RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

MARIA ELVIRA MANCINO

Quantum stochastic differential equations driven by free noises and dilations of markovian semigroups

Rendiconti del Seminario Matematico della Università di Padova, tome 91 (1994), p. 199-223

<http://www.numdam.org/item?id=RSMUP_1994__91__199_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 91 (1994)

Quantum Stochastic Differential Equations Driven by Free Noises and Dilations of Markovian Semigroups.

MARIA ELVIRA MANCINO (*)

1. Introduction.

The Feynman-Kac formula allows to represent solutions of partial differential equations as expectation of Brownian functionals. This formula has been extended to the quantum stochastic case by Accardi (see [1]). The Boson Fock space quantum stochastic calculus has been recently applied to construct dilations of Markovian semigroups and give a quantum stochastic representation of solutions to Feller-Kolmogorov equations via the Feynman-Kac perturbation scheme (see [9]).

A satisfactory quantum stochastic calculus can be developed also with respect to the so called «free noise» introduced by Speicher (see [10]). This has been used to produce different dilations of Markovian semigroups with bounded generators (see [2], [5], [7], [10]). In [6] the stochastic calculus of [5], [7] was extended to the case of infinite creation and annihilation fields and this extension allowed to construct a unitary dilation of the semigroup of every countable state time continuous Markov process.

The aim of this paper is to extend these results to a bigger class of Markovian semigroups. For this end we establish an existence, uniqueness and unitary theorem for quantum stochastic differential equation with unbounded coefficients driven by infinite creation and annihilation and gauge fields. In Section 3 we extend the definitions of the left and right stochastic integrals with respect to the gauge operators too, and examine in detail the corresponding extensions of the left and right

(*) Indirizzo dell'A.: Università di Trento, Dipartimento di Matematica, I-38050 Povo (TN), Italia.

quantum Itô formulae. The case of quantum stochastic differential equation with bounded coefficients is studied in Section 4. The unitary solution gives the cocycle which provides by conjugation, dilations of every uniformly continuous quantum dynamical semigroup. In Section 5 we prove a theorem on quantum stochastic differential equation with unbounded coefficients satisfying the same conditions of the analogue theorem in [4] for Boson noises. These conditions are necessary and sufficient in order the quantum stochastic differential equation has a unique contractive solution. However we will not prove here that these are necessary because the free independence involves some elementary but quite complicated combinatorics. Moreover the main ideas are the same than in the Boson case. We prove then that the unique contractive solution is isometric if and only if the quantum dynamical semigroup is identity preserving. In particular the cocycle we obtain solving the quantum stochastic differential equation always dilates the minimal solution of a Feller-Kolmogorov equation. The technique of time reversal allows to reduce the problem of coisometry to the isometry one as in [6]. Thus, even if the free noise fields satisfy a different Itô table, it turns out that unitary cocycles obtained as solutions of quantum stochastic differential equations driven by free noise fields, can be used to construct unitary dilations of the same class of quantum stochastic differential equations dilated by unitary cocycles in the Boson Fock space.

I wish to thank Prof. F. Fagnola for suggesting me the problem and for his valuable help during the work.

2. Notations.

Let $\mathcal{K}$ be a complex separable Hilbert space, with fixed orthonormal basis $\{e_i\}_{i\in \boldsymbol{Z}^*}$ and let $\mathcal{F}$ be the full Fock space over $L^2(\boldsymbol{R}_+)\otimes\mathcal{K}$. For all functions u in $L^2(\boldsymbol{R}_+)\otimes\mathcal{K}$ and, for all $t\in\boldsymbol{R}_+$, $j\in\boldsymbol{Z}^*$ define $u^j(t)$ as $\langle e_j, u(t)\rangle_{\mathcal{K}}$ and let $\mathcal{M}$ be the set of step functions u in $L^2(\boldsymbol{R}_+)\otimes\mathcal{K}$ such that $\|u\|\leq 1$ and for all $j\in\boldsymbol{Z}^*$, $u^j(t)=0$ eventually and $\sup_s |u^j(s)|\leq 1$. For all element u of $\mathcal{M}$ let

$$N(u)=\max\{|i|: u^i(\cdot) \text{ is a nonzero function in } L^2(R_+)\}.$$

Let h_0 be a Hilbert space and consider the Hilbert space $\mathcal{H}=h_0\otimes\mathcal{F}$. Let $\mathcal{D}_0$ be a dense linear submanifold of h_0 and let $\mathcal{D}$ be the set of elements of $\mathcal{H}$ which can be written in the form $x\otimes\xi$, where x is an element of $\mathcal{D}_0$ and ξ is the k-fold tensor product of k

elements of $\mathcal{M}$ for some positive integer k. The vector $x \otimes \xi$ will be denoted also by $x\xi$.

For every $g \in L^2(\boldsymbol{R}_+, \boldsymbol{C})$ we consider the annihilation, creation and gauge operators $\{p_j^i\}_{i,j \in \boldsymbol{Z}}$ defined by:

$$p_j^0(g)\, u_1 \otimes \ldots \otimes u_k = g e_j \otimes u_1 \otimes \ldots \otimes u_k \qquad \text{for all } j \in \boldsymbol{Z}^*,$$

$$p_0^i(g)\, u_1 \otimes \ldots \otimes u_k = \langle g^i, u_1^i \rangle_{L^2(\boldsymbol{R}_+)} u_2 \otimes \ldots \otimes u_k \qquad \text{for all } i \in \boldsymbol{Z}^*,$$

$$p_j^i(g)\, u_1 \otimes \ldots \otimes u_k = (g u_1^i)\, e_j \otimes u_2 \otimes \ldots \otimes u_k \qquad \text{for all } i, j \in \boldsymbol{Z}^*.$$

We recall now some notations introduced in [6] and generalize for our purposes. Denote by $\mathcal{A}_{t]}$ (resp. $\mathcal{A}_{[t}$) with $t \in [0, +\infty]$, the $\star$-subalgebra of $\mathcal{B}(\mathcal{F})$ generated by $\{p_j^i(g)\}_{i,j \in \boldsymbol{Z}}$, where g is an element of $L^2(0, t)$ (resp. $L^2(t, +\infty)$).

Consider then the class $\mathcal{L}_{t]}(\mathcal{D}, \mathcal{H})$ of operators with domain containing $\mathcal{D}$, which are weak limits on $\mathcal{D}$ of finite sums of operators of the form $L \otimes A$, where L is a bounded operator on h_0 and A is an element of $\mathcal{A}_{t]}$. Analogous definitions for the class $\mathcal{L}_{[t}(\mathcal{D}, \mathcal{H})$. We write $\mathcal{L}(\mathcal{D}, \mathcal{H})$ when $t = +\infty$. Then a family $(X(t))_{t \geq 0}$ of operators in $\mathcal{L}(\mathcal{D}, \mathcal{H})$ is called a process.

For all $z \in \mathcal{H}$ with chaos decomposition with respect to Fock space $z = \sum_{k=0}^{\infty} z_k$, let $[z \otimes \xi]$ denote the vector in $\mathcal{H}$ with chaos decomposition $\sum_{k=0}^{\infty} z_k \otimes \xi$. We note that, with each $A \in \mathcal{B}(h_0) \otimes \mathcal{A}_{\infty]}$, it is possible to associate two operators A_+, A_- of $\mathcal{B}(h_0) \otimes \mathcal{A}_{\infty]}$ ([5], [7]) characterized by:

$$\text{(2.1)} \quad A_+ x\xi = [(Ax\Omega) \otimes \xi], \qquad \langle A_- y\eta, x\xi \rangle = \langle y\eta, [(A^* x\Omega) \otimes \xi] \rangle.$$

Remark that, for all $t \in \boldsymbol{R}_+$ and all $X \in \mathcal{L}_{t]}(\mathcal{D}, \mathcal{H})$, the first formula (2.1) uniquely defines an operator $X_+ \in \mathcal{L}_{t]}(\mathcal{D}, \mathcal{H})$.

Now for all $t \in \boldsymbol{R}_+$ and $i, j \in \boldsymbol{Z}$ define the processes

$$p_j^i(t) = p_j^i(\chi_{(0,t)}).$$

The deterministic process $(tI)_{t \geq 0}$ will be denoted by $(p_0^0(t))_{t \geq 0}$. The relations between the p_j^i yield the Itô table ([5], [7]), for all i, i', $j, j' \in \boldsymbol{Z}$

$$\text{(2.2)} \qquad dp_{i'}^{j'}(t)\, dp_j^i(t) = \hat{\delta}_j^{j'}\, dp_{i'}^i(t)$$

where $\hat{\delta}_j^{j'}$ is zero if $j = 0$ or $j' = 0$ and is equal to the Kronecher delta $\delta_j^{j'}$ otherwise.

In order to extend the definition of left and right stochastic integrals with respect to the processes $\{p_j^i\}_{i,j \in \boldsymbol{Z}}$, we consider the two classes of processes $\mathfrak{I}_2$ and $\mathfrak{I}_-$ introduced in [6].

