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On the Algebraic and Arithmetical Structure of Generalized Polynomial Algebras.

FRANZ HALTER-KOCH (*)

ABSTRACT - We introduce a new kind of polynomial rings in infinitely many indeterminates (called large polynomial rings). The large polynomial ring over a factorial or a Krull domain is itself factorial or a Krull domain. The algebra of polynomial functions on an abelian group turns out to be essentially a large polynomial ring.

Introduction.

The classical notion of a polynomial function permits far-reaching generalizations, see [3], Ch. IV, [7], [13] and only recently [12]. In this paper we deal with polynomial functions defined on a module over a commutative ring R with values in an R -algebra. These polynomial functions form a commutative ring, whose algebraic structure is determined by means of a new kind of formal polynomial rings (called *large polynomial rings*). These large polynomial rings have nice arithmetical properties: They are factorial resp. Krull domains if the base ring is a factorial resp. a Krull domain.

1. Large polynomials and power series.

Throughout this paper, let $I \neq \emptyset$ be a set, denote by $\mathcal{S}(I)$ the set of all finite subsets of I , and let $\leq$ be a total order on I . For $n \in \mathbb{N}_0$, we set

$$I_{\leq}^n = \{(i_1, \dots, i_n) \in I^n \mid i_1 \leq i_2 \leq \dots \leq i_n\};$$

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in particular, $I_\leq^0$ is the singleton consisting of the empty sequence.

Let R be a commutative ring (always with $1 \neq 0$) and $X = (X_i)_{i \in I}$ a family of (algebraic independent) indeterminates over R . For a subset $J \subset I$, we set $X_J = (X_i)_{i \in J}$. Let

$$R[X] = R[(X_i)_{i \in I}] = R[\mathcal{F}(X)]$$

be the total algebra of the free abelian monoid $\mathcal{F}(X)$ with basis X ; see [3], ch. III, § 2, no. 11, 12. We call $R[X]$ the *large power series ring in X over R* ; it coincides with the ring A_1 investigated in [2] and with the ring $R[(X_i)_{i \in I}]_3$ investigated in [9].

PROPOSITION 1. *Let R be a domain.*

- i) $R[X]$ is a domain.
- ii) Suppose that all power series rings $R[X_1, \dots, X_m]$ in finitely many indeterminates over R are factorial; then $R[X]$ is also factorial. In particular, if R is a regular factorial ring, then $R[X]$ is factorial.
- iii) If R is a Krull domain, then $R[X]$ is a Krull domain.

PROOF. i) [3], ch. IV, § 4, no. 8 or [6] or [15].

ii) [2], [6] or [15].

iii) [9].

In [15] a more general class of rings is dealt with. ■

Every $f \in R[X]$ has a unique representation in the form

$$f = \sum_{P \in \mathcal{F}(X)} \lambda_P P = \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in I_\leq^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdots X_{i_n}$$

with coefficients $\lambda_P, \lambda_{i_1, \dots, i_n} \in R$; addition and multiplication in $R[X]$ are defined in the usual way. For f as above and $J \subset I$, we set

$$f_J = \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in J_\leq^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdots X_{i_n} \in R[X_J],$$

and we define $\pi_J: R[X] \rightarrow R[X_J]$ by $\pi_J(f) = f_J$. For $J' \subset J \subset I$ we define $\pi_{J, J'}: R[X_J] \rightarrow R[X_{J'}]$ by

$$\begin{aligned} \pi_{J, J'} \left(\sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in J_\leq^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdots X_{i_n} \right) &= \\ &= \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in J'_{\leq}^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdots X_{i_n}. \end{aligned}$$

π_J and $\pi_{J, J'}$ are ring epimorphisms satisfying $\pi_{J, J'} \circ \pi_J = \pi_{J'}$ and $\pi_{J', J''} \circ \pi_{J, J'} = \pi_{J, J''}$ whenever $J'' \subset J' \subset J \subset I$. With the mappings $\pi_{J, J'}$, the system $(R[X_J])_{J \in \mathcal{E}(I)}$ becomes a projective system of R -algebras, and

$$\pi = \varprojlim_{J \in \mathcal{E}(I)} \pi_J: R[X] \rightarrow \varprojlim_{J \in \mathcal{E}(I)} R[X_J]$$

is an isomorphism of R -algebras; if $f \in R[X]$, then $\pi(f) = (f_J)_{J \in \mathcal{E}(I)}$.

