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FEDERICO MENEGAZZO

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Automorphisms of p -Groups with Cyclic Commutator Subgroup.

FEDERICO MENEGAZZO (*)

ABSTRACT - We study the automorphism groups of finite, non abelian, 2-generated p -groups with cyclic commutator subgroup, for odd primes p . We exhibit presentations of the relevant groups, and compute the orders of $\text{Aut } G$, $O_p(\text{Aut } G)$, and of the linear group induced on the factor group $G/\Phi(G)$.

In this paper we give a systematic account of the automorphism groups of finite, non abelian, 2-generated p -groups with cyclic commutator subgroup, for odd primes p .

Special cases of this problem have of course been studied in connection with many questions, with the aim of providing examples and counterexamples; still, the general information available is remarkably scarce.

It is a remark by Ying Cheng[2] that in such groups G the central factor group $G/2(G)$ is metacyclic, hence modular; it follows that $|G|$ divides the order of $\text{Aut } G$ [3]. Another known fact is that in any metabelian 2-generated p -group $G = \langle a, b \rangle$, for all choices of $x, y \in G'$, there is an automorphism α mapping a to ax and b to by [1]; moreover, if G' is cyclic and p is odd, such automorphisms are inner [2]. This implies that the order of $\text{Inn } G$ is $|G'|^2$. $\text{Aut } G$ naturally induces a group of linear transformations of the $\mathbb{Z}/p\mathbb{Z}$ -vector space $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup; we denote this group—which in our case is a subgroup of $GL(2, \mathbb{Z}/p\mathbb{Z})$ —by $\text{Aut}_l G$, the l beeing a reminder of «linear». The kernel of this action, i.e.

(*) Indirizzo dell'A.: Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Belzoni 7, 35131 Padova (Italy).

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$\{\alpha \in \text{Aut } G \mid g^z \Phi(G) = \Phi(G), \forall g \in G\}$, is sometimes denoted by $\text{Aut}^\Phi(G)$; for every p -group $\text{Inn } G \leq \text{Aut}^\Phi(G) \leq O_p(\text{Aut } G)$.

We found it necessary to analyse separately several cases; indeed, what we got is almost a classification (a classification of finite, non abelian, 2-generated p -groups with cyclic commutator subgroup, for odd primes p , has been given by Miech in [4]). For each case, we will exhibit presentations of the relevant groups and compute the orders of $\text{Aut } G$, $O_p(\text{Aut } G)$, $\text{Aut}_i G$. In many instances, we have been able to display the effect of $\text{Aut } G$ on two chosen generators of G ; we hope that also in the remaining cases the information we provide is helpful.

1. In this section we will deal with metacyclic groups. Accordingly, we suppose that a group G has a cyclic normal subgroup $N = \langle b \rangle$ of order p^m , say, with cyclic factor group $G/N = \langle aN \rangle$ of order p^l . We may choose a such that $b^a = b^{1+p^s}$ for some s , $1 \leq s < m$. Since the order of $1 + p^s \bmod p^m$ is p^{m-s} we also have $m - s \leq l$. The centre is $Z(G) = \langle a^{p^{m-s}}, b^{p^{m-s}} \rangle$.

Suppose that G splits over N . Then $G = \langle a, b \mid a^{p^l} = b^{p^m} = 1, b^a = b^{1+p^s} \rangle$ is a presentation of G . If $\bar{b} = a^z b^w$ is a candidate for an $\text{Aut } G$ -image of b , we must have $\langle \bar{b}^{p^s} \rangle = G' = \langle b^{p^s} \rangle$ and $a^z = \bar{b} b^{-w} \in C_G(G')$. It follows that $(a^z b^w)^{p^s} = a^{zp^s} b^{wp^s} [b^w, a^z]^{(p^s)} \in \langle b^{p^s} \rangle$, and then $a^{zp^s} = 1$, $[b^w, a^z]^{(p^s)} = 1$, $(a^z b^w)^{p^s} = b^{wp^s}$, and $p \nmid w$. We also have $(a^z b^w)^a = a^z b^w b^{wp^s} = (a^z b^w)^{1+p^s}$: such a $\bar{b}$ is in fact in the $\text{Aut } G$ -orbit of b , and an automorphism mapping b to $\bar{b}$ and a to $\bar{a}$ exists if and only if $\bar{a}$ has order p^l and $\bar{b}^{\bar{a}} = \bar{b}^{1+p^s} = \bar{b}^a$, i.e. $\bar{a} \in a\Omega_l(C_G(\bar{b}))$, where $C_G(\bar{b}) = \langle \bar{b} \rangle \times \langle a^{p^{m-s}} \rangle$. We summarize our results:

$$(A.1) \quad G = \langle a, b \mid a^{p^l} = b^{p^m} = 1, b^a = b^{1+p^s} \rangle$$

where $1 \leq s < m$ and $m - s \leq l$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} b \mapsto a^z b^w, \\ a \mapsto a a^{\lambda p^{m-s}} (a^z b^w)^\mu, \end{cases}$$

where $zp^s \equiv 0 \pmod{p^l}$, $w \not\equiv 0 \pmod{p}$, $\mu p^l \equiv 0 \pmod{p^m}$.

— If $l \geq m$:

$$|\text{Aut } G| = (p-1)p^{l+m+2s-1}, \quad |\text{Inn } G| = p^{2(m-s)},$$

$$|O_p(\text{Aut } G)| = p^{l+m+2s-1}, \quad |\text{Aut}_i G| = p(p-1).$$

— If $m > l > s$:

$$|\text{Aut } G| = (p - 1)p^{2l+2s-1}, \quad |\text{Inn } G| = p^{2(m-s)},$$

$$|O_p(\text{Aut } G)| = p^{2l+2s-1}, \quad |\text{Aut}_l G| = p - 1.$$

— If $s \geq l$:

$$|\text{Aut } G| = (p - 1)p^{3l+s-1}, \quad |\text{Inn } G| = p^{2(m-s)},$$

$$|O_p(\text{Aut } G)| = p^{3l+s-1}, \quad |\text{Aut}_l G| = p(p - 1).$$

Suppose now that in our metacyclic p -group G there is no cyclic normal subgroup N having a cyclic complement. G has a presentation $G = \langle a, b \mid b^{p^m} = 1, b^a = b^{1+p^s}, a^{p^l} = b^{p^h} \rangle$. $1 \neq b^{p^h} \in Z(G)$ yields $m > h \geq m - s$; b cannot have maximum order, so $l > h$; for the same reason, $G' \not\leq \langle a \rangle$, which means $s < h$. As a first approximation to the $\text{Aut } G$ -orbit of b , we look for elements $g = a^z b^w$ generating normal subgroups N of order p^m containing $G' = \langle b^{p^s} \rangle$. This happens if and only if $\langle g^{p^s} \rangle = \langle b^{p^s} \rangle$, i.e. $a^{2p^s} \in \langle a \rangle \wedge \langle b \rangle = \langle a^{p^l} \rangle$, or $a^z \in \langle a^{p^{l-s}} \rangle$, and $w \not\equiv 0 \pmod{p}$. We have $[\langle a^{p^{l-s}} \rangle : \langle a^{p^l} \rangle] \phi(p^m) = p^s \phi(p^m)$ such elements g , where ϕ is the Euler function, which generate p^s subgroups N as above. For every choice of N , $G = \langle a, N \rangle$ and the automorphism group induced on N by conjugation in G is the group generated by the power $1 + p^s$; it is therefore possible to choose $\bar{a} \in G$ so that $G/N = \langle \bar{a}N \rangle$ and $g\bar{a} = g^{1+p^s}$ for every generator g of N . The choice of $\bar{a}$ is not unique: all the elements in the coset $\bar{a}C_G(N)$ (and they only) share the same properties. The order of $C_G(N) = NC_{\langle a \rangle}(N) = N\langle a^{p^{m-s}} \rangle$ is p^{l+s} : once N is given, we have p^{l+s} possible choices for $\bar{a}$. Comparing the orders, for every such $\bar{a}$ we find $\langle \bar{a} \rangle \wedge N = \langle \bar{a}^{p^l} \rangle = N^{p^h}$: it is then possible to choose a generator $\bar{b}$ of N such that $\bar{a}^{p^l} = \bar{b}^{p^h}$. Again, the choice of $\bar{b}$ is not unique: the possible choices are the elements of the coset $\bar{b}\Omega_h(N)$. All told, the number of pairs $(\bar{a}, \bar{b})$ satisfying the given presentation of G is $(p^s \text{ choices for } N)$ times $(p^{l+s} \text{ choices of } \bar{a})$ times $(p^h \text{ choices of } \bar{b})$. In summary

$$(A.2) \quad G = \langle a, b \mid b^{p^m} = 1, b^a = b^{1+p^s}, a^{p^l} = b^{p^h} \rangle$$

where $1 \leq s < h < m$ and $m - s \leq h < l$.

$$|\text{Aut } G| = p^{l+h+2s}, \quad |\text{Inn } G| = p^{2(m-s)}, \quad |\text{Aut}_l G| = p.$$

REMARK. For any $n \geq 6$, fix m such that $3 \leq m \leq n/2$, $l = n - m$, $s = 1$, $h = m - 1$: we get a group G of order p^n with $|\text{Aut } G| = p^{n+1}$.

2. In this section we will deal with groups G which are not metacyclic, and have nilpotency class 2. We first fix the notation: a, b are generators for G such that $G/G' = \langle aG' \rangle \times \langle bG' \rangle$, aG' has order p^l and bG' has order p^m . Then $u = [b, a]$ generates G' , u has order p^n with $1 \leq n \leq l$, $n \leq m$, and of course $u \in Z(G)$, $Z(G) = \langle a^{p^n}, b^{p^n}, u \rangle$.