$\mathfrak{I}_2$ is the class of processes F strongly measurable on $\mathcal{D}$, such that $F(t) \in \mathcal{L}_{t]}(\mathcal{D}, \mathcal{H})$ and

$$\int_0^t \|F(s)\, x\xi\|^2\, ds < +\infty$$

for all $t \in \boldsymbol{R}_+$. We note that the elements of $\mathfrak{I}_2$ can be integrated from the right with respect to p_j^i for all $i, j \in \boldsymbol{Z}$ and from the left with respect to p_j^0 for all $j \in \boldsymbol{Z}$ thanks to (2.1) and the Proposition 3.1 in [5].

$\mathfrak{I}_-$ is the class of processes F with the following property: there exists a sequence $\{F^{(n)}\}_{n \geq 0}$ in $\mathcal{B}(h_0) \otimes \mathcal{A}_{\infty]}$ such that, for all $t \in \boldsymbol{R}_+$, $F^{(n)}(t)$ is a member of $\mathcal{B}(h_0) \otimes \mathcal{A}_{t]}$, the operator $F^{(n)*}(t)$ converges strongly on h_0 and, for all $t \in \boldsymbol{R}_+$ and all vector $x\xi \in \mathcal{D}$, it holds:

$$\lim_{m, n \to \infty} \int_0^t \|(F_-^{(n)}(s) - F_-^{(m)}(s))\, x\xi\|^2\, ds = 0\,.$$

For all $F \in \mathfrak{I}_-$ one can define the process F_- as a strong limit on $\mathcal{D}$ of processes $\{F_-^{(n)}\}_{n \geq 0}$. Note that, for elements in the class $\mathfrak{I}_-$, the integrability from the left with respect to dp_j^i for all $i \in \boldsymbol{Z}^*, j \in \boldsymbol{Z}$ is guaranteed in virtue of Proposition 3.1 in [5].

3. Stochastic calculus preliminaries.

In this section we compute some quantum Itô formula, we will need to find the unitary solution to the quantum stochastic differential equation we deal with.

For all $\xi = u_1 \otimes \ldots \otimes u_k$ where $k \in \boldsymbol{Z}$ let η_k be the element $u_1 \otimes \ldots \otimes u_{h-1} \otimes u_{h+1} \otimes \ldots \otimes u_k$ if $h \in \{1, \ldots, k\}$, the vacuum vector Ω if $k = 1$, and 0 otherwise. If G is a process and α is a positive integer, we denote with $G^{(\alpha)}$ the process $\sum_{\beta \geq 0} \Pi_\beta G \Pi_{\beta + \alpha}$, and Π_β is the orthogonal projection onto the β-th chaos in $\mathcal{F}$.

LEMMA 3.1. *Let G, F be processes in $\mathfrak{I}_2$. Then, for all positive real numbers s, t with $s < t$, for all $i, j \in \boldsymbol{Z}^*$ and for all $x\xi$, $x'\xi' \in \mathcal{D}$, with*

$\xi = u_1 \otimes \ldots \otimes u_k$, $\xi' = u_1' \otimes \ldots \otimes u_{k'}'$, *we have:*

(i) $$\langle G(s)x'\xi', F(s)p_0^i(s,t)x\xi\rangle = \int_s^t u_1^i(r)\,dr\langle G(s)x'\xi', F(s)x\eta_1\rangle,$$

(ii) $$\langle G_+(s)x'\xi', F_+(s)p_j^0(s,t)x\xi\rangle = \\ = \int_s^t \overline{u_{k'-k}'^j(r)}\,dr\langle G_+(s)x'\eta_{k'-k}', F_+(s)x\xi\rangle,$$

(iii) $$\langle G_+(s)x'\xi', F_+(s)p_j^i(s,t)x\xi\rangle = \\ = \int_s^t \overline{u_{k'-k+1}'^i(r)}\,u_1^j(r)\,dr\langle G_+(s)x'\eta_{k'-k+1}', F_+(s)x\eta_1\rangle.$$

PROOF. The identity (i) follows immediately from the definition of p_0^i, with $i \in \mathbf{Z}^*$.

Let us now prove (ii):

$$\langle G_+(s)x'\xi', F_+(s)p_j^0(s,t)x\xi\rangle = \sum_{p,q}\langle G_+^{(p)}(s)x'\xi', F_+^{(q)}(s)p_j^0(s,t)x\xi\rangle =$$

$$= \sum_q \langle G_+^{(q+k-k'+1)}(s)x'u_1' \otimes \ldots \otimes u_{k'}', F_+^{(q)}(s)p_j^0(s,t)xu_1 \otimes \ldots \otimes u_k\rangle =$$

$$= \begin{cases} \sum_q \langle G_+^{(q+k-k'+1)}(s)x'\eta_{k'-k}', F_+^{(q)}(s)x\xi\rangle \int_s^t \overline{u_{k'-k}'^j(r)}\,dr & \text{if } k' \geq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Now (ii) follows immediately.

The proof of (iii) is obtained in the same way, and we omit it. ■

This lemma is a slight extension of Lemma 2.2 in [6], because in our case we are interested in computing scalar products of processes involving the gauge operators, too. We observe that we have not the same results as in [6]: infact, if G and F containg also gauge operators p_j^i, with $i, j \in \mathbf{Z}$, then the scalar product $\langle G(s)x'\xi', F_+(s)p_j^i(s,t)x\xi\rangle$ is more complicated. Anyway our result will be enough for our later purposes.

LEMMA 3.2. *Let G be a process in $\mathfrak{Z}_2$ and F be a process in $\mathfrak{Z}_-$, then, for all positive real number s, t with $s < t$, for all $i \in \mathbf{Z}^*$ and for all*

$x\xi$, $x'\xi' \in \mathcal{D}$, with $\xi = u_1 \otimes \ldots \otimes u_k$, $\xi' = u_1' \otimes \ldots \otimes u_{k'}'$, we have:

(i) $\langle G(s)x'\xi', p_0^i(s,t)F_-(s)x\xi\rangle =$

$$= \sum_{\alpha=0}^{k-1} \int_s^t u_{\alpha+1}^i(r)\,dr \langle G(s)x'\xi', F_-^{(\alpha)}(s)x\eta_{\alpha+1}\rangle.$$

Let G, F be processes in $\mathfrak{I}_-$, then for all $x\xi$, $x'\xi' \in \mathcal{D}$, for all positive real number s, t with $s < t$ and for all $i,j \in \mathbf{Z}^*$, we have:

(ii) $\langle G_-(s)x'\xi', p_j^i(s,t)F_-(s)x\xi\rangle =$

$$= \sum_{\alpha=(k-k')\vee 0}^{k-1} \int_s^t \overline{u_{k'-k+\alpha+1}'^{j}(r)}\, u_{\alpha+1}^i(r)\,dr \,\cdot$$

$$\cdot \langle G_-^{(k'-k+\alpha)}(s)x'\eta_{k'-k+\alpha+1}', F_-^{(\alpha)}(s)x\eta_{\alpha+1}\rangle.$$

PROOF. We will show only (i), the proof of (ii) being analogous. Since F is in $\mathfrak{I}_-$, it is strong limit on $\mathcal{D}$ of processes $\{F^{(n)}\}_{n\geq 0}$ in $\mathcal{B}(h_0)\otimes \mathcal{A}_{\infty]}$. Then we can verify (i) for such simple processes. We have:

$$\langle G(s)x'\xi', p_0^i(s,t)F_-(s)x\xi\rangle = \sum_{\alpha=0}^{k-1}\langle G(s)x'\xi', p_0^i(s,t)F_-^{(\alpha)}(s)x\xi\rangle =$$

$$= \sum_{\alpha=0}^{k-1}\langle G(s)x'\xi', [F_-^{(\alpha)}(s)xu_1\otimes\ldots\otimes u_\alpha]\otimes p_0^i(s,t)u_{a+1}\otimes\ldots\otimes u_k\rangle =$$

$$= \sum_{\alpha=0}^{k-1}\int_s^t u_{\alpha+1}^i(r)\,dr\langle G(s)x'\xi', F_-^{(\alpha)}(s)x\eta_{\alpha+1}\rangle.$$

Then (i) is proved. ■

Now using Lemmas 3.1, 3.2 and the identities (2.2) we can compute the left Itô formula for all processes $F \in \mathfrak{I}_-$, and the right Itô formula for all processes F_+ with $F \in \mathfrak{I}_2$. We study in detail only two cases; the others are obtained with similar arguments. Suppose that F is a process in $\mathfrak{I}_2 \cap \mathfrak{I}_-$, and, for all $t \in \mathbf{R}_+$, $i,j \in \mathbf{Z}$, let

$$\mathcal{L}_j^i(F,t) = \int_0^t dp_j^i(s)F(s).$$

Note that $\mathcal{L}_j^i(F,t)_-$ is identically zero for all $i,j \in \mathbf{Z}^*$. Then, for all