If $\mathfrak{S} \subset \mathcal{E}(I)$ is cofinal, we identify

$$\varprojlim_{J \in \mathcal{E}(I)} R[X_J] = \varprojlim_{J \in \mathfrak{S}} R[X_J],$$

and we shall in the sequel simply write $\varprojlim$ to denote the inverse limit over $\mathcal{E}(I)$ or some cofinal subset.

The constructions performed so far suggest to endow $R[X]$ with a topology as follows; for the topological concepts used in the sequel we refer to [4].

For every $J \in \mathcal{E}(I)$, we give $R[X_J]$ the discrete topology. We endow $\varprojlim R[X_J]$ with the topology of the projective limit and shift this topology to $R[X]$ by means of π . This topology on $R[X]$ (which makes π into a homeomorphism) will be called the *limit topology*; it is obviously different from the usual topology on power series rings, and it is discrete if I is finite.

The limit topology makes $R[X]$ into a separated complete topological R -algebra. For $f \in R[X]$ and $J \in \mathcal{E}(I)$, we set

$$\mathcal{U}_J(f) = \{g \in R[X] \mid g_J = f_J\};$$

then $\{\mathcal{U}_J(f) \mid J \in \mathcal{E}(I)\}$ is a fundamental system of neighbourhoods of f , and the family $(f_J)_{J \in \mathcal{E}(I)}$ converges to f in the limit topology.

As an R -module, $R[X]$ is of the form

$$R[X] = \prod_{d \geq 0} R[X]_d, \quad \text{where} \quad R[X]_d = \prod_{(i_1, \dots, i_d) \in I^d_{\leq}} R X_{i_1} \cdots X_{i_d};$$

in particular, $R[X]_0 = R$, and the elements of $R[X]_d$ are of the form

$$\sum_{(i_1, \dots, i_d) \in I^d_{\leq}} \lambda_{i_1, \dots, i_d} X_{i_1} \cdots X_{i_d} \quad \text{where} \quad \lambda_{i_1, \dots, i_d} \in R;$$

they are called *large forms of degree d* ; if $f \in R[X]_d$ and $g \in R[X]_e$, then $fg \in R[X]_{d+e}$.

The ring $R[\mathbb{X}]$ contains the usual polynomial ring $R[X]$, consisting of all elements

$$\sum_{P \in \mathcal{F}(X)} \lambda_P P = \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \dots \cdot X_{i_n} \in R[\mathbb{X}]$$

where only finitely many of the coefficients λ_P resp. $\lambda_{i_1, \dots, i_n}$ are different from zero.

The main purpose of this paper is to investigate the subring

$$R[\langle X \rangle] = R[\langle (X_i)_{i \in I} \rangle] = \coprod_{d \geq 0} R[\mathbb{X}]_d \subset R[\mathbb{X}]$$

consisting of all $f \in R[\mathbb{X}]$ of the form

$$f = \sum_{n=0}^N \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \dots \cdot X_{i_n}$$

for some $N \in \mathbb{N}_0$; we call $R[\langle X \rangle]$ the *large polynomial ring* and its elements *large polynomials (in X over R)*. Any $f \in R[\langle X \rangle]$ has a unique representation in the form

$$f = \sum_{d \geq 0} f_d,$$

where $f_d \in R[\mathbb{X}]_d$ are large forms of degree d , and $f_d = 0$ for all but finitely many $d \geq 0$. As in the classical case, we call

$$\deg(f) = \sup \{d \geq 0 \mid f_d \neq 0\} \in \mathbb{N}_0 \cup \{-\infty\}$$

the *degree* of f .

