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A Lattice of Homomorphs.

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Preliminary notes.

In this paper all groups are finite and soluble. The homomorph \( h(\mathcal{B}) \)
for a boundary \( \mathcal{B} \) consists of all \( <\mathcal{B}\)-perfect groups\), namely all those
groups that have no \( \mathcal{B}\)-groups among their epimorphic images. The boundary \( b(\mathcal{K}) \)
for a homomorph \( \mathcal{K} \) consists of all groups \( G \) such that
\( G \notin \mathcal{K} \) and if \( 1 \neq N \leq G \), then \( G/N \notin \mathcal{K} \). The maps \( h \) and \( b \) are mutually
inverse bijections between the set of non-empty homomorphs and the
set of boundaries. Let \( \mathcal{K} \) be a homomorph. We recall from [4] that the
class \( D_{\mathcal{K}} \) of \( \mathcal{K} \) comprises all groups \( G \) such that \( \text{Cov}_{\mathcal{K}}(G) \neq \emptyset \) namely all
those groups that have \( \mathcal{K}\)-covering subgroups. \( D_{\mathcal{K}} \) is also a homo-
morph. We study in [6] the set

\[ H(\mathcal{U}) = \{ \mathcal{K} | D_{\mathcal{K}} = \mathcal{U} \}, \text{ where } \mathcal{U} \text{ is a homomorph}. \]

Those homomorphs \( \mathcal{K} \) such that \( D_{\mathcal{K}} = \mathcal{U} \) behave with regard to \( \mathcal{U} \) in
a somewhat similar way to the Schunck classes with regard to the
whole universe of soluble groups. The class \( \mathcal{C}(\mathcal{U}) \) (see (2.1) of [6]) is in-
troduced in order to characterize the homomorphs \( \mathcal{K} \) of \( H(\mathcal{U}) \), when
\( H(\mathcal{U}) = \emptyset \) or \( |H(\mathcal{U})| = 1 \) and to study the relation of usual containment
in \( H(\mathcal{U}) \). The class \( \mathcal{C}(\mathcal{U}) \) consists of those primitive groups \( G \) in \( \mathcal{U} \) that satisfy:

If \( M \triangleleft X \) and \( X/\text{core}_XM = G \), we have \( M \in \mathcal{U} \) if and only if
\( X \in \mathcal{U} \).

Let \( \mathcal{P} \) denote the class of finite soluble primitive groups.

If \( H(\mathcal{U}) \neq \emptyset \), the minimum in \( H(\mathcal{U}) \) with regard to the relation of
containment is \( \mathcal{K} = h((b(\mathcal{U}) - \mathcal{P}) \cup \mathcal{C}(\mathcal{U})) \) (see [6], (3.3)).

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In this paper we study the relation of strong containment in $H(\mathcal{U})$ given by

1 DEFINITION. Let $\mathcal{U}$ be a homomorph. Let $\mathcal{X}, \mathcal{Y} \in H(\mathcal{U})$. We say that $\mathcal{X}$ is strongly contained in $\mathcal{Y}$, and write $\mathcal{X} \ll \mathcal{Y}$ if, for each $G \in \mathcal{U}$ an $\mathcal{X}$-covering subgroup of $G$ is contained in some $\mathcal{Y}$-covering subgroup of $G$.

For a homomorph $\mathcal{X}$, we denote $\mathcal{X} := h(b(\mathcal{X}) \cap \mathcal{P})$. For every group $G \in D\mathcal{X}$ we have: $\text{Cov}_{\mathcal{X}}(G) = \text{Cov}_{\mathcal{X}}(G)$ (see [6], (1.8)).

2 LEMMA. Let $\mathcal{X}$ be a homomorph. We denote $a(\mathcal{X}) := \{G \in D\mathcal{X} | \text{ if } H \in \text{Cov}_{\mathcal{X}}(G), H \cap \text{Soc } G = 1\}$. We have:

a) $a(\mathcal{X}) = a(\mathcal{X}) \cap D\mathcal{X}$.

b) $\mathcal{X} = h(a(\mathcal{X}))$.

PROOF. a) It is evident by the definition.

b) Since $b(\mathcal{X}) = b(\mathcal{X}) \cap \mathcal{P}$, we have $b(\mathcal{X}) \subseteq a(\mathcal{X}) \cap D\mathcal{X} = a(\mathcal{X})$ and therefore $h(a(\mathcal{X})) \subseteq h(b(\mathcal{X})) = \mathcal{X}$. Since $\mathcal{X} = h(a(\mathcal{X}))$ (see [2], VI (1.4)) and $a(\mathcal{X}) \subseteq a(\mathcal{X})$, we have $\mathcal{X} = h(a(\mathcal{X})) \subseteq h(a(\mathcal{X}))$.