Suppose first that $\langle b \rangle \wedge G' = \langle a \rangle \wedge G' = 1$. G has a presentation $G = \langle a, b, u \mid a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^b = u \rangle$, and the elements of G can be uniquely written in the form $a^x b^y u^z$, $x \in \mathbb{Z}/p^l \mathbb{Z}$, $y \in \mathbb{Z}/p^m \mathbb{Z}$, $z \in \mathbb{Z}/p^n \mathbb{Z}$. We can also assume $l \geq m$. If θ is any automorphism of G/G' and we fix elements $\bar{a} \in (aG')^\theta$, $\bar{b} \in (bG')^\theta$ and set $\bar{u} = [\bar{b}, \bar{a}]$, it is immediately seen that $\langle \bar{a}, \bar{b} \rangle = G$ and $\bar{a}, \bar{b}, \bar{u}$ satisfy the relations, so that the assignment $a \mapsto \bar{a}, b \mapsto \bar{b}, u \mapsto \bar{u}$ extends to an automorphism of G . This means that the obvious homomorphism $\text{Aut } G \rightarrow \text{Aut}(G/G')$ is onto; its kernel is $\text{Inn } G$, of order p^{2n} . The structure of $\text{Aut}(G/G')$ is well known: if $l = m$ it is isomorphic with $GL(2, \mathbb{Z}/p^l \mathbb{Z})$, while if $l > m$ it is isomorphic to the group of all matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ where $x \in \mathbb{Z}/p^l \mathbb{Z}$, $y, w \in \mathbb{Z}/p^m \mathbb{Z}$, $z \in p^{l-m} \mathbb{Z}/p^n \mathbb{Z}$, x and $w \not\equiv 0 \pmod{p}$.

We obtained:

$$(B.1) \quad G = \langle a, b, u \mid a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^b = u \rangle$$

with $l \geq m \geq n \geq 1$.

— If $l = m$:

$$|\text{Aut } G| = p^{2n+4(l-1)+1}(p^2-1)(p-1), \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{2n+4(l-1)}, \quad |\text{Aut}_l G| = p(p^2-1)(p-1).$$

— If $l > m$:

$$|\text{Aut } G| = p^{2n+3m+l-2}(p-1)^2, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{2n+3m+l-2}, \quad |\text{Aut}_l G| = p(p-1)^2.$$

[Notice that, if $G/G' = \langle \bar{a}G' \rangle \times \langle \bar{b}G' \rangle$, then $\{\bar{a}, \bar{b}\} = \{a^\alpha, b^\alpha\}$ for some $\alpha \in \text{Aut } G$, so in particular $\langle \bar{a} \rangle \wedge G' = \langle \bar{b} \rangle \wedge G' = 1$. This fact will avoid any overlapping with the discussion that follows.]

Suppose now that for no direct decomposition $G/G' = \langle aG' \rangle \times \langle bG' \rangle$ it is possible to choose a and b such that $\langle a \rangle \wedge G' = \langle b \rangle \wedge G' = 1$, but there is one such decomposition with $\langle a \rangle \wedge G' = 1$ (and $\langle b \rangle \wedge G' \neq 1$). G has a presentation $G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1, b^{p^m} = u^{p^k}, b^a = bu, u^a = u^b = u \rangle$, where $1 \leq k < n$ (we are no longer assuming $l \geq m$). If $\bar{b} = a^r b^s u^t$ is in the $\text{Aut } G$ -orbit of b , then $\langle \bar{b}^{p^m} \rangle = (G')^{p^k} = \langle u^{p^k} \rangle$; this happens if and only if $a^{rp^m} = 1$ and $s \not\equiv 0 \pmod{p}$. For such a $\bar{b}$ we have $[\bar{b}, a] = u^s$, $\bar{b}^{p^m} = b^{sp^m} = u^{sp^k} = [\bar{b}, a]^{p^k}$ and $G = \langle a, \bar{b} \rangle$, so that there is in fact an automorphism of G mapping b to $\bar{b}$ and fixing a . If θ is any automorphism of G mapping b to $\bar{b}$, then $[\bar{b}, a^\theta]^{p^k} = \bar{b}^{p^m} = [\bar{b}, a]^{p^k}$ and $(a^\theta)^{p^l} = 1$; so the condition on a^θ is $a^\theta \in \alpha\Omega_l(C_G(\bar{b}^{p^k}))$, where $C_G(\bar{b}^{p^k}) = \langle a^{p^{n-k}}, \bar{b}, G' \rangle$, hence $\Omega_l(C_G(\bar{b}^{p^k})) = \langle a^{p^{n-k}} \rangle \Omega_l(\langle \bar{b} \rangle G')$. We may now state

$$(B.2) \quad G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1, b^{p^m} = u^{p^k}, b^a = bu, u^a = u^b = u \rangle$$

where $l \geq n, m \geq n, 1 \leq k < n$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} b \mapsto a^r b^s u^t, \\ a \mapsto aa^{\lambda p^{n-k}} c, \end{cases}$$

where $a^r \in \Omega_m(\langle a \rangle)$, $s \not\equiv 0 \pmod{p}$, $c \in \Omega_l(\langle a^r b^s u^t \rangle G')$.

— If $l \leq m$:

$$|\text{Aut } G| = (p-1)p^{3l+m+2k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{3l+m+2k-1}, \quad |\text{Aut}_l G| = p(p-1).$$

— If $m < l < m+n-k$:

$$|\text{Aut } G| = (p-1)p^{2l+2m+2k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{2l+2m+2k-1}, \quad |\text{Aut}_l G| = p-1.$$

— If $m+n-k \leq l$:

$$|\text{Aut } G| = (p-1)p^{l+3m+n+k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{l+3m+n+k-1}, \quad |\text{Aut}_l G| = p(p-1).$$

To finish with the class 2 case, we have to deal with the following situation: for any choice of generators a, b of G such that $G/G' =$

$= \langle aG' \rangle \times \langle bG' \rangle$ we have $\langle a \rangle \wedge G' \neq 1$ and $\langle b \rangle \wedge G' \neq 1$. G will have a presentation $G = \langle a, b, u \mid u^{p^n} = 1, a^{p^l} = u^{p^h}, b^{p^m} = u^{p^k}, b^a = bu, u^a = u^b = u \rangle$, where $1 \leq h < n$, $1 \leq k < n$, and we may assume $l \geq m$. There are further restrictions, namely: $h > k$ and $l > m + h - k$. In fact, from $h \leq k$ it would follow $b^{p^m} = u^{p^k} = (u^{p^h})^{p^{k-h}} = a^{p^{l+k-h}}$, $(b(a^{-1})^{p^{l+k-h-m}})^{p^m} = 1$ and the generators $a, b_0 = b(a^{-1})^{p^{l+k-h-m}}$ would satisfy $G/G' = \langle aG' \rangle \times \langle b_0G' \rangle$ and $\langle b_0 \rangle \wedge G' = 1$. Similarly, if $h > k$ but $l \leq m + h - k$, we could set $a_0 = a(b^{-1})^{p^{m+h-k-l}}$ and obtain $G/G' = \langle a_0G' \rangle \times \langle bG' \rangle$ and $\langle a_0 \rangle \wedge G' = 1$. We then assume $h > k$ and $l > m + h - k$ and look for subgroups $C = \langle c \rangle$ where $c = a^x b^y u^z$ and $D = \langle d \rangle$ where $d = a^r b^s u^t$, such that $C^{p^l} = (G')^{p^h}$ and $D^{p^m} = (G')^{p^k}$. Now $c^{p^l} = a^x b^y u^z = u^{xp^h} (u^{yp^k})^{p^{l-m}} = u^{p^h(x+yp^{l-m+k-h})}$ generates $\langle u^{p^h} \rangle$ if and only if $x \not\equiv 0 \pmod{p}$: there are $\phi(p^l)p^{m+n}$ such elements, which generate $\phi(p^l)p^{m+n}/\phi(p^{l+n-h}) = p^{m+h}$ subgroups C . And $d^{p^m} = a^{rp^m} b^{sp^k} = a^{rp^m} u^{sp^k}$ generates $\langle u^{p^k} \rangle$ if and only if $a^{rp^m} \in \langle a \rangle \wedge G' = \langle a^{p^l} \rangle = \langle u^{p^h} \rangle$ and $s \not\equiv 0 \pmod{p}$: there are $\phi(p^m)p^{m+n}$ such elements, which generate $\phi(p^m)p^{m+n}/\phi(p^{m+n-k}) = p^{m+k}$ subgroups D . It is also clear that, for any choice of C and D as above, $G = \langle C, D \rangle$ and arbitrary generators c of C (in place of a) and d of D (in place of b) satisfy, together with $v = [d, c]$ in place of u , relations very similar to the original ones, the difference being that some coefficients λ, μ might appear in $c^{p^l} = v^{\lambda p^h}$, $d^{p^m} = v^{\mu p^k}$ ($\lambda, \mu \not\equiv 0 \pmod{p}$). But there certainly are particular generators $\bar{a}, \bar{b}$ of C, D respectively (a convenient choice is c^μ, d^λ), which satisfy, with $\bar{u} = [\bar{b}, \bar{a}]$, all the relations (e.g., ab, b is not a «good pair», but $ab, b^{1+p^{l-m+k-h}}$ is one). For such a choice of $\bar{a}, \bar{b}$, we find that $\bar{a}^i, \bar{b}^j$ ($i \not\equiv 0, j \not\equiv 0 \pmod{p}$) will again satisfy the relations if and only if $[\bar{b}^j, \bar{a}^i] = \bar{u}^{ij}$ is such that $\bar{u}^{ip^h} = \bar{a}^{ip^l} = \bar{u}^{ijp^h}$ and $\bar{u}^{jp^k} = \bar{b}^{jp^m} = \bar{u}^{ijp^k}$, i.e. if and only if $j \equiv 1 \pmod{p^{n-h}}, i \equiv 1 \pmod{p^{n-k}}$: and we have p^{l+m} such pairs. So the number of automorphisms of G is: (number of C 's) times (number of D 's) times p^{l+m} . We can now state:

$$(B.3) \quad G = \langle a, b, u \mid u^{p^n} = 1, a^{p^l} = u^{p^h}, b^{p^m} = u^{p^k}, b^a = bu, u^a = u^b = u \rangle$$

where $m \geq n > h > k \geq 1, l > m + h - k$.