Clearly, an element $f \in R[\mathbb{X}]$ belongs to $R[\langle X \rangle]$ if and only if $f_J \in R[\mathbb{X}_J]$ for all $J \in \mathcal{E}(I)$ and $\sup \{\deg(f_J) \mid J \in \mathcal{E}(I)\} < \infty$; in this case, $\deg(f) = \max \{\deg(f_J) \mid J \in \mathcal{E}(I)\}$.

We introduce a more general class of polynomial rings, containing $R[X]$ and $R[\langle X \rangle]$ as special cases, as follows. Let $\aleph$ be an infinite cardinal, and let $R[\langle X \rangle]_{\aleph}$ be the set of all large polynomials

$$f = \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \dots \cdot X_{i_n}$$

for which

$$\text{card} \{(i_1, \dots, i_n) \in I_{\leq}^n \mid \lambda_{i_1, \dots, i_n} \neq 0\} < \aleph;$$

large polynomials with this property will be called *large $\aleph$ -polynomials*. Clearly, $R[\langle X \rangle]_{\aleph}$ is a subring of $R[\mathbb{X}]$, $R[\langle X \rangle]_{\aleph_0} = R[X]$, and $\aleph \geq \max \{\aleph_0, \text{card}(I)\}$ implies $R[\langle X \rangle]_{\aleph} = R[\langle X \rangle]$.

We say that an indeterminate X_j occurs in a large polynomial

$f \in R[\langle X \rangle]$, if $f \notin R[(X_i)_{i \in I \setminus \{j\}}]$ and we set

$$I_f = \{j \in I \mid X_j \text{ occurs in } f\}.$$

For any $f \in R[\langle X \rangle]$, we have $f \in R[\langle X \rangle]_\kappa$ if and only if $\text{card}(I_f) < \kappa$.

PROPOSITION 2. *Let R be a domain.*

- i) $R[\langle X \rangle]_\kappa$ is a domain, and $R[\langle X \rangle]_\kappa^\times = R^\times$.
- ii) If $f, g \in R[\langle X \rangle]$ and $0 \neq fg \in R[\langle X \rangle]_\kappa$, then $f \in R[\langle X \rangle]_\kappa$ and $g \in R[\langle X \rangle]_\kappa$.
- iii) If $f, g \in R[\langle X \rangle]$, then $\deg(fg) = \deg(f) + \deg(g)$.

PROOF. Since $R[\langle X \rangle]_\kappa \subset R[\![X]\!]$, it is a domain by Proposition 1; i) and iii) are proved as in the classical case, see [3], Ch. IV, §9, no. 5. Since R is a domain, we have $I_{fg} = I_f \cup I_g$ for all $f, g \in R[\![X]\!] \setminus \{0\}$, which implies ii). ■

We endow $R[\langle X \rangle]$ with the subspace topology induced from the limit topology on $R[\![X]\!]$. If I is infinite, $R[\langle X \rangle]$ is not closed in $R[\![X]\!]$ and hence it is not complete. Its closure $\overline{R[\langle X \rangle]}$ consists of all $f \in R[\![X]\!]$ such that $f_J \in R[\![X_J]\!]$ for all $J \in \mathcal{E}(I)$. The ring $\overline{R[\langle X \rangle]}$ coincides with the ring A_2 investigated in [2]; it was proved there, that this ring does not even satisfy the ascending chain condition for principal ideals (if I is infinite).

The ring $R[\langle X \rangle]$ has the following universal mapping property.