$x\xi$, $x'\xi' \in \mathcal{D}$, i, j, $i', j' \in \mathbf{Z}^*$ and for all processes G, F in the space $\mathfrak{I}_2 \cap \mathfrak{I}_-$, we have:

$$\langle \mathcal{L}_{j'}^{i'}(G, t)\, x'\xi', \mathcal{L}_j^i(F, t)\, x\xi \rangle =$$

$$= \hat{\delta}_j^{j'} \sum_{\alpha = (k-k') \vee 0}^{k-1} \int_0^t \langle G_-^{(k'-k+\alpha)}(s)\, x'\eta'_{k'-k+\alpha+1}, F_-^{(\alpha)}(s)\, x\eta_{\alpha+1} \rangle \cdot$$

$$\cdot\, \overline{u'^{i'}_{k'-k+\alpha+1}(s)}\; u^i_{\alpha+1}(s)\, ds .$$

Now suppose G, F are processes in $\mathfrak{I}_2$ and for all $t \in \mathbf{R}_+$, $i, j \in \mathbf{Z}$ denote by $\mathcal{R}_j^i(F, t)$ the right integral of F with respect to p_j^i. We have then, for all $x\xi$, $x'\xi' \in \mathcal{D}$, $i, i', j, j' \in \mathbf{Z}^*$:

$$\langle \mathcal{R}_{j'}^{i'}(G_+, t)\, x'\xi', \mathcal{R}_j^i(F_+, t)\, x\xi \rangle =$$

$$= \int_0^t \langle \mathcal{R}_{j'}^{i'}(G_+, s)\, x'\eta'_{k'-k+1}, F_+(s)\, x\eta_1 \rangle\, \overline{u'^{i}_{k'-k+1}(s)}\; u_1^j(s)\, ds +$$

$$+ \int_0^t \langle G_+(s)\, x'\eta'_1, \mathcal{R}_j^i(F_+, s)\, x\eta_{k-k'+1} \rangle\, \overline{u'^{j'}_1(s)}\; u^{i'}_{k-k'+1}(s)\, ds +$$

$$+ \hat{\delta}_j^{j'} \int_0^t \langle G_+(s)\, x'\Omega, F_+(s)\, x\Omega \rangle\, \overline{u'^{i'}_1(s)}\; u^i(s) \langle \eta'_1, \eta_1 \rangle\, ds .$$

We conclude this Section with a proposition whose proof is essentially an application of the Itô formula.

In the following proposition F_{-1}, F_0, F_{+1} denote respectively the processes F_-, F, F_+. Moreover for all $\xi = u_1 \otimes \ldots \otimes u_k \in \mathcal{D}$, let $J(\xi)$ be the subset of $\mathcal{D}$ consisting of Ω and vectors of the form $u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(h)}$ with $h \in \{1, \ldots, k\}$ and σ: $\{1, \ldots, h\} \to \{1, \ldots, k\}$ increasing.

PROPOSITION 3.3. *Let $\{F_i^j\}_{i,j \in \mathbf{Z}}$ be processes in $\mathfrak{I}_2 \cap \mathfrak{I}_-$, satisfying*

$$\sum_{j \in \mathbf{Z}} \int_0^t \| (F_i^j)_\varepsilon(s)\, x\xi \|^2 ds < +\infty \tag{3.1}$$

for all $t \in \mathbf{R}_+$, $x\xi \in \mathcal{D}$, $i \in \mathbf{Z}$ and $\varepsilon \in \{-1, 0, +1\}$.. Define, for all $\mathrm{n} \in \mathbf{N}$, *the processes*

$$X^{(n)}(t) = \sum_{|i|,\ |j| \leq n} \int_0^t dp_j^i(s) F_i^j(s).$$

Then there exists a process X such that:

(i) $$\lim_{n \to \infty} \sup_{0 \leq s \leq t} \|((X^{(n)} - X)_\varepsilon(s)) x\xi\| = 0$$

for all $t \in \boldsymbol{R}_+$, $x\xi \in \mathcal{D}$ *and* $\varepsilon \in \{-1, 0, +1\}$

(ii) $$\|X(t) x\xi\|^2 \leq c_{t,\xi} \sum_{\eta \in J(\xi)} \sum_{|i| \leq N(\xi)} \sum_{j \in \boldsymbol{Z}} \int_0^t (\|(F_i^j)_-(s) x\eta\|^2 + \|F_i^j(s) x\eta\|^2)\, ds$$

for all $t \in \boldsymbol{R}_+$, $x\xi \in \mathcal{D}$, *where* $c_{t,\xi} = 3(t+1) e^{2t + 2N(\xi) + 1}(2N(\xi) + 1)^2 2^k$.

PROOF. First observe that (3.1) in the case $\varepsilon = +1$ follows from the case $\varepsilon = 0$ using the first of the fundamental relations (2.1). For the same reason it is enough to show (i) in the cases $\varepsilon = 0, -1$. Consider, for all $t \in \boldsymbol{R}_+$, and $m < n$

$$(X^{(n)} - X^{(m)})(t) = \sum_{m < \max\{|i|,\ |j|\} \leq n} \int_0^t dp_j^i(s) F_i^j(s).$$

Now, using the left Itô formula, we obtain for all $m, n > N(\xi)$:

$$\|(X^{(n)} - X^{(m)})(t) x\xi\|^2 = \sum_{m < |j| \leq n} \int_0^t \|F_0^j(s) x\xi\|^2 ds.$$

Therefore from (3.1) it follows that, for all $t \in \boldsymbol{R}_+$, $x\xi \in \mathcal{D}$:

(3.2) $$\lim_{m, n \to \infty} \sup_{0 \leq s \leq t} \|(X^{(n)} - X^{(m)})(s) x\xi\| = 0.$$

It is easily verified by the left Itô formula that if $m, n > N(\xi)$, then $X_-^{(n)}(t) x\xi = X_-^{(m)}(t) x\xi$ for all $t \in \boldsymbol{R}_+$, $x\xi \in \mathcal{D}$. This implies

(3.3) $$\lim_{m, n \to \infty} \sup_{0 \leq s \leq t} \|(X^{(n)} - X^{(m)})_-(s) x\xi\| = 0.$$

Then, for all $t \in \boldsymbol{R}_+$, let $X_\varepsilon(t)$ be the strong limit on $\mathcal{D}$ of $X_\varepsilon^{(n)}(t)$. Now from (3.2) and (3.3) follows (i). To get (ii) we need first an inequality for

$X_-(t)$. Using Lemma 3.2 we have:

$$\|X_-^{(n)}(t)x\xi\|^2 =$$

$$= 2\,\Re e \sum_{0<|i|\leq n\wedge N(\xi)} \sum_{\alpha=0}^{k-1} \int_0^t \langle X_-^{(n)}(s)x\xi, (F_i^0)_-^{(\alpha)}(s)x\eta_{\alpha+1}\rangle u_{\alpha+1}^i(s)\,ds +$$

$$+ \int_0^t \langle X_-^{(n)}(s)x\xi, (F_0^0)_-(s)x\xi\rangle\,ds \leq \int_0^t \|X_-^{(n)}(s)x\xi\|^2\,ds +$$

$$+ \int_0^t \left\| \sum_{0<|i|\leq n\wedge N(\xi)} \sum_{\alpha=0}^{k-1} (F_i^0)_-^{(\alpha)} x\eta_{\alpha+1} + (F_0^0)_-(s)x\xi \right\|^2 ds\,.$$

Hence, by Gronwall's lemma, we obtain

$$\|X_-^{(n)}(t)x\xi\|^2 \leq e^t \int_0^t \left\| \sum_{0<|i|\leq n\wedge N(\xi)} \sum_{\alpha=0}^{k-1} (F_i^0)_-^{(\alpha)} x\eta_{\alpha+1} + (F_0^0)_-(s)x\xi \right\|^2 ds \leq$$

$$\leq e^t(2N(\xi)+1) \sum_{|i|\leq N(\xi)} \sum_{\eta\in J(\xi)} \int_0^t \|(F_i^0)_- x\eta\|^2\,ds\,.$$

The letting n go to infinity we conclude

$$\|X_-(t)x\xi\|^2 \leq e^t(2N(\xi)+1) \sum_{|i|\leq N(\xi)} \sum_{\eta\in J(\xi)} \int_0^t \|(F_i^0)_- x\eta\|^2\,ds\,.$$

In analogous way if we compute

$$\|X^{(n)}(t)x\xi\|^2$$

using the left Itô formula, then by Schwarz's inequality and the inequality obtained above for $\|X_-(t)x\xi\|^2$ we have

$$\|X^{(n)}(t)x\xi\|^2 \leq (2N(\xi)+1)\int_0^t \|X^{(n)}(s)x\xi\|^2\,ds + \varphi(t)$$

where

$$\varphi(t) \leq 3(2N(\xi)+1)^2\,|J(\xi)|\,e^t(t+1)\,\cdot$$

$$\cdot \sum_{\eta\in J(\xi)} \sum_{|i|\leq N(\xi)} \sum_{|j|\leq n} (\|(F_i^j)_-(s)x\eta\|^2 + \|(F_i^j)(s)x\eta\|^2)\,ds\,.$$

Hence, by Gronwall's lemma, we conclude that

$$\|X^{(n)}(t)\,x\xi\|^2 \leqslant c_{t,\,\xi} \sum_{|i| \leqslant N(\xi)} \sum_{\eta \in J(\xi)} \sum_{|j| \leqslant n} \int_0^t (\|(F_i^j)_-\,x\eta\|^2 + \|F_i^j\,x\eta\|^2)\,ds\,.$$

