s))(T/k + \eta_x(s)) = c_x$. Since β is even and $x \in E$, we have that $\eta_x(s + 1/2k) = \eta_x(s)$, and then it is easy to check, by using (1.6), that $\langle f'(x), z_2 \rangle = -\langle f'(x), z \rangle$, and the lemma is proved. ■

PROOF OF THEOREM 1.1. Let us suppose that (1.1)-(1.6) hold, let $k_0 \in \mathbb{N}$ be as in Lemma 2.6, $\delta > 0$ as in Lemma 2.7, and fix $k \in \mathbb{N}$ such that $k \geq k_0$ and $k\delta - \beta_0 T^2 / k^2 > 0$. From Lemma 2.6, the functional $f|_E$ satisfies the PSC condition on E . Let $w(s) = r(\cos(2k\pi s), \sin(2k\pi s), 0)$; clearly $w \in E$, and since $\beta(w(s)) \geq ar^p + b$, we have

$$\int_0^{1/k} \beta(w)(T/k + \eta_w)^2 ds \geq (ar^p + b) T^2 / k^3,$$

so that (see Remark 1.2) $f(w) \leq 4k\pi^2 c_2 r^{q+2} - (ar^p + b) T^2 / k^3 + \beta_0 T^2 / k^3$, and $f(w) < 0$ for r large enough (we recall that $q + 2 < p$). Set

$\Gamma = \{\gamma \in C([0, 1], E) \mid \gamma(0) = 0, \gamma(1) = w\}$, and let

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} f(\gamma(t)).$$

Let ρ be as in Lemma 2.7; since $f(0) = 0$ and we can assume $\|w\| > \rho$, we have $\delta \leq c < +\infty$. From the mountain pass lemma (see [1]), we have that c is a critical value for the functional $f|_E$. From Lemma 2.9 we get a critical point $x \in H$ of f with $f(x) = c$. Since $c > 0$, we have $\dot{x} \neq 0$. Because of Remark 2.1, $z(s) = (x(s), t(s))$, where $t(s) = Ts/k + \int_0^s \eta_x(\tau) d\tau$, is a 1-periodic T -trajectory on $(\mathbf{R}^4, g)$.

Finally, in order to prove that z is spacelike, we observe that

$$\begin{aligned} E_z &= \int_0^1 \alpha(x)[\dot{x}, \dot{x}] ds - \int_0^1 \beta(x) \dot{t}^2 ds = \\ &= k \left(\int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] ds - \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 ds \right) = \\ &= k \left(f(x) - \frac{\beta_0 T^2}{k^3} \right) \geq k\delta - \frac{\beta_0 T^2}{k^2} > 0, \end{aligned}$$

so that $E_z > 0$, and Theorem 1.1 is proved. ■

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