PROPOSITION 3. *Let $\varphi: R \rightarrow S$ be a homomorphism of commutative rings, $(x_i)_{i \in I} \in S^{(I)}$, and give S the discrete topology. Then there exists a unique continuous ring homomorphism $\phi: R[\langle X \rangle] \rightarrow S$ satisfying $\phi|R = \varphi$ and $\phi(X_i) = x_i$ for all $i \in I$.*

PROOF. Clearly there exists exactly one ring homomorphism $\varphi^*: R[\![X]\!] \rightarrow S$ satisfying $\varphi^*|R = \varphi$ and $\varphi^*(X_i) = x_i$ for all $i \in I$. For any $f \in R[\![X]\!]$, we have $(\varphi^*)^{-1}(\varphi^*f) \supset \mathcal{U}_J(f)$, where $J = \{i \in I \mid x_i \neq 0\}$; hence φ^* is continuous and has a unique extension to a continuous homomorphism ϕ as asserted. ■

2. Arithmetical properties of the large polynomial ring.

In this section we shall prove that the large polynomial ring $R[\langle X \rangle]_\kappa$ is a factorial domain resp. a Krull domain if R is so (Theorems 1 and 2).

First we recall from [1] the notation of a *finite factorization domain* (FFD). An integral domain R is called an FFD, if every $a \in R \setminus (R^\times \cup \{0\})$ is a product of irreducible elements of R and possesses (up to associates) only finitely many divisors in R . If R is an FFD, then every polynomial ring $R[X_1, \dots, X_m]$ is an FFD by [1, Prop. 5.3; every Krull domain is an FFD by [10, Theorem 5.

PROPOSITION 4. *Let R be an FFD and $f \in R[\langle X \rangle] \setminus R$. Then f is irreducible in $R[\langle X \rangle]$ if and only if there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is irreducible in $R[X_J]$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$.*

PROOF. If f is reducible in $R[\langle X \rangle]$, then $f = gh$, where $g, h \in R[\langle X \rangle] \setminus R^\times$. This implies $f_J = g_J h_J$, and if J is sufficiently large; then $g_J, h_J \notin R^\times$. Therefore f_J is irreducible in $R[X_J]$ for all sufficiently large $J \in \mathcal{E}(I)$.

For the converse, suppose that, for any $J_0 \in \mathcal{E}(I)$, there exists some $J \in \mathcal{E}(I)$ such that $J \supset J_0$ and f_J is reducible in $R[X_J]$. We set $n = \deg(f) \in \mathbb{N}$, and we shall prove that f is reducible in $R[\langle X \rangle]$. By assumption, the set

$$\mathfrak{S} = \{J \in \mathcal{E}(I) \mid \deg(f_J) = n, f_J \text{ is reducible in } R[X_J]\}$$

is cofinal in $\mathcal{E}(I)$, and for $J \in \mathfrak{S}$ the set

$$E_J = \{\varphi \in R[X_J] \setminus R^\times \mid \deg(\varphi) < n, \varphi \mid f_J\}$$

is not empty. If $J, J' \in \mathfrak{S}$, $J' \subset J$ and $\varphi \in E_J$, then there exists some $\psi \in R[X_{J'}] \setminus R$ such that $f_J = \varphi\psi$, and consequently $f_{J'} = \pi_{J,J'}(f_J) = \pi_{J,J'}(\varphi)\pi_{J,J'}(\psi)$. This implies $\pi_{J,J'}(\varphi) \mid f_{J'}$ and $n = \deg(f_{J'}) = \deg(\pi_{J,J'}(\varphi)) + \deg(\pi_{J,J'}(\psi)) \leq \deg(\varphi) + \deg(\psi) = \deg(f_J) = n$, whence $\deg(\pi_{J,J'}(\varphi)) = \deg(\varphi) < n$. If $\pi_{J,J'}(\varphi) \in R$, then $0 = \deg(\pi_{J,J'}(\varphi)) = \deg(\varphi)$, which implies $\varphi \in R$ and thus $\pi_{J,J'}(\varphi) = \varphi \notin R^\times$. In any case, we obtain $\pi_{J,J'}(\varphi) \notin R^\times$ and therefore $\pi_{J,J'}(\varphi) \in E_{J'}$.

Now we consider the projective system $((E_J)_{J \in \mathfrak{S}}, (\pi_{J,J'} \mid E_J: E_J \rightarrow E_{J'})_{J' \subset J})$, and we assert that $\varprojlim E_J \neq \emptyset$.