$$|\text{Aut } G| = p^{l+3m+h+k}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

3. From now on, we suppose G is not metacyclic, and its nilpotency class is greater than two. Let us fix some notation. G' is cyclic of order p^n , and $G/C_G(G')$ is cyclic and non-trivial, so that G is generated by

two elements a, b such that $b \in C_G(G')$, a acts on G' as the power $1 + p^s$ ($0 < s < n$) and $G' = \langle [b, a] \rangle$; we denote by p^m the order of bG' and by p^l the order of $a\langle b, G' \rangle$. The $\text{Aut } G$ -orbit of b is contained in $C_G(G') = \langle a^{p^{n-s}}, b, G' \rangle$ and the $\text{Aut } G$ -orbit of a is contained in the coset $aC_G(G')$. We set $u = [b, a]$; since $\langle b, G' \rangle$ is abelian and normal, $[b^y u^z, a] = u^{y+zp^s}$. We will show that if $i \geq 0$ then $[b, a^{p^i}] \equiv u^{p^i} \pmod{\langle u^{p^{s+i}} \rangle}$. This is true for $i = 0$; so, suppose $i > 0$ and, by induction, $[b, a^{p^{i-1}}] = u^{p^{i-1} + \lambda p^{s+i-1}} = u^{p^{i-1}(1 + \lambda p^s)}$. $a^{p^{i-1}}$ acts on G' as some power $1 + \mu p^{s+i-1}$, and $1 + (1 + \mu p^{s+i-1}) + \dots + (1 + \mu p^{s+i-1})^{p-1} = p(1 + \nu p^{s+i-1})$ for some ν , so that (on G') the endomorphism $1 + a^{p^{i-1}} + \dots + (a^{p^{i-1}})^{p-1}$ is the power $p(1 + \nu p^{s+i-1})$. And then $[b, a^{p^i}] = [b, (a^{p^{i-1}})^p] = u^{p^{i-1}(1 + \lambda p^s)p(1 + \nu p^{s+i-1})} \in u^{p^i} \langle u^{p^{s+i}} \rangle$, which establishes our claim. So, in particular, $[b, a^{p^{n-s}}] = u^{p^{n-s}}$. And if $a^{xp^{n-s}} b^y u^z$, $a^{1+\lambda p^{n-s}} b^\mu u^\nu$ are arbitrary elements of $C_G(G')$ and $aC_G(G')$, respectively, we can compute the useful formula

$$[a^{xp^{n-s}} b^y u^z, a^{1+\lambda p^{n-s}} b^\mu u^\nu] = u^{y+pn^{-s}(y\lambda - x\mu) + zp^s}.$$

We can now easily compute $\langle a \rangle \wedge C_G(b) = \langle a \rangle \wedge Z(G) = \langle a^{p^n} \rangle \geq \langle a \rangle \wedge \langle [b, G'] \rangle = \langle a^{p^l} \rangle$, which gives $l \geq n$; $\langle b \rangle \wedge C_G(a) = \langle b \rangle \wedge Z(G) = \langle b^{p^n} \rangle$; $C_G(b) = \langle a^{p^n}, b, u \rangle$ and $Z(G) = \langle a^{p^n} \rangle C_{\langle b, u \rangle}(a) = \langle a^{p^n}, b^{p^n}, b^{-p^s} u \rangle$. Since $\langle aZ(G) \rangle \wedge \langle bZ(G) \rangle = Z(G)$, we see that $G/Z(G)$ is metacyclic of order p^{2n} , and $bZ(G)^{aZ(G)} = bZ(G)^{1+p^s}$.

In this section we deal with the following special case: $C_G(G')/G'$ contains a direct factor $\langle bG' \rangle$ of G/G' ; and $\langle bG' \rangle$ has a complement $\langle aG' \rangle$ in G/G' such that $\langle a \rangle \wedge G' = 1$.

To begin, we suppose, in addition, that b can be chosen to satisfy $\langle b \rangle \wedge G' = 1$. Then G has a presentation $G = \langle a, b, u \mid a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u \rangle$, where $l \geq n, m \geq n$. If $\bar{b} = a^x b^y u^z$ is to be in the $\text{Aut } G$ -orbit of b , then $\bar{b} \in C_G(G')$, $\bar{b} \notin \Phi(G)$, $\bar{b}^{p^m} = 1$, so that $a^x \in \Omega_m(\langle a^{p^{n-s}} \rangle)$, $y \not\equiv 0 \pmod{p}$. And if $\bar{a}$ is in the $\text{Aut } G$ -orbit of a , then $\bar{a} = ac$ for some $c \in \Omega_l(C_G(G')) = \langle a^{p^{n-s}} \rangle \Omega_l(\langle [b] \rangle \langle u \rangle)$. Conversely, for any choice of $\bar{a}, \bar{b}$ in agreement with these requirements, if we set $\bar{u} = [\bar{b}, \bar{a}]$, we get a generating triple for G which satisfies the defining relations. Hence we can state:

$$(C.1) \quad G = \langle a, b, u \mid a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u \rangle$$

where $l \geq n, m \geq n, 0 < s < n$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} a \mapsto a^{1+\lambda p^{n-s}} b^\mu u^\nu, \\ b \mapsto a^x b^y u^z, \end{cases}$$

where $b^\mu \in \Omega_l(\langle b \rangle)$, $a^x \in \Omega_m(\langle a^{p^{n-s}} \rangle)$, $y \not\equiv 0 \pmod{p}$.

— If $m \leq l - n + s$:

$$|\text{Aut } G| = (p-1)p^{l+3m+n+s-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{l+3m+n+s-1}, \quad |\text{Aut}_l G| = p(p-1).$$

— If $l - n + s < m \leq l$:

$$|\text{Aut } G| = (p-1)p^{2l+2m+2s-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{2l+2m+2s-1}, \quad |\text{Aut}_l G| = p(p-1).$$

— If $l < m$:

$$|\text{Aut } G| = (p-1)p^{3l+m+2s-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{3l+m+2s-1}, \quad |\text{Aut}_l G| = p-1.$$

In this second part of the section we still assume that G has generators a, b such that $G/G' = \langle aG' \rangle \times \langle bG' \rangle$ and $\langle a \rangle \wedge G' = 1$ as in the first part, but now $\langle b \rangle \wedge G' \neq 1$. G can be presented in the form: $G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, b^{p^m} = u^{rp^k} \rangle$ with $l \geq n$, $m > k$, $r \not\equiv 0 \pmod{p}$, $0 < k < n$, $p^m \equiv rp^{k+s}(p^n)$; conversely, from the presentation above one can easily read that no element $g \in C_G(G')$ exists such that $\langle gG' \rangle$ is a non-trivial direct factor of G/G' and $\langle g \rangle \wedge G' = 1$.

If $\bar{b} = a^{xp^{n-s}} b^y u^z \in C_G(G')$ is in the $\text{Aut } G$ -orbit of b , then $\langle \bar{b}^{p^m} \rangle = (G')^{p^k}$; the group $C_G(G')$ has class ≤ 2 and b^{p^m} belongs to its centre, so that $\bar{b}^{p^m} = a^{xp^{n-s+m}} b^{yp^m} u^{zp^m}$, and $\langle \bar{b}^{p^m} \rangle = (G')^{p^k}$ if and only if $a^{xp^{n-s+m}} = 1$ and $y \not\equiv 0 \pmod{p}$. On the other hand, suppose $a^{xp^{n-s}} \in \Omega_m(\langle a^{p^{n-s}} \rangle)$ and $y \not\equiv 0 \pmod{p}$; set $\bar{b} = a^{xp^{n-s}} b^y u^z$ and $\bar{u} = [\bar{b}, a] = [b^y u^z, a] = u^{y+zp^s}$. Then $G = \langle a, \bar{b} \rangle$, all the relations except maybe the last one hold, and $\bar{b}^{p^m} = b^{yp^m} u^{zp^m} = u^{yrp^k+zp^m} = u^{rp^k(y+zp^s)} = \bar{u}^{rp^k}$: hence, there is an automorphism of G mapping b to $\bar{b}$ and fixing a . For a given $\bar{b}$ as above, the conditions on $\bar{a} = ac$, where $c \in C_G(G')$, in order that some $\theta \in \text{Aut } G$ exists which satisfies $a^\theta = \bar{a}$, $b^\theta = \bar{b}$ are the following: $\bar{a}^{p^l} = 1$, and $[\bar{b}, \bar{a}]^{p^k} =$

$= [\bar{b}, a]^{p^k}$, i.e. $c \in \Omega_l(C_G(G')) = \langle a^{p^{n-s}} \rangle \Omega_l(\langle \bar{b} \rangle) G'$ and $c \in C_G(\bar{b}^{p^k}) = \langle a^{p^{n-k}} \rangle \langle \bar{b}, G' \rangle$, which means $\bar{a} = a a^\lambda \bar{b}^\mu u^\nu$ with $a^\lambda \in \langle a^{p^{n-s}} \rangle \wedge \langle a^{p^{n-k}} \rangle$, $\bar{b}^\mu \in \Omega_l(\langle \bar{b} \rangle)$. To compute the orders explicitly, we now only have to make the necessary distinctions, according to the relative sizes of k and s , $|\bar{b}| = p^{m+n-k}$ and p^l , $|a^{p^{n-s}}| = p^{l-n+s}$ and p^m . We get the following summary:

(C.2) $G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, b^{p^m} = u^{rp^k} \rangle$
 where $l \geq n$, $m > k$, $r \neq 0 \pmod{p}$, $0 < k < n$, $0 < s < n$, $p^m \equiv rp^{k+s} \pmod{p^n}$.