Now letting n to infinity we get the inequality (ii) and by hypothesis (3.1) the series is convergent. ■

4. A class of quantum stochastic differential equations with bounded coefficients.

PROPOSITION 4.1. *Let $\{L_i^j\colon\, i, j \in \boldsymbol{Z}\}$ be bounded operators on h_0 and suppose that there exist positive constants c_i, $i \in \boldsymbol{Z}$ such that, for all $x \in h_0$,*

$$\sum_{j \in \boldsymbol{Z}} \|L_i^j\,x\|^2 \leqslant c_i^2\,\|x\|^2\,. \tag{4.1}$$

Then, for each $X_0 \in \mathcal{B}(h_0)$, there exists a process $X \in \mathfrak{J}_2 \cap \mathfrak{J}_-$ satisfying, for all $t \in \boldsymbol{R}_+$

$$X(t) = X_0 + \sum_{i,\,j \in \boldsymbol{Z}} \int_0^t dp_j^i(s)\,L_i^j\,X(s)\,. \tag{4.2}$$

PROOF. We set up the following iterative scheme

$$X^{(0)}(t) = X_0\,,$$

$$X^{(n+1)}(t) = \sum_{i,\,j \in \boldsymbol{Z}} \int_0^t dp_j^i(s)\,L_i^j\,X^{(n)}(s) \qquad \text{for } n \geqslant 0\,.$$

Each $X^{(n)}$ is well defined and for all $t \in \boldsymbol{R}_+$, $x\xi \in \mathcal{D}$:

$$\|X^{(n)}(t)\,x\xi\|^2 \leqslant \frac{C(\xi)^n\,e^{2nt}\,(t+1)^n\,t^n}{n!}\,\|X_0\|^2\,\|x\|^2 \tag{4.3}$$

where $C(\xi) = 3(2N(\xi)+1)^3\,|J(\xi)|^2\,e^{2N(\xi)+1} \max_{|i| \leqslant N(\xi)} \{c_i^2\}$.

This inequality follows immediately by induction using the inequality (ii) in Proposition 3.3 and the hypothesis (4.1). In the same way we obtain an analogous inequality for $X_-^{(n)}$:

$$\|X_-^{(n)}(t)\,x\xi\|^2 \leqslant \frac{\widehat{C}(\xi)^n\,e^{nt}\,t^n}{n!}\,\|x\|^2 \tag{4.4}$$

where $\widehat{C}(\xi) = (2N(\xi) + 1)|J(\xi)| \max_{|i| \leq N(\xi)} \|L_i^0\|^2$. Therefore for all $x\xi \in \mathcal{D}$, $t \in \boldsymbol{R}_+$ the process X defined by

$$X(t)\, x\xi = \sum_{n \geq 0} X^{(n)}(t)\, x\xi$$

satisfies (4.2). ■

We state now the main result of this Section.

THEOREM 4.2. *Let* $\{L_i^j\}_{i,j \in \boldsymbol{Z}}$ *be bounded operators on* h_0 *with the property that, for each* $j \in \boldsymbol{Z}$, *there exists a positive constant* c_j *such that*

$$\sum_{i \in \boldsymbol{Z}^*} \|(L_i^j)^* x\|^2 \leq c_j^2 \|x\|^2 \quad \text{for all } x \in h_0 . \tag{4.5}$$

Consider also the following conditions:

$$L_0^0 + (L_0^0)^* + \sum_{l \in \boldsymbol{Z}^*} (L_0^l)^* L_0^l \leq 0 , \tag{4.6}$$

$$L_j^0 + (L_0^j)^* + \sum_{l \in \boldsymbol{Z}^*} (L_0^l)^* L_j^l = 0 \quad \text{for all } j \in \boldsymbol{Z}^* , \tag{4.7}$$

$$L_i^j + (L_j^i)^* + \sum_{l \in \boldsymbol{Z}^*} (L_j^l)^* L_i^l = 0 \quad \text{for all } i, j \in \boldsymbol{Z}^* . \tag{4.8}$$

If (4.6), (4.7), (4.8) *hold, then there exists a unique contractive process* U *such that*

$$U(t) = 1 + \sum_{i,j \in \boldsymbol{Z}} \int_0^t U(s) L_i^j \, dp_j^i(s) . \tag{4.9}$$

Suppose moreover that the lefthand side of (4.6) *vanishes and the following identity holds*:

$$L_i^j + (L_j^i)^* + \sum_{l \in \boldsymbol{Z}^*} L_l^j (L_l^i)^* = 0 \quad \text{for all } i, j \in \boldsymbol{Z}^* . \tag{4.10}$$

then U *is unitary and the adjoint process* V *is solution of*

$$V(t) = 1 + \sum_{i,j \in \boldsymbol{Z}} \int_0^t dp_i^j(s) (L_i^j)^* V(s) . \tag{4.11}$$

REMARK 4.3. The conditions (4.6), (4.7) imply that also the following conditions hold:

$$L_0^0 + (L_0^0)^* + \sum_{l \in \boldsymbol{Z}^*} L_l^0 (L_l^0)^* \leq 0\,, \tag{4.12}$$

$$L_j^0 + L_0^j)^* + \sum_{l \in \boldsymbol{Z}^*} L_l^0 (L_l^j)^* = 0 \qquad \text{for all } j \in \boldsymbol{Z}^*\,, \tag{4.13}$$

REMARK 4.4. Using (4.13) the property (4.5) also implies that there exists a positive constant c such that

$$\sum_{j \in \boldsymbol{Z}^*} \|L_i^j x\|^2 \leq c^2 \|x\|^2 \qquad \text{for all } x \in h_0,\ i \in \boldsymbol{Z}\,.$$

We will prove the result of Theorem 4.2 in several steps.

PROPOSITION 4.5. *If hypothesis* (4.5) *and conditions* (4.6), (4.7), (4.10) *of Theorem* 4.2 *are fulfilled, then there exists a contractive solution of the equation* (4.11). *Moreover if the lefthand side of* (4.6) *vanishes, there is a unique isometric solution.*

PROOF. From Proposition 4.1, it follows that there exists a solution of (4.11). Then using the left Itô formula we compute, for all $t \in \boldsymbol{R}_+$, $x\xi$, $x'\xi' \in \mathcal{D}$ the derivative of the scalar product:

$$d\langle V(t)\, x'\, \xi',\, V(t)\, x\xi \rangle\,.$$

Grouping similar terms we obtain four addends:

$$\Big\langle V(t)\, x'\, \xi', \big[L_0^0 + (L_0^0)^* + \sum_{j \in \boldsymbol{Z}^*} L_j^0 (L_j^0)^* \big] V(t)\, x\xi \Big\rangle$$

$$\sum_{0 < |j| \leq N(\xi)} \sum_{\alpha=0}^{k-1} \Big\langle V(t)\, x'\, \xi', \big[L_j^0 + (L_0^j)^* + \sum_{l \in \boldsymbol{Z}^*} L_l^0 (L_l^j)^* \big] V_-^{(\alpha)}(t)\, x\eta_{\alpha+1} \Big\rangle\, u_{\alpha+1}^j(t)$$

$$\sum_{0 < |j| \leq N(\xi)} \sum_{\alpha=0}^{k'-1} \Big\langle V_-^{(\alpha)}(t)\, x'\, \eta'_{\alpha+1}, \big[L_0^j + (L_j^0)^* + \sum_{l \in \boldsymbol{Z}^*} L_l^j (L_l^0)^* \big] V(t)\, x\xi \Big\rangle\, \overline{u_{\alpha+1}^{\prime j}(t)}$$

$$\sum_{0 < |i|,\, |j| \leq N(\xi)} \sum_{\alpha=(k-k')\vee 0}^{k-1} \overline{u_{k'-k+\alpha+1}^{\prime j}(t)}\, u_{\alpha+1}^i(t)\,\cdot$$

$$\cdot \Big\langle V_-^{(k'-k+\alpha)}(t)\, x'\, \eta'_{k'-k+\alpha+1}, \big[L_i^j + (L_j^i)^* + \sum_{l \in \boldsymbol{Z}^*} L_l^j (L_l^i)^* \big] V_-^{(\alpha)}(t)\, x\eta_{\alpha+1} \Big\rangle.$$

Using (4.10), (4.12), (4.13) we conclude that $V(t)$ is a contraction. Moreover, if (4.12) vanishes, the first addend vanishes also, hence we have:

$$\langle V(t)\,x'\,\xi',\, V(t)\,x\xi\rangle = \langle x'\,\xi',\, x\xi\rangle\,.$$

Clearly $V(t)$ is the only isometric solution. ■

LEMMA 4.6. *Let* $\{F_i^j\}_{i,j\in\boldsymbol{Z}}$ *be bounded operators on* h_0 *with the property that, for each* $j\in\boldsymbol{Z}$, *there exists a positive constant* c_j *such that*

$$\sum_{i\in\boldsymbol{Z}^*}\|F_i^j x\|^2\leq c_j^2\|x\|^2\qquad \textit{for all } x\in h_0\,.$$

If Y *is a solution of the equation*

$$Y(t)=\sum_{i,j\in\boldsymbol{Z}}\int_0^t Y(s)\,F_i^j\,dp_j^i(s)$$

with the property that there exists a constant C *such that:* $\sup_{s\leq t}\|Y(s)\,x\xi\|\leq C\|x\|\,\|\xi\|$ *for all* $t\in\boldsymbol{R}_+$, $x\xi\in\mathcal{D}$, *then* $Y=0$.