Since $R[X_J]$ is an FFD, the set

$$\overline{E}_J = \{\varphi R^\times \mid \varphi \in E_J\}$$

of classes of associates in E_J is finite. For $J, J' \in \mathfrak{S}$, $J' \subset J$, we define $\overline{\pi}_{J,J'}: \overline{E}_J \rightarrow \overline{E}_{J'}$, by $\overline{\pi}_{J,J'}(\varphi R^\times) = \pi_{J,J'}(\varphi) R^\times$; then we have $\varprojlim \overline{E}_J \neq \emptyset$ by [5], (Ch. III, § 7, no. 4, Ex. II). Let $(\varphi'_J)_{J \in \mathfrak{S}}$ be a family of polynomials $\varphi'_J \in R[X_J]$ such that $(\varphi'_J R^\times)_{J \in \mathfrak{S}} \in \varprojlim \overline{E}_J$, and fix some $J_0 \in \mathfrak{S}$. If $J \in \mathfrak{S}$, $J \supset J_0$, then $\pi_{J,J_0}(\varphi'_J) = u_J \varphi'_{J_0}$ for some $u_J \in R^\times$, and we set $\varphi_J =$

$= u_J^{-1} \varphi'_J \in E_J$; this implies $\pi_{J, J_0}(\varphi_J) = \varphi_{J_0} = \varphi'_{J_0}$ for all $J \supset J_0$. If $J, J' \in \mathfrak{S}$, $J \supset J' \supset J_0$, then $\pi_{J, J'}(\varphi_J) = v\varphi'_{J'}$ for some $v \in R^\times$, and $\varphi_{J_0} = \pi_{J, J_0}(\varphi_J) = \pi_{J', J_0} \circ \pi_{J, J'}(\varphi_J) = \pi_{J', J_0}(v\varphi'_{J'}) = v\varphi_{J_0}$ implies $v = 1$; therefore $(\varphi_J)_{J \in \mathfrak{S}, J \supset J_0} \in \varprojlim E_J$.

If $\varphi = (\varphi^{(J)})_{J \in \mathfrak{S}} \in \varprojlim E_J \subset \varprojlim R[X_J]$ and $g = \pi^{-1}(\varphi) \in R[X]$, then $g_J = \varphi^{(J)} \in R[X_J]$ for all $J \in \mathfrak{S}$ and $\deg(g_J) = \deg(\varphi^{(J)}) < n$, which implies $g \in R[\langle X \rangle]$ and $\deg(g) < n$. If $g \in R$, then $\varphi^{(J)} = g_J = g$ for all $J \in \mathfrak{S}$, and consequently $g \notin R^\times$.

For any $J \in \mathfrak{S}$, we have $f_J = \varphi^{(J)} \psi^{(J)}$ for some polynomial $\psi^{(J)} \in R[X_J]$; this implies $\psi = (\psi^{(J)})_{J \in \mathfrak{S}} \in \varprojlim R[X_J]$ and (as above) $h = \pi^{-1}(\psi) \in R[\langle X \rangle]$. Since $f_J = g_J h_J = (gh)_J$ for all $J \in \mathfrak{S}$, we obtain $f = gh$. Since $g \notin R^\times$ and $\deg(g) < n$, f is reducible in $R[\langle X \rangle]$. ■

PROPOSITION 5. *Let R be a domain, $f \in R[\langle X \rangle]$ and suppose that there exists some $J_0 \in \mathcal{E}(I)$ such that, for any $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$, f_J is a prime element of $R[X_J]$. Then f is a prime element of $R[\langle X \rangle]$.*