— If $m \leq l - n + s$ and $s \leq k$:

$$|\text{Aut } G| = (p-1)p^{l+3m+n+s-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{l+3m+n+s-1}, \quad |\text{Aut}_l G| = p(p-1).$$

— If $m \leq l - n + s$, $k < s$ and $m + n - k \leq l$:

$$|\text{Aut } G| = (p-1)p^{l+3m+n+k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{l+3m+n+k-1}, \quad |\text{Aut}_l G| = p(p-1).$$

— If $m \leq l - n + s$, $k < s$ and $m + n - k > l$:

$$|\text{Aut } G| = (p-1)p^{2l+2m+2k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{2l+2m+2k-1}, \quad |\text{Aut}_l G| = p-1.$$

— If $m > l - n + s$, $s < k$ and $m + n - k \leq l$:

$$|\text{Aut } G| = (p-1)p^{2l+2m+s+k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{2l+2m+s+k-1}, \quad |\text{Aut}_l G| = p(p-1).$$

— If $m > l - n + s$, $s < k$ and $m + n - k > l$:

$$|\text{Aut } G| = (p-1)p^{3l+m-n+s+2k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{3l+m-n+s+2k-1}, \quad |\text{Aut}_l G| = p-1.$$

— If $m > l - n + s$ and $s \geq k$:

$$|\text{Aut } G| = (p-1)p^{3l+m-n+2s+k-1}, \quad |\text{Inn } G| = p^{2n},$$

$$|O_p(\text{Aut } G)| = p^{3l+m-n+2s+k-1}, \quad |\text{Aut}_l G| = p-1.$$

4. In this section we retain the general hypotheses and the notation established at the beginning of section 3, and we still assume that $C_G(G')/G'$ contains a direct factor $\langle bG' \rangle$ of G/G' , but we now suppose that for all complements $\langle aG' \rangle$ of $\langle bG' \rangle$ in G/G' we have $\langle a \rangle \wedge G' \neq 1$.

The easy case in this context is when $\langle b \rangle \wedge G' = 1$. G has then a presentation $G = \langle a, b, u \mid a^{p^l} = u^{p^h}, b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u \rangle$ with the usual inequalities $m \geq n, 0 < s < n; [a^{p^l}, b] = 1$ gives $l \geq n$, and of course $n > h > 0$; moreover, $[u^{p^h}, a] = 1$ yields $h \geq n - s$. If $c = a^{\lambda p^{n-s}} b^\mu u^\nu \in C_G(G')$ and $\bar{a} = ac$ is in the $\text{Aut } G$ -orbit of a , then $c^{p^l} = a^{\lambda p^{n-s+l}} b^{\mu p^l} = (u^{p^h})^{\lambda p^{n-s}} b^{\mu p^l} \in G'$, so that $b^{\mu p^l} = 1$ and $\bar{a}^{p^l} = u^{p^h(1+\lambda p^{n-s})}$. And if $\bar{b} = a^{xp^{n-s}} b^y u^z$ is in the $\text{Aut } G$ -orbit of b then $y \not\equiv 0 \pmod{p}$, and $\bar{b}^{p^m} = 1$ gives $a^{xp^{n-s+m}} = 1$. Also, $\bar{u} = [\bar{b}, \bar{a}] = u^{y+p^{n-s}(y\lambda-x\mu)+zp^s}$, and an automorphism θ of G mapping a to $\bar{a}$ and b to $\bar{b}$ exists if and only if $\bar{a}^{p^l} = \bar{u}^{p^h}$: we have to study the solutions of the congruence

$$1 + \lambda p^{n-s} + x\mu p^{n-s} \equiv y(1 + \lambda p^{n-s}) \pmod{p^{n-h}}$$

with the conditions $\mu p^l \equiv 0 \pmod{p^m}$ and $xp^{n-s+m} \equiv 0 \pmod{p^{l+n-h}}$. This congruence has solutions in y (precisely p^{m-n+h} solutions mod p^m) for any given λ, μ, ν, x, z satisfying the conditions above. So we have

$$(D.1) \quad G = \langle a, b, u \mid a^{p^l} = u^{p^h}, b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u \rangle$$

with $l \geq n, m \geq n > h \geq n - s, 0 < s < n$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} a \mapsto a^{1+\lambda p^{n-s}} b^\mu u^\nu, \\ b \mapsto a^{xp^{n-s}} b^y u^z, \end{cases}$$

where $\mu p^l \equiv 0 \pmod{p^m}$, $xp^{n-s+m} \equiv 0 \pmod{p^{l+n-h}}$ and $y(1 + \lambda p^{n-s}) \equiv 1 + (\lambda + x\mu)p^{n-s} \pmod{p^{n-h}}$.

— If $m \leq l$ and $m \geq l + s - h$:

$$|\text{Aut } G| = p^{2l+2m-n+2s+h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

— If $m \leq l$ and $m < l + s - h$:

$$|\text{Aut } G| = p^{l+3m-n+s+2h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

— If $m > l$ and $m \geq l + s - h$:

$$|\text{Aut } G| = p^{3l+m-n+2s+h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = 1.$$

— If $m > l$ and $m < l + s - h$:

$$|\text{Aut } G| = p^{2l+2m-n+s+2h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = 1.$$

5. Suppose now that $C_G(G')/G'$ contains a direct factor of G/G' , but for all generating pairs a, b (with $b \in C_G(G')$) for which $G/G' = \langle aG' \rangle \times \langle bG' \rangle$ one has $\langle a \rangle \wedge G' \neq 1$, $\langle b \rangle \wedge G' \neq 1$. With our usual notation ($u = [b, a]$ of order p^n , $u^a = u^{1+p^s}$ etc.) we find that G has a presentation $G = \langle a, b, u \mid u^{p^h} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, a^{p^r} = u^{p^h}, b^{p^m} = u^{p^k} \rangle$, with $0 < h < n \leq l$, $n - s \leq h$, $r \not\equiv 0 \pmod{p}$, $0 < k < n$, $k < m$ and $p^m \equiv rp^{k+s} \pmod{p^n}$. Some cases have to be excluded. If $k < h$ then $l + k > m + h$, otherwise $(b^{tp^{m+h-k-l}})^{p^l} = (b^{tp^m})^{p^{h-k}} = (u^{p^k})^{p^{h-k}} = u^{p^l}$ (for the inverse t of $r \pmod{p^n}$); hence with $a_0 = ab^{-tp^{m+h-k-l}}$ we get $a_0^{p^l} = 1$ and the pair a_0, b satisfies $G/G' = \langle a_0G' \rangle \times \langle bG' \rangle$ and $\langle a_0 \rangle \wedge G' = 1$. Similarly, if $k \geq h$ we must have $m + n - s > l + k - h$, since otherwise $a^{rp^{l+k-h}} = (u^{p^h})^{rp^{k-h}} = b^{p^m} = (a^{rp^{l+k-h-m}})^{p^m}$ with $a^{rp^{l+k-h-m}} \in \langle a^{p^{n-s}} \rangle \leq C_G(G')$ and we could substitute $b_0 = a^{-rp^{l+k-h-m}}b$ for b ; but $\langle b_0 \rangle \wedge G' = 1$.

To determine $\text{Aut } G$, the point is to find all pairs $\bar{a}, B$ with $\bar{a} \in \langle aC_G(G') \rangle$, B cyclic, $B \leq C_G(G')$ such that $B^{p^m} = (G')^{p^k}$, $\langle \bar{a}^{p^l} \rangle = (G')^{p^h}$, $G = \langle \bar{a}, B \rangle$ and $g^{p^m} = [g, \bar{a}]^{p^k}$ for some generator g of B : in fact, if this is true, then for every generator g^i of B we have $[g^i, \bar{a}]^{p^k} = (g^i)^{p^m}$, and it is clear that there is one particular generator $\bar{b}$ satisfying $\bar{a}^{p^l} = [\bar{b}, \bar{a}]^{p^h}$. Moreover, we can then easily determine all the «good» generators: $\bar{b}^j$ ($j \not\equiv 0 \pmod{p}$) is one if and only if $[\bar{b}^j, \bar{a}]^{p^h} = [\bar{b}, \bar{a}]^{p^h}$, i.e. $[\bar{b}^{p^h(j-1)}, \bar{a}] = 1$. This means $\bar{b}^{p^h(j-1)} \in \langle \bar{b} \rangle \wedge Z(G) = \langle \bar{b}^{p^n} \rangle$, $j \equiv 1 \pmod{p^{n-h}}$, so there are precisely p^{m+h-k} «good» generators of B .

Now, take $\bar{a} = a^{1+\lambda p^{n-s}}b^\mu u^\nu$; we have $\bar{a}^{p^l} = a^{p^l}a^{\lambda p^{n-s+l}}b^{\mu p^l}u^{\nu p^l} \in G'$ implies $\mu p^l \equiv 0 \pmod{p^m}$ which in turn gives $\langle \bar{a}^{p^l} \rangle = \langle a^{p^l} \rangle = (G')^{p^h}$ if $k > h$; but if $k \leq h$ we have $l > m + h - k$, and then $b^{\mu p^l} = (u^{p^k})^{p^{l-m}} \in \langle u^{p^{h+1}} \rangle$ and again $\langle \bar{a}^{p^l} \rangle = (G')^{p^h}$.