Let us recall now the following

3 DEFINITION ([5] and [3] (8.2)). Let $\mathcal{B} \subset \mathcal{P}$. We define $B_0 = \mathcal{B}$, and if $B_i$ has already been defined, let $B_{i+1} = \{(X/C_X(V)[V]) | H \leq X \leq K < G = KF(G) \in B_i, H \in \text{Cov}_{h(\mathcal{B}_i)}(K)\}$.

We denote by $\mathcal{B}^\ast$ the union of all class $\mathcal{B}_i$ previously defined.

In a similar way to (8.3) from [3] we have

4 PROPOSITION. Let $\mathcal{X}$ be a homomorph and $\mathcal{B} \subset \mathcal{P}$ such that $\mathcal{B} \subset a(\mathcal{X})$. We have that $\mathcal{B}^\ast \subseteq a(\mathcal{X})$ (in particular $a(\mathcal{X})^\ast = a(\mathcal{X})$).

PROOF. Let us prove that $B_i \subseteq a(\mathcal{X})$ for every $i \in \mathbb{N}$. We proceed by induction on $i$. We have that $\mathcal{B} = B_0 \subseteq a(\mathcal{X})$. Suppose $B_i \subseteq a(\mathcal{X})$. Let $B \in B_{i+1}$. There exists $G \in B_i \subseteq a(\mathcal{X})$, $Y \leq X \leq K$, $K$ complement of $F(G)$, $H \in \text{Cov}_{h(\mathcal{B}_i)}(K)$, $V$, $W$, $X$-subgroups of $F(G)$, $V/W$, $X$-composition
of $F(G)$ such that $B = X/C_X(V/W)[(V/W)]$. Since $B_j \subseteq a(\mathcal{C}) \subseteq a(\mathcal{K})$, by [1] (2.2), we have $\mathcal{K} \ll h(B_j)$, hence there exists $H \in \text{Cov}_{\mathcal{K}}(K)$ such that $H \leq Y$. As $G \in a(\mathcal{K}) \subseteq \mathcal{D},$ we have $H \in \text{Cov}_{\mathcal{N}}(K) \subseteq \text{Cov}_{\mathcal{N}}(G)$. Besides, it can be confirmed that

$$B = X/C_X(V/W)[(V/W)] \cong XV/C_X(V/W)W.$$ 

By the properties of covering subgroups $H \in \text{Cov}_{\mathcal{N}}(XV)$ and

$$HC_X(V/W)W/C_X(V/W)W \in \text{Cov}_{\mathcal{N}}(XV/C_X(V/W)W),$$

therefore $B \in \mathcal{D}$. We know from [3] (8.3), that $B \in a(\mathcal{K}),$ so we can deduce that $B \in a(\mathcal{K}) \cap \mathcal{D} = a(\mathcal{K}).$

Below we study the relation $\ll$ in $H(\mathcal{U}).$

5 Proposition. Let $\mathcal{X}, \mathcal{Y} \in H(\mathcal{U}).$ We have $\mathcal{X} \ll \mathcal{Y}$ if and only if $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Y}}$.

Proof. $\Rightarrow$ It is evident from that comment before Lemma 2. 

$\Rightarrow$ We have $b(\tilde{\mathcal{Y}}) = b(\mathcal{Y}) \cap S$. By definition of $\ll$ and $a(\mathcal{X})$, we have that $b(\mathcal{Y}) \cap S = b(\mathcal{Y}) \cap \mathcal{D} \subseteq a(\mathcal{X})$. Moreover, $a(\mathcal{X}) \subseteq a(\tilde{\mathcal{X}})$, hence $b(\tilde{\mathcal{Y}}) \subseteq b(\tilde{\mathcal{X}})$ and by [1] (2.2), $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Y}}$.

Since the mapping $\mathcal{K} \rightarrow \tilde{\mathcal{Y}}$ from $H(\mathcal{U})$ to the set of Schunck classes is injective (see [6], 3.1), $H(\mathcal{U})$ can be considered a subset of the Schunck classes ordered by $\ll$.

In the examples described in [6] (1.9), (3.8), (3.9), $(H(\mathcal{U}), \ll)$ has a lattice structure. In these examples we have $a(\mathcal{U}) = a(\mathcal{K})$. In this respect, we can say:

6 Proposition. Let $\mathcal{U}$ be a homomorph and $\mathcal{M}$ the minimum for $\zeta$ in $H(\mathcal{U})$. The following statements are equivalent:

a) $a(\mathcal{U}) = a(\mathcal{M})$;

b) $a(\mathcal{U})^\zeta = a(\mathcal{U})$.

Proof. $a) \Rightarrow b)$ It follows immediately from Proposition 4.