PROOF. Suppose that $f | gh$ for some $g, h \in R[\langle X \rangle]$. For any $J \in \mathcal{E}(I)$, this implies $f_J | g_J h_J$, and if $J \supset J_0$, then either $f_J | g_J$ or $f_J | h_J$. We set

$$\mathfrak{S}' = \{J \in \mathcal{E}(I) | f_J | g_J\}, \quad \mathfrak{S}'' = \{J \in \mathcal{E}(I) | f_J | h_J\},$$

and we obtain

$$\{J \in \mathcal{E}(I) | J \supset J_0\} \subset \mathfrak{S}' \cup \mathfrak{S}''$$

which implies that either $\mathfrak{S}'$ or $\mathfrak{S}''$ is cofinal in $\mathcal{E}(I)$. Without restriction, let $\mathfrak{S}'$ be cofinal in $\mathcal{E}(I)$. For $J \in \mathfrak{S}'$, there exists some polynomial $\varphi^{(J)} \in R[X_J]$ such that $g_J = f_J \varphi^{(J)}$. If $J, J' \in \mathfrak{S}'$ and $J \supset J'$, then $g_{J'} = \pi_{J, J'}(g_J) = \pi_{J, J'}(f_J) \pi_{J, J'}(\varphi^{(J)}) = f_{J'} \pi_{J, J'}(\varphi^{(J)}) = f_{J'} \varphi^{(J')}$ implies $\pi_{J, J'}(\varphi^{(J)}) = \varphi^{(J')}$ and hence $\varphi = (\varphi^{(J)})_{J \in \mathfrak{S}'} \in \varprojlim R[X_J]$. If $q = \pi^{-1}(\varphi)$, then $q_J = \varphi^{(J)} \in R[X_J]$ and $\deg(q_J) = \deg(\varphi^{(J)}) \leq \deg(g_J) \leq \deg(g)$ for all $J \in \mathfrak{S}'$; this implies $q \in R[\langle X \rangle]$. Since $g_J = f_J q_J = (fq)_J$ for all $J \in \mathfrak{S}'$, we obtain $g = fq$, whence $f | g$ in $R[\langle X \rangle]$. ■

Next we adopt Gauss' Lemma for large polynomials. An element $f \in R[\langle X \rangle]$ is called *primitive*, if $f = \lambda f^*$ where $\lambda \in R$ and $f^* \in R[\langle X \rangle]$ implies $\lambda \in R^\times$ (i.e., 1 is a g.c.d. of all coefficients of f in R). Hence an element of R is primitive if and only if it lies in $R^\times$.

PROPOSITION 6. *Let R be an FFD and $f \in R[\langle X \rangle] \setminus R$. Then the following assertions are equivalent:*

- a) *f is primitive.*
- b) *f_J is primitive for some $J \in \mathcal{E}(I)$.*
- c) *There exists some $J_0 \in \mathcal{E}(I)$ such that f_J is primitive for all $J \supset J_0$.*

PROOF. Obviously, $c) \Rightarrow b) \Rightarrow a)$. Now set

$$f = \sum_{P \in \mathcal{F}(X)} \lambda_P P \in R[\langle X \rangle]$$

and suppose that f is primitive, i.e., 1 is a g.c.d. of $\{\lambda_P \mid P \in \mathcal{F}(X)\}$. Since R is an FFD, there exists a finite subset $\mathcal{P} \subset \mathcal{F}(X)$ such that 1 is a g.c.d. of $\{\lambda_P \mid P \in \mathcal{P}\}$. If $J_0 \in \mathcal{E}(I)$ is such that $\mathcal{P} \subset \mathcal{F}(X_{J_0})$, then $\mathcal{P} \subset \mathcal{F}(X_J)$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$; therefore 1 is a g.c.d. of $\{\lambda_P \mid P \in \mathcal{F}(X_J)\}$ for any such J , which means that

$$f_J = \sum_{P \in \mathcal{F}(X)} \lambda_P P \in R[X_J]$$

is primitive. ■

PROPOSITION 7 (Gauss' lemma). *Let R be a factorial domain and K a quotient field of R .*

- i) *If $f, g \in R[\langle X \rangle]$ are primitive, then fg is also primitive.*
- ii) *If $f \in R[\langle X \rangle]$ is primitive, $g \in K[\langle X \rangle]$ and $fg \in R[\langle X \rangle]$, then already $g \in R[\langle X \rangle]$.*