If $g = a^{xp^{n-s}}b^y u^z$ ($y \not\equiv 0 \pmod{p}$), we find $g^{p^m} = a^{xp^{n-s+m}}b^{yp^m}u^{zp^m}$ (since $m \geq s$). In case $h \leq k$ we have $n - s + m > l + k - h$, and so $a^{xp^{n-s+m}} \in (G')^{p^{k+1}}$, which implies $\langle g^{p^m} \rangle = \langle b^{p^m} \rangle = (G')^{p^k}$. If instead $k < h$, then $g^{p^m} \in G'$ forces $a^{xp^{n-s+m}} \in G'$, i.e. $xp^{n-s+m} \equiv 0 \pmod{p^l}$, and again $\langle g^{p^m} \rangle = (G')^{p^k}$.

Finally, we study the condition $g^{p^m} = [g, \bar{a}]^{p^k}$ for $g, \bar{a}$ as above, with $\mu p^l \equiv 0 \pmod{p^m}$ and $xp^{n-s+m} \equiv 0 \pmod{p^l}$. The condition is

$$a^{xp^{n-s+m}} = u^{rp^{n-s+k}(y\lambda - x\mu)}.$$

At this point, we have found all pairs $\bar{a}, g$: $g = a^{xp^{n-s}}b^y u^z$ with $a^{xp^{n-s+m}} \in \langle u^{p^h} \rangle \wedge \langle u^{p^{n-s+k}} \rangle$ and $y \not\equiv 0 \pmod{p}$, $\bar{a} = a^{1+\lambda p^{n-s}}b^\mu u^\nu$ where $\mu p^l \equiv 0$

(p^m) and λ is determined (modulo the order of $u^{p^{n-s+k}}$) by

$$(*) \quad (u^{rp^{n-s+k}})^\lambda = a^{xp^{n-s+m}} u^{rx\mu p^{n-s+k}}.$$

As we expected, the conditions on λ, μ depend only on B , and not on the particular generator g . The number of subgroups B is then $|A/\langle a^{p^l} \rangle| p^n \phi(p^m) / \phi(p^{m+n-k}) = p^k |A/\langle a^{p^l} \rangle|$, where ϕ is the Euler function and $A = \langle c \in \langle a^{p^{n-s}} \rangle \mid c^{p^m} \in \langle u^{p^{n-s+k}} \rangle \rangle$ (since $h+m \geq n-s+m > n$ we have $a^{p^{l+m}} = u^{p^{h+m}} = 1$, so that $a^{p^l} \in A$).

The possibilities for $\alpha = |A/\langle a^{p^l} \rangle|$ are the following (we use the fact that $|a^{p^{n-s}}| = p^{l-h+s}$):

— if $k \geq s$ then $A = \Omega_m(\langle a^{p^{n-s}} \rangle)$; if $l-h+s < m$ then $\alpha = p^{l-n+s}$, while if $l-h+s \geq m$ then $\alpha = p^{m+h-n}$;

— if $n > n-s+k \geq h$ we have $u^{p^{n-s+k}} = (u^{p^h})^{p^{n-s+k-h}} = a^{p^{l+n-s+k-h}}$, so that $A/\langle a^{p^l} \rangle = \Omega_m(\langle a^{p^{n-s}} \rangle / \langle a^{p^{l+n-s+k-h}} \rangle)$: if $l+k-h < m$ then again $\alpha = p^{l-n+s}$, while if $l+k-h \geq m$ then $\alpha = p^{m+h+s-n-k}$;

— if $h > n-s+k$, then $A = \{c \in \langle a^{p^{n-s}} \rangle \mid c^{p^m} \in \langle a^{p^l} \rangle\}$, so that $A/\langle a^{p^l} \rangle = \Omega_m(\langle a^{p^{n-s}} \rangle / \langle a^{p^l} \rangle)$: if $l-n+s < m$ then $\alpha = p^{l-n+s}$, while if $l-n+s \geq m$ then $\alpha = p^m$.

For a given choice of B , the number of $\bar{a}$'s is $\beta p^n |\Omega_l(\langle bG' \rangle)|$ where β stands for the number of distinct cosets $a^{\lambda p^{n-s}} \langle a^{p^l} \rangle$ when λ satisfies (*). In case $k \geq s$ λ is arbitrary, hence $\beta = p^{l-n+s}$; on the other hand, when $k < s$ the coset $\lambda + p^{s-k} \mathbb{Z}$ is fixed, which splits into p^{l-n+k} cosets mod p^{l-n+s} : in this case $\beta = p^{l-n+k}$. In the end we get:

$$(D.2) \quad G = \langle a, b, u \mid u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s},$$

$$u^b = u, a^{p^l} = u^{p^h}, b^{p^m} = u^{rp^k} \rangle$$

where $0 < h < n \leq l$, $0 < s < n$, $n-s \leq h$, $r \not\equiv 0 \pmod{p}$, $0 < k < n$, $k < m$, $p^m \equiv rp^{k+s} \pmod{p^n}$, and if $k < h$ then $l+k > m+h$, while if $k \geq h$ then $m+n-s > l+k-h$.

$\text{Aut } G$ is a p -group, $|\text{Inn } G| = p^{2n}$, $|\text{Aut}_l G| = p$ if $l \geq m$ and $|\text{Aut}_l G| = 1$ if $l < m$.

— If $k \geq s$ and $l-h+s < m$:

$$|\text{Aut } G| = p^{2l+m-n+h+2s+\min\{l,m\}}.$$

— If $k \geq s$ and $l-h+s \geq m$:

$$|\text{Aut } G| = p^{l+2m-n+2h+s+\min\{l,m\}}.$$

— If $k < s$, $n - s + k \geq h$ and $l + k < m + h$:

$$|\text{Aut } G| = p^{2l+m-n+h+s+k+\min\{l,m\}}.$$

— If $k < s$, $n - s + k \geq h$ and $l + k \geq m + h$:

$$|\text{Aut } G| = p^{l+2m-n+2h+s+\min\{l,m\}}.$$

— If $h > n - s + k$ and $l - n + s < m$:

$$|\text{Aut } G| = p^{2l+m-n+h+s+k+\min\{l,m\}}.$$

— If $h > n - s + k$ and $l - n + s \geq m$:

$$|\text{Aut } G| = p^{l+2m+h+k+\min\{l,m\}}.$$

6. From now on, we shall deal with the case when no nontrivial direct factor of G/G' is contained in $C_G(G')/G'$. With our usual symbols, G is generated by a and b , $b \in C_G(G')$, $u = [b, a]$ has order p^n , $u^a = u^{1+p^s}$ ($0 < s < n$), $|bG'| = p^m$, $|a\langle b, G' \rangle| = p^l$ and $a^{p^l}G' = b^{p^h}G'$ for some h , $0 < h < m$. G/G' does not split over $\langle bG' \rangle$, hence $l > h$; and $[b, a^{p^l}] = 1$ implies $l \geq n$. We also have $n - s > l - h$, because from $n - s \leq l - h$ it would follow $a^{p^{l-h}} \in C_G(G')$ and $G/G' = \langle aG' \rangle \times \langle a^{p^{l-h}}b^{-1}G' \rangle$. And conversely, if $n - s > l - h$, then $\langle a^{\lambda p^{n-s}}bG' \rangle \wedge \langle ab^\mu G' \rangle \geq \langle a^{p^l}G' \rangle$ for all λ, μ and indeed $C_G(G')/G'$ does not contain any direct factor of G/G' .

In this section we further assume that $\langle b, G' \rangle = \langle b \rangle \times G'$, i.e. $b^{p^m} = 1$; the remaining cases will be discussed from section 7 onwards.

The simplest instance of this class is when a, b can be chosen so that $\langle a \rangle \langle b \rangle \cap G' = 1$. G has then a presentation $G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h} \rangle$ where $0 < s < n$, $0 < h < m$, $0 < l - h < n - s$, $h \geq n$ (the last inequality coming from the fact that $b^{p^h} \in Z(G)$). If we take $\bar{a} = a^{1+\lambda p^{n-s}}b^\mu u^\nu \in aC_G(G')$, $\bar{b} = a^{xp^{n-s}}b^y u^z \in C_G(G')$ with $y \not\equiv 0 \pmod{p}$, and $\bar{u} = [\bar{b}, \bar{a}]$, then $G = \langle \bar{a}, \bar{b} \rangle$, $\bar{b}^{p^m} = 1$ because $|a^{p^{n-s}}| = p^{m+l-h-n+s} < p^m$, $\bar{u}^{p^n} = 1$; it only remains to check the relation $\bar{a}^{p^l} = \bar{b}^{p^h}$, i.e. $a^{(1+\lambda p^{n-s})p^l + \mu p^{2l-h}} = a^{xp^{n-s}+h+yp^l}$. In other terms, x, y, λ, μ have to be solutions of

$$(**) \quad xp^{n-s+h-l} + y \equiv 1 + \lambda p^{n-s} + \mu p^{l-h} \pmod{p^{m-h}};$$

x, λ, μ can be taken at will, and then $y + p^{m-h}\mathbb{Z}$ is determined. In particular $y \equiv 1 \pmod{p}$, so $\text{Aut } G$ is a p -group. We have $|C_G(G')| = |\langle a^{p^{n-s}} \rangle / \langle a^{p^l} \rangle| p^{m+n} = p^{l+m+s}$ choices for $\bar{a}$. Once $\bar{a}$ is given, according to $(**)$ we can choose $\bar{b}$ arbitrarily in the coset

$b^{1+\lambda p^{n-s}+\mu p^{l-h}} \langle a^{p^{n-s}} b^{-p^{n-s+h-l}}, b^{p^{m-h}} \rangle G'$. Since $\langle a^{p^{n-s}} b^{-p^{n-s+h-l}} G' \rangle$ has order p^{l-n+s} , $|\langle b^{p^{m-h}} G' \rangle| = p^h$, and these subgroups of G/G' are independent, we have p^{l+s+h} choices for $\bar{b}$. We obtained

$$(E.1) \quad G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h} \rangle,$$

where $0 < s < n$, $0 < h < m$, $0 < l - h < n - s$, $h \geq n$.