$b) \Rightarrow a)$ By $b)$ we obviously have $a(\mathcal{U})^\zeta \cap h(a(\mathcal{U})) = \emptyset$. By [3] (8.4), we have $a(\mathcal{U}) \subseteq a(h(a(\mathcal{U})))$. By [6] (3.3), $h(a(\mathcal{U})) = \mathcal{K}$ and therefore $a(\mathcal{U}) \subseteq a(\mathcal{K})$. 

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Besides, \( \mathfrak{c}(U) \subseteq U = D \mathfrak{M} \) implies \( \mathfrak{c}(U) \subseteq \mathfrak{c}(\mathfrak{M}) \cap D \mathfrak{M} = a(\mathfrak{M}) \). By \([6] (1.7)\), we have \( a(\mathfrak{M}) \subseteq \mathfrak{c}(U) \) and therefore the equality.

7 THEOREM. Let \( U \) be a homomorph such that \( b(U) \cap \mathfrak{P} = \emptyset \). (These homomorphs are known as totally unsaturated).

\((H(U), \ll)\) is a lattice if and only if \( \mathfrak{c}(U)^\ast = \mathfrak{c}(U) \).

PROOF. \( \Rightarrow \) By the proposition above and \([6] (1.7)\), it suffices to prove that \( \mathfrak{c}(U) \subseteq a(\mathfrak{M}) \). Let \( G \in \mathfrak{c}(U) \). Let \( \mathfrak{K} = h(b(U) \cup \{G\}) \). By \([6] (2.3)\), \( \mathfrak{K} \in H(U) \). Since \( \ll \) implies \( \ll \), the infimum of \( \{\mathfrak{K}, \mathfrak{M}\} \) must be \( \mathfrak{M} \). Thus \( \mathfrak{M} \ll \mathfrak{K} \), therefore \( \mathfrak{M} \ll \mathfrak{M} \mathfrak{K} \) and consequently \( b(\mathfrak{K}) \subseteq a(\mathfrak{M}) \). As \( \{G\} = b(\mathfrak{K}) \), we have that

\[ G \in \mathfrak{a}(\mathfrak{M}) \cap U = \mathfrak{a}(\mathfrak{M}) \cap D \mathfrak{M} = a(\mathfrak{M}) \].

\( \Leftarrow \) Let \( x, y \in H(U) \). Recall from \([5] \) Theorem A that

\[ \check{x} \land \check{y} = h((b(\check{x}) \cup b(\check{y}))^\ast) \].

By Proposition 6 we have \( \mathfrak{a}(U)^\ast = \mathfrak{a}(U) = a(\mathfrak{M}) \). Since \( b(\check{x}) \cup b(\check{y}) \subseteq a(\mathfrak{M}) \), by Proposition 4, we have that \( (b(\check{x}) \cup b(\check{y}))^\ast \subseteq a(\mathfrak{M}) \) and therefore \( b(\check{x} \land \check{y}) \subseteq \mathfrak{c}(U) \). By \([6] (2.3)\), we have that \( \mathfrak{K} = h(b(U) \cup U b(\check{x} \land \check{y})) \in H(U) \) and it can easily be confirmed that \( \mathfrak{K} = x \lor (\check{x} \land \check{y}) \).

Now let, \( j = h(a(\check{x}) \cap a(\check{y})) \). Again by the characterization in \([6] (2.3)\) and \( (3.1)\), of the homomorphs in \( H(U) \) we have that \( z = j \cap U \in H(U) \), and \( j = \check{z} \). It can be confirmed that \( z = x \lor \check{y} \).

8 PROPOSITION. Let \( U \) be a totally unsaturated homomorph such that \( (H(U), \ll) \) is a lattice. For every \( x, y \in H(U) \) we have:

a) \( x \land y = x \land \check{y} \).

b) \( x \ll z \neq U \) implies \( x = z \) if and only if \( |b(x) \cap \mathfrak{P}| = 1 \).

PROOF. a) It is clear from the previous proof that

\[ b(x \land y) \cap \mathfrak{P} = b(x \land \check{y}). \]

b) \( \Rightarrow \) If \( |b(x) \cap \mathfrak{P}| \neq 1 \), we can have \( \emptyset \neq b \subset b(x) \cap \mathfrak{P} \subseteq \mathfrak{c}(U) \). Now \( z = h(b(U) \cup b) \in H(U) \), \( z \neq x \) and \( x \ll z \neq U \) in contradiction with the hypothesis.

\( \Leftarrow \) As \( \check{x} = h(b(x) \cap \mathfrak{P}) \), \( \check{x} \) is maximal, hence \( x \ll \check{z} \neq \emptyset \) implies \( \check{x} = \check{z} \) and by Proposition 5 we have the thesis.
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