PROOF. Using Proposition 8.1 in [5] we can prove that, for all $x'\,\xi'$, $x\xi\in\mathcal{D}$, $t\in\boldsymbol{R}_+$, it holds:

$$\langle x'\,\xi',\, Y(t)\,x\xi\rangle = 0\,.$$

This identity is proved by induction on $N=k+k'$, where $\xi'=$ $=u_1'\otimes\ldots\otimes u_{k'}'$ and $\xi=u_1\otimes\ldots\otimes u_k$. ■

Finally we conclude the proof of Theorem 4.2:

PROPOSITION 4.7. *Under the hypothesis of Theorem* 4.2, $V^*=U$ *is the unique solution of the stochastic differential equation:*

$$U(t)=1+\sum_{i,j\in\boldsymbol{Z}}\int_0^t U(s)\,L_i^j\,dp_j^i(s) \tag{4.14}$$

the series $\sum_{i,j\in\boldsymbol{Z}}\int_0^t U(s)\,L_i^j\,dp_j^i(s)$ *being strongly convergent on* $\mathcal{D}$. *Moreover* U *isometric.*

PROOF. For all $x\xi\in\mathcal{D}$ the function $x\mapsto U(s)\,x\xi$ is strongly measurable (because it is weakly measurable), and

$$\int_0^t \|U(s)\,x\xi\|^2\,ds < +\infty\,.$$

This ensures that $U(\cdot)$ is integrable from the right with respect to dp_j^i, for all $i,j \in \mathbf{Z}$. Now using Proposition 8.1 in [5] it follows that U satisfies (4.14). Moreover U is the unique solution by Lemma 4.6.

Now we want to prove that the series is strongly convergent Ⓓ. Observe first that, for all $i,j \in \mathbf{Z}$ with $i > N(\xi)$ we have:

$$\int_0^t U(s)\,L_i^j\,dp_j^i(s)\,x\xi = 0\,.$$

Moreover, for each $m, n \in \mathbf{N}$ with $m < n$:

$$\left\|\left\{\sum_{|i|\leqslant N(\xi)}\sum_{|j|\leqslant n}\int_0^t U(s)L_i^j\,dp_j^i(s) - \sum_{|i|\leqslant N(\xi)}\sum_{|j|\leqslant m}\int_0^t U(s)L_i^j\,dp_j^i(s)\right\}x\xi\right\|^2 \leqslant \tag{4.15}$$

$$\leqslant (2N(\xi))\left\{\left\|\sum_{m<|j|\leqslant n}\int_0^t U_+(s)L_0^j\,dp_j^0(s)\,x\xi\right\|^2 + \right.$$

$$\left. + \sum_{|i|\leqslant N(\xi)}\left\|\sum_{m<|j|\leqslant n}\int_0^t U_+(s)L_i^j\,dp_j^i(s)\,x\xi\right\|^2\right\}.$$

Consider now the second sum in (4.15): computing by the right Itô formula we see that if $m > N(\xi)$ this sum is equal to zero. Then computing again by the right Itô formula, from the first sum in (4.15) we obtain:

$$\left\|\sum_{m<|j|\leqslant n}\int_0^t U_+(s)\,L_0^j\,dp_j^0(s)\,x\xi\right\|^2 = \sum_{m<|j|\leqslant n}\int_0^t \|U_+(s)\,L_0^j\,x\xi\|^2\,ds\,.$$

Remark now that the solution $U(t)$ to (4.14) satisfies, for all $x', x \in h_0$

$$d\langle U(t)\,x'\Omega,\ U(t)\,x\Omega\rangle = 0\,; \tag{4.16}$$

this follows computing the derivative by Itô formula and the hypothesis that (4.6) vanishes. Therefore last remark and (2.1) imply:

$$\|U_+(s)L_0^j x\xi\|^2 \leq \|U(s)L_0^j x\Omega\|^2\|\xi\|^2 \leq \|L_0^j x\|^2\|\xi\|^2 .$$

Now Remark 4.4 allows us to conclude that the series converges strongly on $\mathcal{D}$ uniformly in each bounded interval of $\boldsymbol{R}_+$.

Let us now prove the isometricity of U.

Let $x\xi$, $x'\xi' \in \mathcal{D}$ and $\xi = u_1 \otimes \ldots \otimes u_k$, $\xi' = u_1' \otimes \ldots \otimes u_{k'}'$. The proof proceeds by induction on $k + k'$. Start with $k + k' = 0$ i.e. $\xi = \xi' = \Omega$; this is immediate because of (4.16). Suppose now that

$$d\langle U(t)x'\xi', U(t)x\xi\rangle = 0$$

if $k + k' \leq N$ and show it for $k + k' = N + 1$. We have

$$\begin{aligned}d\langle U(t)x'\xi', U(t)x\xi\rangle &= \sum_{l\in \boldsymbol{Z}^*}\sum_{i,j\in \boldsymbol{Z}}\langle U(t)L_j^l x'\Omega, U(t)L_i^l x\Omega\rangle\langle\xi', dp_j^i(t)\xi\rangle + \\ &+ \langle U(t)x'\xi', U(t)L_0^0 x\xi\rangle dt + \sum_{i\in \boldsymbol{Z}^*}\langle U(t)x'\xi', U(t)L_i^0 dp_0^i(t)x\xi\rangle + \\ &+ \sum_{j\in \boldsymbol{Z}^*}\langle x'\xi', (U^*U)_+(t)L_0^j dp_j^0(t)x\xi\rangle + \sum_{i,j\in \boldsymbol{Z}^*}\langle x'\xi', (U^*U)_+(t)L_i^j dp_j^i(t)x\xi\rangle + \\ &+ \langle U(t)L_0^0 x'\xi', U(t)x\xi\rangle dt + \sum_{i\in \boldsymbol{Z}^*}\langle U(t)L_i^0 dp_0^i(t)x'\xi', U(t)x\xi\rangle + \\ &+ \sum_{j\in \boldsymbol{Z}^*}\langle (U^*U)_+(t)L_0^j dp_j^0(t)x'\xi', x\xi\rangle + \sum_{i,j\in \boldsymbol{Z}^*}\langle (U^*U)_+(t)L_i^j dp_j^i(t)x'\xi', x\xi\rangle .\end{aligned}$$

Observe now that using the inductive hypothesis, for all $i \in \boldsymbol{Z}^*$

$$\langle U(t)x'\xi', U(t)L_i^0 dp_0^i(t)x\xi\rangle = \langle x'\xi', L_i^0 dp_0^i(t)x\xi\rangle$$

and

$$\langle U(t)L_i^0 dp_0^i(t)x'\xi', U(t)x\xi\rangle = \langle L_i^0 dp_0^i(t)x'\xi', x\xi\rangle .$$

Moreover, because of the isometricity of $U(t)$ over h_0, the first term in the sum is equal to

$$\sum_{l\in \boldsymbol{Z}^*}\sum_{i,j\in \boldsymbol{Z}^*}\langle L_j^l x', L_i^l x\rangle\langle\xi', dp_j^i(t)\xi\rangle . \tag{4.17}$$

Remark now that from the isometricity of $U(t)$ on h_0 it follows that

$$(U^*U)_+(t) = 1 . \tag{4.18}$$

Infact using the first identity in (2.1) we obtain, for all $t \in \boldsymbol{R}_+$, $x\xi, x'\xi' \in \mathcal{D}$

$$\langle x'\xi', (U^*U)_+(t)x\xi\rangle = \langle x'\xi', [(U^*U)_+(t)x\Omega\otimes\xi]\rangle =$$
$$= \langle x'\xi', [x\Omega\otimes\xi]\rangle = \langle x'\xi', x\xi\rangle .$$