$$|\text{Aut } G| = p^{2l+m+2s+h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

In the setting recorded at the beginning of this section, we suppose now that the generators a, b satisfy $\langle a \rangle \wedge G' = \langle b \rangle \wedge G' = 1$, but $\langle a \rangle \langle b \rangle \cap G' \neq 1$. G has now a presentation $G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h} u^{tp^j} \rangle$, where $t \not\equiv 0 \pmod{p}$, $0 \leq j < n$, and the usual conditions $0 < s < n$, $l \geq n$, $0 < h < m$, $0 < l - h < n - s$, $m \geq n$ hold, and $p^h + tp^{s+j} \equiv 0 \pmod{p^n}$ (which means $b^{p^h} u^{tp^j} \in Z(G)$). We also assume $j < h$, since otherwise $b_0 = bu^{tp^{j-h}}$ would satisfy $b_0^{p^h} = a^{p^l}$ and $\langle a \rangle \langle b_0 \rangle \cap G' = 1$. We also have $|aG'| = p^{l+m-h}$, $\langle a \rangle \wedge G' = \langle a^{p^{l+m-h}} \rangle = \langle (b^{p^h} u^{tp^j})^{p^{m-h}} \rangle = \langle u^{tp^{j+m-h}} \rangle$, and $\langle a \rangle \wedge G' = 1$ gives the further condition $j + m - h \geq n$.

Here (and in the rest of the paper) it is expedient to use the direct decomposition $C_G(G')/G' = \langle cG' \rangle \times \langle bG' \rangle$, where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$. The order $|cG'| = p^{l-n+s}$ is easily computed. The natural candidates for a^θ, b^θ ($\theta \in \text{Aut } G$) can be written as $ac^\lambda b^\mu u^\nu$ and $c^x b^y u^z$, respectively ($y \not\equiv 0 \pmod{p}$). Some tedious but elementary calculations give:

$$(1) \quad [c^x b^y u^z, ac^\lambda b^\mu u^\nu] = u^{p^{n-s}(y\lambda - x\mu) - xp^{n-s-l+h} + y + zp^s}.$$

If we take the relation $a^{p^l} = b^{p^h} u^{tp^j}$ into account, we can also compute

$$(2) \quad (ac^\lambda b^\mu u^\nu)^{p^l} = b^{p^h(1+\mu p^{l-h})} u^{tp^j(1+\lambda p^{n-s})},$$

$$(3) \quad (c^x b^y u^z)^{p^h} = b^{yp^h} u^{xtp^{j+n-s+h-l} + zp^h}.$$

We now set $\bar{a} = ac^\lambda b^\mu u^\nu$, $\bar{b} = c^x b^y u^z$, $\bar{u} = [\bar{b}, \bar{a}]$. From (3) we get $\bar{b}^{p^m} = b^{yp^m} u^{xtp^{j+n-s+m-l} + zp^m}$; in our case $m \geq n$ and $j + n - s + m - l > j + m - h \geq n$, so $\bar{b}^{p^m} = 1$. It remains to check the relation $\bar{a}^{p^l} = \bar{b}^{p^h} \bar{u}^{tp^j}$: by (1), (2) and (3) the condition reads

$$b^{p^h(1+\mu p^{l-h})} u^{tp^j(1+\lambda p^{n-s})} = b^{yp^h} u^{xtp^{j+n-s+h-l} + zp^h}.$$

Now $\langle b, u \rangle = \langle b \rangle \times \langle u \rangle$ and $p^h + tp^{s+j} \equiv 0 \pmod{p^n}$; so, we are looking for the solutions of the system

$$(I) \quad y \equiv 1 + \mu p^{l-h} \pmod{p^{m-h}},$$

$$(II) \quad 1 + \lambda p^{n-s} \equiv p^{n-s}(y\lambda - x\mu) + y \pmod{p^{n-j}}.$$

We also have $n - j \leq m - h$. Substituting y from (I) into (II) gives

$$\mu p^{l-h}(1 + \lambda p^{n-s} - x p^{n-s-l+h}) \equiv 0 \pmod{p^{n-j}};$$

if (λ, μ, x, y) is a solution of the system, then $\mu p^{l-h} \equiv 0 \pmod{p^{n-j}}$ and $y \equiv 1 + \mu p^{l-h} \pmod{p^{m-h}}$; and conversely any 4-tuple $(\lambda, \mu, x, 1 + \mu p^{l-h} + \gamma p^{m-h})$ with $\mu p^{l-h} \equiv 0 \pmod{p^{n-j}}$ is a solution.

To determine $|\text{Aut } G|$, we need to compute the orders of $\{c^\lambda b^\mu u^\nu \mid \mu p^{l-h} \equiv 0 \pmod{p^{n-j}}\}$, which is p^{l+m+s} if $l - h \geq n - j$ and $p^{2l+m-n+s+j-h}$ if $l - h < n - j$, and of $\langle c, b^{p^{m-h}}, G' \rangle$, which is p^{l+s+h} . We conclude:

$$(E.2) \quad G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1, b^a = bu,$$

$$u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h} u^{tp^j} \rangle$$

with $0 < s < n$, $l \geq n$, $0 < h < m$, $0 < l - h < n - s$, $m \geq n$, $t \not\equiv 0 \pmod{p}$, $0 \leq j < h$, $j < n$, $p^h + tp^{s+j} \equiv 0 \pmod{p^n}$ and $n \leq j + m - h$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} a \mapsto ac^\lambda b^\mu u^\nu, \\ b \mapsto c^x b^{1+\mu p^{l-h}+\gamma p^{m-h}} u^z, \end{cases}$$

where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$ and $\mu p^{l-h} \equiv 0 \pmod{p^{n-j}}$.

— If $l - h \geq n - j$:

$$|\text{Aut } G| = p^{2l+2s+m+h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

— If $l - h < n - j$:

$$|\text{Aut } G| = p^{3l+2s+m-n+j}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

To conclude this section, we now study the groups given by a presentation $G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h} u^{tp^j} \rangle$ as in (E.2), but with $n > j + m - h$ (we retain the other conditions on the parameters). This means that $C_G(G')/G'$ does not contain direct factors of G/G' , there is $b \in C_G(G') \setminus \Phi(G)$ such that $\langle b, G' \rangle = \langle b \rangle \times G'$, and for all elements $a \in G$ which, together with b , generate G , we have $\langle a \rangle \wedge G' \neq 1$. As in the previous case, we set $\bar{a} = ac^\lambda b^\mu u^\nu$, $\bar{b} = c^x b^y u^z$ where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$, $\bar{u} = [\bar{b}, \bar{a}]$ and check whether they satisfy the relations. Since $\bar{b}^{p^m} = u^{xtp^{j+n-s+m-l}}$, we see that $\bar{b}^{p^m} = 1$ for all choices of x if $j + m \geq l + s$; on the other hand, if $j + m < l + s$ we must take $x \equiv 0 \pmod{p^{l+s-m-j}}$. Exactly as in the discus-

sion leading to (E.2), $\bar{a}^{p^l} = \bar{b}^{p^h} \bar{u}^{tp^j}$ is equivalent to the system

$$(I) \quad y \equiv 1 + \mu p^{l-h} \quad (p^{m-h}),$$

$$(II) \quad 1 + \lambda p^{n-s} \equiv p^{n-s} (y\lambda - x\mu) + y \quad (p^{n-j}),$$

but in this case $n - j > m - h$. If we multiply (I) through $1 + \lambda p^{n-s}$, write (II) in the form $y(1 + \lambda p^{n-s}) \equiv 1 + \lambda p^{n-s} + x\mu p^{n-s} (p^{n-j})$, and substitute into (I) we get $\mu p^{l-h} (1 + \lambda p^{n-s} - x p^{n-s-l+h}) \equiv 0 \quad (p^{m-h})$: if $l < m$ then μ must be $\equiv 0 \quad (p^{m-l})$. And (II) may also be rewritten as $(y - 1)(1 + \lambda p^{n-s}) \equiv x\mu p^{n-s} (p^{n-j})$, or $y \equiv 1 + x\mu p^{n-s} \sigma (p^{n-j})$, where σ is the inverse of $1 + \lambda p^{n-s}$ in $\mathbb{Z}/p^m \mathbb{Z}$.

Hence, the solutions of our system are all the 4-tuples $(\lambda, \mu, x, 1 + x\sigma\mu p^{n-s} + \eta p^{n-j})$, where $\mu \equiv 0 \quad (p^{m-l})$ if $l < m$ and $x \equiv 0 \quad (p^{l+s-m-j})$ if $j + m < l + s$. To compute the order of $\text{Aut } G$, we note that for $\mu \equiv 0 \quad (p^{m-l})$ (if $l < m$; and for any μ if $l \geq m$) $|b^{\sigma\mu p^{n-s}} G'| \leq p^{l+s-n} = |cG'|$, so that $C_G(G')/G' = \langle cb^{\sigma\mu p^{n-s}} G' \rangle \times \langle bG' \rangle$ and the sets $\{(cb^{\sigma\mu p^{n-s}})^x b^{\eta p^{n-j}}\} G'$ and $\{(cb^{\sigma\mu p^{n-s}})^x b^{\eta p^{n-j}} | x \equiv 0 \quad (p^{l+s-m-j})\} G'$ have orders $p^{l+m-n+s+j}$ and, respectively, $p^{2m-n+2j}$. We state our results:

$$(E.3) \quad G = \langle a, b, u | u^{p^n} = b^{p^m} = 1, b^a = bu, \quad u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h} u^{tp^j} \rangle$$

with $0 < s < n$, $l \geq n$, $0 < h < m$, $0 < l - h < n - s$, $m \geq n$, $t \neq 0 \quad (p)$, $0 \leq j < h$, $j < n$, $p^h + tp^{s+j} \equiv 0 \quad (p^n)$ and $n > j + m - h$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} a \mapsto ac^\lambda b^\mu u^\nu, \\ b \mapsto b(cb^{\sigma\mu p^{n-s}})^x b^{\eta p^{n-j}} u^z, \end{cases}$$

where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$, $\mu \equiv 0 \quad (p^{m-l})$ if $l < m$, $x \equiv 0 \quad (p^{l+s-m-j})$ if $j + m < l + s$, and σ is the inverse of $1 + \lambda p^{n-s}$ in $\mathbb{Z}/p^m \mathbb{Z}$; $|\text{Inn } G| = p^{2n}$.

— If $l \geq m$ and $j + m \geq l + s$:

$$|\text{Aut } G| = p^{2l+2m-n+2s+j}, \quad |\text{Aut}_l G| = p.$$

— If $l \geq m$ and $j + m < l + s$:

$$|\text{Aut } G| = p^{l+3m-n+s+2j}, \quad |\text{Aut}_l G| = p.$$

— If $l < m$ and $j + m \geq l + s$:

$$|\text{Aut } G| = p^{3l+m-n+2s+j}, \quad |\text{Aut}_l G| = 1.$$

— If $l < m$ and $j + m < l + s$:

$$|\text{Aut } G| = p^{2l+2m-n+s+2j}, \quad |\text{Aut}_l G| = 1.$$

7. In order to complete our analysis, we still have to consider the following situation: for every $b \in C_G(G') \setminus \Phi(G)$ we have

— $\langle bG' \rangle$ is not a direct factor of G/G' ; and

— $\langle b \rangle$ is not a direct factor of $\langle b, G' \rangle$.

We again use our standard notation: $G = \langle a, b \rangle$, $b \in C_G(G')$, $u = [b, a]$ has order p^n , $|bG'| = p^m$, $u^a = u^{1+p^s}$ with $0 < s < n$, $|a\langle b, G' \rangle| = p^l$. As we saw at the beginning of the previous section, the first condition is equivalent to: $a^{p^l} = b^{p^h} u^{tp^j}$ with $l > h$, $m > h$ and $n - s > l - h$, $t \not\equiv 0 \pmod{p}$ (at least for the moment, we are not excluding the possibility that $j \geq n$); $[a^{p^l}, b] = 1$ implies $p^h + tp^{j+s} \equiv 0 \pmod{p^n}$. And the second condition says $b^{p^m} = u^{rp^k}$ for some $r \not\equiv 0 \pmod{p}$, $0 < k < n$; $[b^{p^m}, a] = [u^{rp^k}, a]$ then implies $p^m \equiv rp^{k+s} \pmod{p^n}$; so, in particular, $m > k$ and $m > s$.

Any cyclic subgroup of $C_G(G')$, not contained in $\Phi(G)$, is generated by some element $g = a^{xp^{n-s}} bu^z$; an easy calculation gives

$$\begin{aligned} g^{p^m} &= a^{xp^{m+n-s}} b^{p^m} u^{zp^m} = (b^{p^h} u^{tp^j})^{xp^{m+n-s-l}} u^{rp^k} u^{zp^m} = \\ &= u^{x(rp^{k+n-s-l+h} + tp^{j+m+n-s-h}) + rp^k + zp^m}. \end{aligned}$$

If $j + m + n \leq l + s + k$, the congruence $x(rp^{k+n-s-l+h} + tp^{j+m+n-s-h}) + rp^k + zp^m \equiv 0 \pmod{p^n}$ in the unknowns x, z has a solution $(x_1, 0)$, and then $b_1 = a^{x_1 p^{n-s}} b$ satisfies $\langle b_1 \rangle \wedge G' = 1$. On the other hand, if $j + m + n > l + s + k$, then for all choices of x, z g as above satisfies $\langle g^{p^m} \rangle = \langle u^{p^k} \rangle \neq 1$. Hence our second condition is equivalent to $b^{p^m} = u^{rp^k}$, $r \not\equiv 0 \pmod{p}$, $0 < k < n$, $p^m \equiv rp^{k+s} \pmod{p^n}$ and $j + m + n > l + s + k$.

Next, we determine $\langle a \rangle \wedge G'$. Since $|aG'| = p^{l+m-h}$, we have $\langle a \rangle \wedge G' = \langle a^{p^{l+m-h}} \rangle$, and $a^{p^{l+m-h}} = (b^{p^h} u^{tp^j})^{p^{m-h}} = b^{p^m} u^{tp^{j+m-h}} = u^{rp^k + tp^{j+m-h}}$.

In this section we study the special case in which $\langle a \rangle \wedge G' = 1$, i.e. $rp^k + tp^{j+m-h} \equiv 0 \pmod{p^n}$. Since $k < n$, this implies $h + k = j + m$, and the inequality $j + m + n > l + s + k$ reduces to $n - s > l - h$. Once more, we set $\bar{a} = ac^\lambda b^\mu u^\nu$, $\bar{b} = c^x b^y u^z$ ($y \not\equiv 0 \pmod{p}$), $\bar{u} = [\bar{b}, \bar{a}]$ (where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$) and check the relations. Using (1), (2), (3), it is easily seen that $\bar{b}^{p^m} = \bar{u}^{rp^k}$ translates into

$$u^{y rp^k + x tp^{j+n-s+m-l+zp^m}} = u^{rp^k(p^{n-s}(y\lambda - x\mu) - xp^{n-s-l+h} + y) + z rp^{s+k}},$$

i.e.

$$(4) \quad (a^{p^{l+m-h}})x^{p^{n-s+h-l}} = u^{rp^{k+n-s}(y\lambda - x\mu)}.$$

Similarly, we may write $\bar{a}^{p^l} = \bar{b}^{p^h} \bar{u}^{tp^j}$ both as

$$(5) \quad b^{p^{h(1+\mu p^{l-h}-y)}} = u^{tp^j(p^{n-s}(y\lambda - x\mu) + y - 1 - \lambda p^{n-s})}$$

and as

$$(6) \quad a^{p^{l(1+\mu p^{l-h}-y)}} = u^{tp^j(p^{n-s}(y\lambda - x\mu) - \lambda p^{n-s} + \mu p^{l-h})}.$$

In our case $\langle a \rangle \wedge G' = \langle a^{p^{l+m-h}} \rangle = 1$, so these conditions are equivalent to the system

$$(I) \quad p^{k+n-s}(y\lambda - x\mu) \equiv 0 \quad (p^n),$$

$$(II) \quad 1 + \mu p^{l-h} - y \equiv 0 \quad (p^{m-h}),$$

$$(III) \quad p^{n-s}(y\lambda - x\mu) - \lambda p^{n-s} + \mu p^{l-h} \equiv 0 \quad (p^{n-j});$$

note that $m - h = k - j < n - j$.

Suppose first that $m - h \geq s - j$, so that $k - s = m - h + j - s \geq 0$: the first congruence is trivial. Since $n + s + m - h \geq n - j$, from (II) and (III) we get

$$p^{n-s}(\lambda(y - 1) - x\mu) + \mu p^{l-h} \equiv \mu p^{l-h}(1 - xp^{n-s-l+h}) \equiv 0 \quad (p^{n-j})$$

and then $\mu p^{l-h} \equiv 0 \quad (p^{n-j})$, $y \equiv 1 \quad (p^{m-h})$. Conversely, any 4-tuple (λ, μ, x, y) with $\mu p^{l-h} \equiv 0 \quad (p^{n-j})$ and $y \equiv 1 \quad (p^{m-h})$ is a solution of the system.

If, on the other hand, $m - h < s - j$ (i.e. $k < s$), we proceed as follows. If (λ, μ, x, y) is a solution, then $y \equiv 1 \quad (p)$; let y' be the inverse of y in $\mathbb{Z}/p^n\mathbb{Z}$. From (I) and (II) we can write $\lambda = y'x\mu + \sigma p^{s-k}$, $y = 1 + \mu p^{l-h} + \rho p^{m-h}$ for some σ, ρ and then (III) becomes

$$\mu p^{l-h}(1 + \lambda p^{n-s} + \rho y' x p^{n-s+m-l} - x p^{n-s+h-l}) \equiv 0 \quad (p^{n-j}),$$

forcing $\mu p^{l-h} \equiv 0 \quad (p^{n-j})$ (we used the equality $(n-s) + (s-k) + (m-h) = n-j$). And now (II) implies $y \equiv 1 \quad (p^{m-h})$. Moreover, $n-j-l+h > s-j > s-j-m+h = s-k > 0$, so $\mu \equiv 0 \quad (p^{n-j-l+h})$ yields $\lambda \equiv 0 \quad (p^{s-k})$. Vice versa, it is clear that any 4-tuple (λ, μ, x, y) with $\lambda \equiv 0 \quad (p^{s-k})$, $\mu \equiv 0 \quad (p^{n-j-l+h})$, $y \equiv 1 \quad (p^{m-h})$ is a solution of the sys-

tem. So, we have

$$(F.1) \quad G = \langle a, b, u \mid u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s},$$

$$u^b = u, b^{p^m} = u^{rp^k}, a^{p^l} = b^{p^h} u^{tp^j} \rangle$$

with $0 < s < n$, $0 < k < n$, $m > k$, $m > s$, $p^m \equiv rp^{k+s} (p^n)$, $r \not\equiv 0 (p)$, $l > h$, $m > h$, $n - s > l - h$, $t \not\equiv 0 (p)$, $l \geq n$, $p^h + tp^{j+s} \equiv 0 (p^n)$, $rp^k + tp^{j+m-h} \equiv 0 (p^n)$.