Thus by (4.17) and (4.18), grouping similar terms, we can write:

$$(4.19)\qquad d\langle U(t)x'\xi', U(t)x\xi\rangle = \sum_{l\in \boldsymbol{Z}^*}\langle L_0^l x', L_0^l x\rangle\langle\xi',\xi\rangle dt +$$
$$+\langle U(t)x'\xi', U(t)L_0^0 x\xi\rangle dt + \langle U(t)L_0^0 x'\xi', U(t)x\xi\rangle dt +$$
$$+\sum_{i\in \boldsymbol{Z}^*}\sum_{l\in \boldsymbol{Z}^*}\langle L_0^l x', L_0^l x\rangle\langle\xi',\eta_1\rangle u_1^i(t)\,dt +$$
$$+\sum_{j\in \boldsymbol{Z}^*}\sum_{l\in \boldsymbol{Z}^*}\langle L_j^l x', L_0^l x\rangle\langle\eta_1',\xi\rangle \overline{u_1'^j(t)}\,dt +$$
$$+\sum_{i,j\in \boldsymbol{Z}^*}\sum_{l\in \boldsymbol{Z}^*}\langle L_j^l x', L_i^l x\rangle\langle\eta_1',\eta_1\rangle \overline{u_1'^j(t)}\,u_1^i(t)\,dt +$$
$$+\sum_{i\in \boldsymbol{Z}^*}\langle x', L_i^0 x\rangle\langle\xi',\eta_1\rangle u_1^i(t)\,dt + \sum_{j\in \boldsymbol{Z}^*}\langle x', L_0^j x\rangle\langle\eta_1',\xi\rangle \overline{u_1'^j(t)}\,dt +$$
$$+\sum_{i,j\in \boldsymbol{Z}^*}\langle x', L_i^j x\rangle\langle\eta_1',\eta_1\rangle \overline{u_1'^j(t)}\,u_1^i(t)dt + \sum_{i\in \boldsymbol{Z}^*}\langle L_0^i x', x\rangle\langle\xi',\eta_1\rangle u_1^i(t)dt +$$
$$+\sum_{j\in \boldsymbol{Z}^*}\langle L_j^0 x', x\rangle\langle\eta_1',\xi\rangle \overline{u_1'^j(t)}\,dt + \sum_{i,j\in \boldsymbol{Z}^*}\langle L_j^i x', x\rangle\langle\eta_1',\eta_1\rangle \overline{u_1'^j(t)}\,u_1^i(t)dt .$$

Now if we consider together the fourth, seventh and tenth terms in (4.19) we have the following sum of scalar products:

$$\sum_{i\in \boldsymbol{Z}^*}\Big\langle x', \big[L_i^0 + (L_0^i)^* + \sum_{l\in \boldsymbol{Z}^*}(L_0^l)^* L_i^l\big]x\Big\rangle\langle\xi',\eta_1\rangle u_1^i(t)\,dt$$

and condition (4.7) implies that each term in this sum is equal to zero. In the same way, again by condition (4.8) and the adjoint of (4.7), we see that (4.19) reduces to:

$$d\langle U(t)x'\xi', U(t)x\xi\rangle = \sum_{l\in \boldsymbol{Z}^*}\langle L_0^l x', L_0^l x\rangle\langle\xi',\xi\rangle dt +$$
$$+\langle U(t)x'\xi', U(t)L_0^0 x\xi\rangle dt + \langle U(t)L_0^0 x'\xi', U(t)x\xi\rangle dt .$$

Now define the contraction operator $K(t)$ on h_0 by:

$$\langle U(t)x'\xi', U(t)x\xi\rangle = \langle x', K(t)x\rangle .$$

Then $K(t)$ satisfies the following equation with bounded coefficients:

$$(4.20) \qquad \begin{cases} dK(t) = K(t)L_0^0\,dt + (L_0^0)^*K(t)\,dt + \sum_{l\in \boldsymbol{Z}^*}(L_0^l)^*L_0^l\,dt , \\ K(0) = 1 . \end{cases}$$

This equation has 1 as a solution thanks to the hypothesis that (4.6) vanishes, then, because (4.20) has a unique solution, it follows that $K(t) = 1$ for all $t \in \boldsymbol{R}_+$. Therefore we conclude that, for all $x\xi$, $x'\xi' \in \mathcal{D}$

$$d\langle U(t)x'\xi', U(t)x\xi\rangle = 0 .$$

The isometricity of $U(t)$ is so proved. ■

5. Quantum stochastic differential equations driven by free noises with unbounded coefficients and dilation of Feller's minimal solution.

The class of unbounded coefficients we will consider in this Section, can be described as follows:

(A) the operators $\{L_i^j\}_{i,j\in \boldsymbol{Z}}$ satisfy the following properties:

(i) L_0^0 is the infinitesimal generator of a contraction semigroup $\{P(t)\}_{t\geq 0}$, the domain of L_0^0 contains $\mathcal{D}_0$ and $\mathcal{D}_0$ is a core for L_0^0;

(ii) for all $j \in \boldsymbol{Z}^*$, the domain of L_0^j contains the domain of L_0^0 and the domain of L_j^0 contains $\mathcal{D}_0$;

(iii) for all x in the domain of L_0^0 the following inequality holds:

$$\langle x, L_0^0x\rangle + \langle L_0^0x, x\rangle + \sum_{l\in \boldsymbol{Z}^*}\langle L_0^lx, L_0^lx\rangle \leq 0 ;$$

(iv) for all $x \in \mathcal{D}_0$, y in the domain of L_0^0 and for all $j \in \boldsymbol{Z}^*$:

$$\langle y, L_j^0x\rangle + \langle L_0^jy, x\rangle + \sum_{l\in \boldsymbol{Z}^*}\langle L_0^ly, L_j^lx\rangle = 0 ;$$

(v) for all $x, y \in \mathcal{D}_0$ and for all $i,j \in Z^*$:

$$\langle y, L_i^j x\rangle + \langle L_j^i y, x\rangle + \sum_{l \in \boldsymbol{Z}^*} \langle L_j^l y, L_i^l x\rangle = 0\,;$$

(vi) for all x in the domain of L_0^0:

$$\sum_{j \in \boldsymbol{Z}^*} \|L_0^j x\|^2 < +\infty\,.$$

Now let $\widehat{\mathcal{K}}$ be the Hilbert space obtained by direct sum of $\mathcal{K}$ and the two one-dimensional subspaces generated by the unit vectors $e_{-\infty}$, and $e_{+\infty}$: $\widehat{\mathcal{K}} = \boldsymbol{C}e_{-\infty} \oplus \mathcal{K} \oplus \boldsymbol{C}e_{+\infty}$.

We can identify the family of operators $\{L_i^j\}_{i,j \in \boldsymbol{Z}}$ with a matrix operator $[L]$ on $h_0 \otimes \widehat{\mathcal{K}}$

$$[L] = \begin{pmatrix} 0 & L^0 & L_0^0 \\ 0 & L & L_0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix entries are the operators L_0, L^0, L defined as follows:

L_0 is an operator on h_0 to $h_0 \otimes \mathcal{K}$ whose domain contains the domain of L_0^0 and, for all $j \in \boldsymbol{Z}^*$, x in the domain of L_0^0 and $y \in h_0$:

$$\langle ye_j, L_0 x\rangle = \langle y, L_0^j x\rangle;$$

L^0 is an operator on $h_0 \otimes \mathcal{K}$ to h_0 whose domain contains $\mathcal{D}$ and, for all $i \in \boldsymbol{Z}^*$, $x \in \mathcal{D}_0$, $y \in h_0$:

$$\langle y, L^0 xe_i\rangle = \langle y, L_i^0 x\rangle;$$

L is a bounded operator on $h_0 \otimes \mathcal{K}$ to $h_0 \otimes \mathcal{K}$ such that, for all $i,j \in \boldsymbol{Z}^*$, $x \in \mathcal{D}_0$, $y \in h_0$:

$$\langle ye_j, Lxe_i\rangle = \langle y, L_i^j x\rangle.$$

Through this identification it is immediate to see that our condition (A) for the operators $\{L_i^j\}$ is equivalent to the condition (A) in Definition 2.6 [4] for the operator $[L]$. Then Proposition 3.3 in [4] implies that there exist bounded operators $\{(L_i^j)^{(n)}\}_{i,j \in \boldsymbol{Z}}$ satisfying the conditions (4.6), (4.7), (4.8) and the inequality states in Remark 4.4. Moreover, for all $x \in \mathcal{D}_0$ and for all $i,j \in \boldsymbol{Z}$, we have:

$$s\text{-}\lim_{n\to\infty} (L_i^j)^{(n)} x = L_i^j x\,.$$

THEOREM 5.1. *Suppose that the operators $\{L_i^j\}_{i,j \in \boldsymbol{Z}}$ satisfy the con-*

dition (A). *Then there exists a contractive process U which is a solution to the quantum stochastic differential equation*

$$\begin{cases} dU(t) = \sum_{i,j\in \boldsymbol{Z}} U(t) L_i^j \, dp_j^i(t), \\ U(0) = 1. \end{cases} \tag{5.1}$$

Proof. For each $n \in \boldsymbol{N}$ consider the quantum stochastic differential equation bounded coefficients

$$\begin{cases} dU^{(n)}(t) = \sum_{i,j\in \boldsymbol{Z}} U^{(n)}(t)(L_i^j)^{(n)} \, dp_j^i(t), \\ U^{(n)}(0) = 1. \end{cases} \tag{5.2}$$

Theorem 4.2 provides us the existence of a unique contractive solution of (5.2) for all $n \in \boldsymbol{N}$. An easy computation shows that for all $n \in \boldsymbol{N}$, $e_h \xi \in \mathcal{D}$ and $s, t \in \boldsymbol{R}_+$ with $s < t$ it holds:

$$\|(U^{(n)}(t) - U^{(n)}(s)) e_h \xi\|^2 \leq c(t-s) \tag{5.3}$$

where c does not depend on n. Therefore for all $x\xi$, $x'\xi' \in \mathcal{D}$ the functions $\langle x'\xi, U^{(n)}(\cdot) x\xi \rangle$ are equicontinuous and by the Ascoli-Arzelà theorem, using the standard diagonalization argument, we can find a subsequence $\{U^{(n_k)}\}_{k\geq 0}$ and a contraction valued process U such that, for all $t \in \boldsymbol{R}_+$

$$w\text{-}\lim_{k\to\infty} U^{(n_k)}(t) = U(t).$$

In virtue of (5.3) U is strongly continuous on $\mathcal{D}$. A similar argument as in Proposition 4.5 shows that the series in (5.1) is strongly convergent on $\mathcal{D}$. Clearly U is a solution to (5.1). ■

Now we want to study the unitarity of the solution $U(t)$.
Define the following operators: for each $\beta \in [0,1]$ we set

$$(L_j^i)^{(\beta)} = \begin{cases} L_j^i & \text{if } i, j \in \boldsymbol{Z}^*, \\ \beta L_j^0 & \text{if } i = 0,\ j \in \boldsymbol{Z}^*, \\ \beta L_0^i & \text{if } i \in \boldsymbol{Z}^*,\ j = 0, \\ L_0^0 & \text{if } i = j = 0. \end{cases}$$

Clearly the operators $(L_j^i)^{(\beta)}$ satisfy condition (A), hence there exist bounded operators $(L_j^i)^{(n,\beta)}$ satisfying conditions (4.6), (4.7), (4.8) and, for all $\beta \in [0,1]$, $x \in \mathcal{D}_0$:

$$s\text{-}\lim_n (L_i^j)^{(n,\beta)} x = (L_i^j)^{(\beta)} x .$$

PROPOSITION 5.2. *Under the above setting, for each* $\beta \in [0,1]$ *there exists a unique contractive solution of the quantum stochastic differential equation*

$$(5.4) \qquad \begin{cases} dU^{(\beta)}(t) = \sum_{i,j \in \mathbf{Z}} U^{(\beta)}(t)(L_i^j)^{(\beta)} dp_j^i(t), \\ U^{(\beta)}(0) = 1 . \end{cases}$$

Moreover we have:

$$(5.5) \qquad U(t) = w\text{-}\lim_{\beta \to 1} U^{(\beta)}(t) \quad \textit{for all } t \in \mathbf{R}_+ .$$

PROOF. The existence of the solution to (5.4) follows from Theorem 5.1. The unicity can be proved as in Proposition 3.5 [4]. Finally (5.5) follows from the unicity. ■

Consider now, for all $\beta \in [0,1]$ the families $T^{(\beta)} = (T^{(\beta)}(t))_{t \geq 0}$ of weakly (hence strongly) continuous contractions on h_0 satisfying, for all $x, y \in \mathcal{D}_0$ one of the following equations:

$$(5.6) \quad \langle y, T^{(\beta)}(t) x \rangle = \langle y, x \rangle + \\ + \int_0^t \left\{ \langle L_0^0 y, T^{(\beta)}(s) x \rangle + \langle y, T^{(\beta)}(s) L_0^0 x \rangle + \sum_{l \in \mathbf{Z}^*} \langle L_0^l y, T^{(\beta)}(s) L_0^l x \rangle \right\} ds ,$$

$$(5.7) \quad \langle y, T^{(\beta)}(t) x \rangle = \\ = \langle P(t) y, P(t) x \rangle + \int_0^t \sum_{l \in \mathbf{Z}^*} \langle L_0^l P(t) y, T^{(\beta)}(s) L_0^l P(t) x \rangle ds ,$$

where we denote $P(t)$ the semigroup $e^{tL_0^0}$ for $t \geq 0$.

The following proposition can be proved as Propositions 4.1, 4.2, 4.3 in [4].

PROPOSITION 5.3. *Assume that the operators* $\{L_i^j\}_{i,j \in \mathbf{Z}}$ *satisfy condition* (A), *then the following results hold*:

1) *A family* $(T^{(\beta)}(t))_{t \geq 0}$ *of strongly continuous positive contractions on* h_0 *satisfying* (5.6) *also satisfies* (5.7), (*i.e.* (5.7) *is a weak form of the equation* (5.6)).

2) *For all* $\beta \in [0,1]$, *there exists a strongly continuous family* $(T^{(\beta)}(t))_{t \geq 0}$ *of contractions on* h_0 *satisfying the equation* (5.7) *and, if* $\beta_1 \leqslant \beta_2 \leqslant 1$, *then* $T^{(\beta_1)}(t) \leqslant T^{(\beta_2)}(t)$ *for all* $t \geqslant 0$.

3) *If* $(T(t))_{t \geqslant 0}$ *is a continuous family of positive contractions satisfying* (5.7), *with* $\beta = 1$, *then for all* $t \geqslant 0$, *we have*

$$T^{(1)}(t) \leqslant T(t) \leqslant I$$

where I *denotes the identity operator.*

4) *For all* $\beta \in [0,1)$, $T^{(\beta)}$ *is the unique solution to* (5.7).

5) *For each* $t \geqslant 0$ *we have* $s\text{-}\lim_{\beta \to 1^-} T^{(\beta)} = T^{(1)}(t)$.

COROLLARY 5.4. *Suppose that the operators* $\{L_i^j\}_{i,j \in Z}$ *satisfy condition* (A) *and the following identity*:

$$\langle y, L_0^0 x\rangle + \langle L_0^0 y, x\rangle + \sum_{l \in Z^*} \langle L_l^0 y, L_l^0 x\rangle = 0 \tag{5.8}$$

for all x, y *in the domain of* L_0^0. *Then the equation* (5.7) *in the case* $\beta = 1$ *has the unique solution* $T(t) = I$ *for all* $t \geqslant 0$ *if and only if* $T^{(1)}(t) = I$ *for all* $t \geqslant 0$. *Moreover in this case the equation* (5.6) *has the unique positive solution* $T(t) = I$ *for all* $t \geqslant 0$.

The following theorem gives a necessary and sufficient condition for uniqueness of the solution to equation (5.7). It can be shown as Theorem 3.4 in [3].

THEOREM 5.5. *Let the operators* $\{L_i^j\}$ *satisfy condition* (A) *and* (5.8). *For each* $\lambda > 0$, *let* $\mathcal{B}_\lambda^+$ *denote the set of positive bounded operators* X *on* h_0 *such that, for all* $x, y \in \mathcal{D}_0$:

$$\langle y, XL_0^0 x\rangle + \langle L_0^0 y, Xx\rangle + \sum_{l \in Z^*} \langle L_l^0 y, XL_l^0 x\rangle = \lambda \langle y, Xx\rangle .$$

Then the following conditions are equivalent:

i) *the equation* (5.7) *has the only solution* $T(t) = I$ *for all* $t \geqslant 0$

ii) *for each* $\lambda > 0$ *the set* $\mathcal{B}_\lambda^+$ *containly only* 0.

In order to prove that U is isometric, we first show that the vacuum expectation of the contractive solution to equation (5.4) coincides with the minimal solution of the associated Feller-Kolmogorov equation (5.7).

PROPOSITION 5.6. *Let* U *be the unique contractive solution to the quantum stochastic differential equation* (5.4) *with* $\beta = 1$. *Then, for all*

$t \in \boldsymbol{R}_+$, $x \in \mathcal{D}_0$, *we have*

$$\langle U(t)x, U(t)x\rangle = \langle x, T^{(1)}(t)x\rangle. \tag{5.9}$$

PROOF. Using the left Itô formula, we can show that, for all $\beta \in [0,1]$, $x \in \mathcal{D}_0$, $t \geqslant 0$ the bilinear form on h_0 given by

$$(y, x) \mapsto \langle U^{(\beta)}(t)y, U^{(\beta)}(t)x\rangle \tag{5.10}$$

defines a continuous family of contractions on h_0 satisfying the equation (5.6), hence (5.7). Then by the result 4) of Proposition 5.3, for all $\beta \in [0,1)$

$$\|U^{(\beta)}(t)x\|^2 = \langle U^{(\beta)}(t)x, U^{(\beta)}(t)x\rangle = \langle x, T^{(\beta)}(t)x\rangle. \tag{5.11}$$

Now by results 2) and 3) of Proposition 5.3, we have:

$$\langle x, T^{(\beta)}(t)x\rangle \leqslant \langle x, T^{(1)}(t)x\rangle \leqslant \langle x, T(t)x\rangle. \tag{5.12}$$

Moreover, because $U(t) = w\text{-}\lim_{\beta\to 1} U^{(\beta)}(t)$ we have:

$$\langle x, T(t)x\rangle = \|U(t)x\|^2 \leqslant \lim_{\beta\to 1^-} \inf \|U^{(\beta)}(t)x\|^2. \tag{5.13}$$

Hence (5.11), (5.12), (5.13) imply:

$$\|U^{(\beta)}(t)x\|^2 \leqslant \|U(t)x\|^2 \leqslant \lim_{\beta\to 1^-} \inf \|U^{(\beta)}(t)x\|^2. \tag{5.14}$$

Finally, because $T^{(1)}(t) = s\text{-}\lim_{\beta\to 1^-} T^{(\beta)}(t)$,

$$\langle x, T^{(1)}(t)x\rangle = \lim_{\beta\to 1^-}\langle x, T^{(\beta)}(t)x\rangle = \lim_{\beta\to 1^-}\|U^{(\beta)}(t)x\|^2. \tag{5.15}$$