The effect of $\text{Aut } G$ on the generators a, b is

$$\begin{cases} a \mapsto ac^\lambda b^\mu u^\nu, \\ b \mapsto bc^x b^{rp^{m-h}} u^z, \end{cases}$$

where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$, $\mu p^{l-h} \equiv 0 (p^{n-j})$, and $\lambda \equiv 0 (p^{s-k})$ in case $k < s$.

— If $s \leq k$ and $l - h \geq n - j$:

$$|\text{Aut } G| = p^{2l+m+n+h+s}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

— If $s \leq k$ and $l - h < n - j$:

$$|\text{Aut } G| = p^{3l+m-n+j+s}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = 1.$$

— If $s > k$:

$$|\text{Aut } G| = p^{3l-n+h+s+2k}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = 1.$$

8. In this final section, we will use the notation established in section 7 to study the only case left, namely: for all $b \in C_G(G') \setminus \Phi(G)$, $\langle bG' \rangle$ is not a direct factor of G/G' and $\langle b \rangle$ is not a direct factor of $\langle b, G' \rangle$, and also $\langle a \rangle \wedge G' \neq 1$ for every $a \in G \setminus \langle b, \Phi(G) \rangle$. We saw that G has a presentation $G = \langle a, b, u \mid u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, b^{p^m} = u^{rp^k}, a^{p^l} = b^{p^h} u^{tp^j} \rangle$ (with some conditions on the numbers $n, s, l, h, t, j, m, r, k$, for which we refer to the previous section). And $\langle a \rangle \wedge G' = \langle a^{p^{l+m-h}} \rangle$, where $a^{p^{l+m-h}} = u^{rp^k + tp^{j+m-h}}$, so that $rp^k + tp^{j+m-h} \not\equiv 0 (p^n)$. We set $\langle a \rangle \wedge G' = \langle u^{p^i} \rangle$; we have $0 < i < n$. Of course, $i = k$ in case $k < j + m - h$, and $i = j + m - h$ in case $k > j + m - h$. If $k = j + m - h$ then $i = k + i'$, where $r + t \equiv 0 (p^{i'})$, $r + t \not\equiv 0 (p^{i'+1})$. We claim that $i < k + l - h$. This is obvious in the first two cases. Suppose $k = j + m - h$; if $i \geq k + l - h$, then $r + t \equiv 0 (p^{l-h})$, and the congruence $r + t + p^{l-h} r \mu \equiv 0 (p^{n-k})$ has a solution μ_0 : using (2) we get $(ab^{\mu_0})^{p^{l+m-h}} = (b^{p^{h(1+\mu_0 p^{l-h})}} u^{tp^j})^{p^{m-h}} =$

$= u^{rp^k(1+\mu_0p^{l-h})+tp^{j+m-h}} = u^{p^k(r+t+\mu_0p^{l-h})} = 1$, contradicting an earlier assumption.

Set once again $\bar{a} = ac^\lambda b^\mu u^\nu$, $\bar{b} = c^x b^y u^z$, $\bar{u} = [\bar{b}, \bar{a}]$, where $c = a^{p^{n-s}} b^{-p^{n-s-i-h}}$ and $y \not\equiv 0 \pmod{p}$. There exists $\theta \in \text{Aut } G$ such that $a^\theta = \bar{a}$, $b^\theta = \bar{b}$ if and only if $\bar{b}^{p^m} = \bar{u}^{rp^k}$ and $\bar{a}^{p^l} = \bar{b}^{p^h} \bar{u}^{tp^j}$, i.e. if and only if (4) and (6) hold. Suppose first that $\theta \in C_{\text{Aut } G}(a)$; then (4) and (6) with $\lambda, \mu = 0$ are, respectively:

$$(a^{p^{l+m-h}})^{xp^{n-s+h-l}} = 1; \quad a^{p^{l(1-y)}} = 1$$

whose solutions are: $x \equiv 0 \pmod{p^{s+l-h}}$ if $i < s + l - h$ (and x arbitrary if $i \geq s + l - h$), $y \equiv 1 \pmod{p^{m-h+n-i}}$. In this way we determine $|C_{\text{Aut } G}(a)| = p^{l-n+h+i}$ if $i \geq s + l - h$, $|C_{\text{Aut } G}(a)| = p^{2h+i-n}$ if $i < s + l - h$ (notice that $u^{p^i} \in Z(G)$, hence $i \geq n - s$ and $h + i - n \geq h - s > 0$)).

We will now use (4) and (6) again, in order to find the $\text{Aut } G$ -orbit of a . If $(\bar{a}, \bar{b}) = (a^\theta, b^\theta)$, $\theta \in \text{Aut } G$, then from (6) we get $a^{p^{l(1+\mu p^{l-h}-y)}} \in \langle a \rangle \wedge \langle u \rangle$, hence $1 + \mu p^{l-h} - y \equiv 0 \pmod{p^{m-h}}$ and $y - 1 = \mu p^{l-h} - \rho p^{m-h}$ for some ρ . Note that $j + n - s + m - h > l + s + k - s - h = k + l - h > i$ implies $u^{tp^{j+h-s}\rho p^{m-h}} \in \langle a \rangle \wedge \langle u \rangle$, so that again (6) yields $u^{tp^j(p^{n-s}(\lambda\mu p^{l-h}-x\mu)+\mu p^{l-h})} \in \langle a \rangle \wedge \langle u \rangle$, i.e. $tp^{j+l-h}(\lambda p^{n-s} - xp^{n-s-l+h} + 1) \equiv 0 \pmod{p^i}$. And we have shown that if $\bar{a}$ is in the $\text{Aut } G$ -orbit of a , then $\mu p^{j+l-h} \equiv 0 \pmod{p^i}$.

For the converse, suppose $\mu p^{j+l-h} = \sigma p^i$ for some σ , and set $\rho p^{m-h} = 1 + \mu p^{l-h} - y$ and $\xi = xp^{n-s-l+h}$. Then (4) and (6) translate into

$$(4^*) \quad qp^i xp^{n-s+h-l} \equiv rp^{k+n-s}(\lambda(1 + \mu p^{l-h} - \rho p^{m-h}) - x\mu) \pmod{p^n},$$

$$(6^*) \quad qp^i \rho \equiv tp^j(p^{n-s}((\mu p^{l-h} - \rho p^{m-h})\lambda - x\mu) + \mu p^{l-h}) \pmod{p^n},$$

where $q \not\equiv 0 \pmod{p}$ is such that $a^{p^{l+m-h}} = u^{qp^i}$.

Since $k + n - s > k + l - h > i$, $k + n - s - i - (n - s - l + h) = k - i - h - l > 0$, $j + n + m - s - h > l + k - h > i$, all the coefficients are divisible by p^i ; hence (4) and (6) are equivalent to the system Σ of congruences (in the unknowns ξ, ρ)

$$\begin{cases} (q + \mu rp^{k+l-i-h})\xi + r\lambda p^{k+n-s-i+m-h}\rho \equiv r\lambda p^{k+n-s-i}(1 + \mu p^{l-h}) \pmod{p^{n-i}}, \\ t\sigma\xi + (q + t\lambda p^{j+n-s+m-h-i})\rho \equiv t\sigma(\lambda p^{n-s} + 1) \pmod{p^{n-i}}. \end{cases}$$

The determinant of Σ is invertible in $\mathbb{Z}/p^{n-i}\mathbb{Z}$ for any choice of λ and μ (with $\mu p^{j+l-h} = \sigma p^i$). Moreover, $k + n - s - i > n - s - l + h$ implies that the solution for ξ (which is unique in $\mathbb{Z}/p^{n-i}\mathbb{Z}$, for given λ and μ) is divisible by $p^{n-s-l+h}$; so we can solve for x , and take $y = 1 + \mu p^{l-h} - \rho p^{m-h}$. And then we conclude that the $\text{Aut } G$ -orbit of a is the set

$\{ac^\lambda b^\mu u^\nu \mid \mu p^{j+l-h} \equiv 0 \pmod{p^i}\}$, whose cardinality is p^{l+m+s} if $j+l-h \geq i$, and $p^{2l+m+s+j-h-i}$ otherwise. We can now state

$$(F.2) \quad G = \langle a, b, u \mid u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s},$$

$$u^b = u, a^{p^l} = b^{p^h} u^{tp^j}, b^{p^m} = u^{rp^k} \rangle$$

where $0 < s < n$, $0 < k < n$, $m > k$, $m > s$, $p^m \equiv rp^{k+s} \pmod{p^n}$, $r \not\equiv 0 \pmod{p}$, $l > h$, $m > h$, $n-s > l-h$, $t \not\equiv 0 \pmod{p}$, $l \geq n$, $p^h + tp^{j+s} \equiv 0 \pmod{p^n}$, $rp^k + tp^{j+m-h} \not\equiv 0 \pmod{p^n}$, $j+m+n > l+s+k$.

Put p^i = the p -part of $rp^k + tp^{j+m-h}$.

— If $i \geq s+l-h$ and $i \leq j+l-h$:

$$|\text{Aut } G| = p^{2l+m-n+s+h+i}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

— If $i \geq s+l-h$ and $i > j+l-h$:

$$|\text{Aut } G| = p^{3l+m-n+s+j}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = 1.$$

— If $i < s+l-h$ and $i \leq j+l-h$:

$$|\text{Aut } G| = p^{l+m-n+s+2h}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = p.$$

— If $i < s+l-h$ and $i > j+l-h$:

$$|\text{Aut } G| = p^{2l+m-n+s+h+j}, \quad |\text{Inn } G| = p^{2n}, \quad |\text{Aut}_l G| = 1.$$

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