Then (5.14) and (5.15) allows to conclude. ∎

PROPOSITION 5.7. *Let U be the unique contractive solution of the quantum stochastic differential equation*

$$\begin{cases} dU(t) = \sum_{i,j\in \boldsymbol{Z}} U(t)L_i^j\, dp_i^j(t), \\ U(0) = 1. \end{cases} \tag{5.16}$$

and suppose that condition (A) *and* (5.8) *hold, then the following conditions are equivalent*:

(i) $\langle U(t)y, U(t)x\rangle = \langle y, x\rangle$ *for all $x, y \in h_0$ and for all $t \geqslant 0$*

(ii) *U is isometric.*

PROOF. Clearly we have only to show that (i) implies (ii). We use an inductive argument on $N = N(\xi) + N(\xi')$, where $\xi, \xi' \in h_0$. The case $N = 0$ is trivially true because of condition (i). Suppose it holds for N, we will show it for $N(\xi) + N(\xi') = N + 1$. Then, as in Proposition 4.5, one can prove that:

$$d\langle U(t)x'\xi', U(t)x\xi\rangle = \langle U(t)L_0^0 x'\xi', U(t)x\xi\rangle dt + \tag{5.17}$$

$$+ \langle U(t)x'\xi', U(t)L_0^0 x\xi\rangle dt + \sum_{l \in \mathbf{Z}^*} \langle L_0^l x'\xi', L_0^l u\xi\rangle dt .$$

Therefore we have

$$\frac{d}{ds}\langle U(s)P(t-s)x'\xi', U(s)P(t-s)x\xi\rangle =$$

$$= \sum_{l \in \mathbf{Z}^*} \langle L_l^0 P(t-s)x', L_l^0 P(t-s)x\rangle\langle\xi', \xi\rangle$$

and, by (5.8), the righthand side is equal to

$$-\{\langle U(s)P(t-s)L_0^0 x', U(s)P(t-s)x\rangle\langle\xi', \xi\rangle +$$

$$+ \langle U(s)P(t-s)x', U(s)P(t-s)L_0^0 x\rangle\langle\xi', \xi\rangle\} .$$

Now integrating on $[0, t]$ we obtain

$$\langle U(t)x'\xi', U(t)x\xi\rangle - \langle P(t)x', P(t)x\rangle\langle\xi', \xi\rangle =$$

$$= -\int_0^t (\langle U(s)P(t-s)L_0^0 x', U(s)P(t-s)x\rangle +$$

$$+ \langle U(s)P(t-s)x', U(s)P(t-s)L_0^0 x\rangle)\langle\xi', \xi\rangle ds =$$

$$= \int_0^t \frac{d}{ds}\langle U(s)P(t-s)x', U(s)P(t-s)x\rangle\langle\xi, \xi\rangle ds =$$

$$= (\langle U(t)x', U(t)x\rangle - \langle P(t)x', P(t)x\rangle)\langle\xi', \xi\rangle .$$

Hence, using (i), we conclude that

$$\langle U(t)x'\xi', U(t)x\xi\rangle = \langle x'\xi', x\xi\rangle .$$

This completes the induction argument and shows that U is isometric. ∎

We can summarize the above results in the following

THEOREM 5.8. *Assume that conditions* (A) *and* (5.8) *hold. Moreover suppose that, for all* $t \geq 0$, *we have* $T^{(1)}(t) = I$. *If* U *is the unique contractive solution to the equation* (5.16), *then* U *is isometric.*

We reduce now the problem of coisometricity of U to the isometricity of the dual cocycle $\tilde{U}$ as done in [6]. For all $t \in \boldsymbol{R}_+$, let $\mathcal{R}_t$ be the unitary operator $\mathcal{F}$ defined by

$$\mathcal{R}_t \Omega = \Omega$$

$$\mathcal{R}_t u_1 \otimes \ldots \otimes u_k = (\rho_t u_1) \otimes \ldots \otimes (\rho_t u_k)$$

where ρ_t is the time reversal on the interval $[0, t]$ defined on $L^2(\boldsymbol{R}_+) \otimes \mathcal{K}$ by

$$(\rho_t f)(s) = \begin{cases} f(t-s) & \text{if } s \leq t, \\ f(s) & \text{if } s > t. \end{cases}$$

Let $\tilde{U}(t)$ denote the operator $\mathcal{R}_t U^*(t) \mathcal{R}_t$ and, for all $n \in \boldsymbol{N}$, let $\tilde{U}^{(n)}(t)$ be the operator $\mathcal{R}_t U^{(n)*}(t) \mathcal{R}_t$, where $U^{(n)}(t)$ is the unitary solution to the quantum stochastic differential equations

$$U^{(n)}(t) = 1 + \sum_{i, j \in \boldsymbol{Z}} \int_0^t U^{(n)}(t) L_i^j \, dp_j^i(s)$$

and $U(t) = w\text{-}\lim_{n \to \infty} U^{(n)}(t)$ for all $t \in \boldsymbol{R}_+$. An analogous proof of Lemma 3.7 in [6] shows that, for all $x'\xi'$, $x\xi \in \mathcal{D}$, $t \in \boldsymbol{R}_+$, which is a continuity point for functions u, u', the operators $\tilde{U}^{(n)}(t)$ satisfy:

$$\begin{aligned}
\frac{d}{dt} \langle x'\xi', \tilde{U}^{(n)}(t) x\xi \rangle &= \langle x'\xi', \tilde{U}^{(n)}(t)(L_0^0)^{(n)*} x\xi \rangle + \\
&+ \sum_{j \in \boldsymbol{Z}^*} \langle x'\xi', \tilde{U}^{(n)}(t)(L_j^0)^{(n)*} x\eta_1 \rangle u_1^j(t) + \\
&+ \sum_{i \in \boldsymbol{Z}^*} \langle x'\eta'_{k'-k}, \tilde{U}_+^{(n)}(t)(L_0^i)^{(n)*} x\xi \rangle \overline{u'^{i}_{k'-k}(t)} + \\
&+ \sum_{i, j \in \boldsymbol{Z}^*} \langle x'\eta'_{k'-k+1}, \tilde{U}_+^{(n)}(t)(L_j^i)^{(n)*} x\eta_1 \rangle \overline{u'^{j}_{k'-k+1}(t)}\, u_1^i(t).
\end{aligned}$$

Hence $\tilde{U}^{(n)}(t)$ is the only isometric solution to

$$\tilde{U}^{(n)}(t) = 1 + \sum_{i, j \in \boldsymbol{Z}} \int_0^t \tilde{U}^{(n)}(t)(L_i^j)^{(n)*} \, dp_j^i(s).$$

Then $\tilde{U}(t)$ satisfies

$$\tilde{U}(t) = 1 + \sum_{i,j \in \mathbf{Z}} \int_0^t \tilde{U}(t)(L_i^j)^* \, dp_j^i(s).$$

Now the proof of the isometricity of $\tilde{U}(t)$ is the same as for $U(t)$, if we suppose that $\tilde{T}^{(1)}(t) = I$ for all $t \geq 0$, where $\tilde{T}^{(1)}(t)$ is the family of contractions on h_0 satisfying, for all $x, y \in \mathcal{D}_0$ the following equation:

$$\langle y, \tilde{T}^{(\beta)}(t)x \rangle = \langle y, x \rangle +$$

$$+ \int_0^t \left\{ \{(L_0^0)^* y, \tilde{T}^{(\beta)}(s)\, x\rangle + \langle y, \tilde{T}^{(\beta)}(s)(L_0^0)^* x\rangle + \sum_{l \in \mathbf{Z}^*} \langle (L_0^l)^* y, \tilde{T}^{(\beta)}(s)(L_0^l)^* x\rangle \right\} ds.$$

REFERENCES

[1] L. Accardi, *On the quantum Feynman-Kac formula*, Rend. Sem. Mat. Fis. Milano, 48 (1978).

[2] L. Accardi - F. Fagnola - J. Quaegebeur, *A representation free quantum stochastic calculus*, J. Funct. Anal., **104** (1992).

[3] A. M. Chebotarev, *The theory of the conservative dynamical semigroups and its applications*, Preprint MIEM n. 1, March 1990.

[4] F. Fagnola, *Characterization of isometric and unitary weakly differentiable cocycles in Fock space*, Quantum Probability, VIII (1993).

[5] F. Fagnola, *On quantum stochastic integration with respect to «free» noises*, Quantum Probability and Related Topics, VI (1991).

[6] F. Fagnola - M. Mancino, *Free noise dilation of semigroups of countable state Markov processes*, Quantum Probability and Related Topics, VII (1992).

[7] B. Kümmerer - R. Speicher, *Stochastic integration on the Cuntz algebra O_∞*, SFB-Preprint n. 602, Heidelberg, 1990.

[8] A. Mohari - K. R. Parthasarathy, *On a class of generalized Evans-Hudson flows related to classical Markov processes*, Preprint, I.S.I. Delhi Centre, 1991.

[9] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, I.S.I. New Delhi, 1990.

[10] R. Speicher, *A new example of «independence» and «white noise»*, Probab. Th. Rel. Fields, 84 (1990), pp. 141-159.

Manoscritto pervenuto in redazione il 2 novembre 1992.