PROOF. For classical polynomials $f \in K[X]$, we use the notation of the content $c(f)$ as in [8], § 8. Then we have $c(fg) = c(f)c(g)$ for all $f, g \in K[X]$; $f \in R[X]$ if and only if $c(f)$ is integral; $f \in R[X]$ is primitive if and only if $c(f) = 1$.

i) If $f, g \in R[\langle X \rangle]$ are primitive, then $f_J, g_J \in R[X_J]$ are primitive for some $J \in \mathcal{E}(I)$ by Proposition 6. Then $(fg)_J = f_J g_J$ is also primitive, and again Proposition 6 implies that fg is primitive.

ii) By Proposition 6, there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is primitive for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$. For such J , $c(f_J g_J) = c(g_J)$ is integral, since $f_J g_J = (fg)_J \in R[X_J]$; this implies $g_J \in R[X_J]$, and consequently $g \in R[\langle X \rangle]$. ■

PROPOSITION 8 *Let R be a factorial domain, K a quotient field of R and $f \in R[\langle X \rangle]_K \setminus R$. Then the following assertions are equivalent:*

- a) f is a prime element of $R[\langle X \rangle]_{\mathbb{K}}$.
- b) f is irreducible in $R[\langle X \rangle]_{\mathbb{K}}$.
- c) f is primitive and irreducible in $K[\langle X \rangle]$.

PROOF. For finite I , this is classical; see [14], Ch. V, § 6.

a) $\Rightarrow$ b) is obvious.

b) $\Rightarrow$ c) If f is irreducible in $R[\langle X \rangle]_{\mathbb{K}}$, then f is irreducible in $R[\langle X \rangle]$ by Proposition 2, ii). By Proposition 4, there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is irreducible in $R[X_J]$ and hence in $K[X_J]$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$. Again by Proposition 4 it follows that f is irreducible in $K[\langle X \rangle]$. Being irreducible in $R[\langle X \rangle]$, f is primitive by definition.

c) $\Rightarrow$ a) By Propositions 4 and 6, there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is primitive and irreducible in $K[X_J]$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$. Hence f_J is a prime element in $R[X_J]$ for all such J , and Proposition 5 implies that f is a prime element of $R[\langle X \rangle]$. By Proposition 2, ii),

$$fR[\langle X \rangle] \cap R[\langle X \rangle]_{\mathbb{K}} = fR[\langle X \rangle]_{\mathbb{K}},$$

and hence f is also a prime element of $R[\langle X \rangle]_{\mathbb{K}}$. ■

THEOREM 1. *Let R be a factorial domain and K a quotient field of R . Then $R[\langle X \rangle]_{\mathbb{K}}$ is a factorial domain; the prime elements of $R[\langle X \rangle]_{\mathbb{K}}$ are the primes of R and the primitive polynomials $f \in R[\langle X \rangle]_{\mathbb{K}} \setminus R$ which are irreducible in $K[\langle X \rangle]$.*

PROOF. If $f \in R[\langle X \rangle]_{\mathbb{K}} \setminus R$ is primitive and irreducible in $K[\langle X \rangle]$, then f is a prime element of $R[\langle X \rangle]_{\mathbb{K}}$ by Proposition 8. If $p \in R$ is a prime element of R , then $R[\langle X \rangle]_{\mathbb{K}}/pR[\langle X \rangle]_{\mathbb{K}}$ is a domain, and thus p is a prime element of $R[\langle X \rangle]_{\mathbb{K}}$.

We must prove that every $f \in R[\langle X \rangle]_{\mathbb{K}} \setminus (R^\times \cup \{0\})$ has a factorization $f = p_1 \cdots p_r f_1 \cdots f_s$, where $p_i \in R$ are prime elements and $f_j \in R[\langle X \rangle]_{\mathbb{K}} \setminus R$ are irreducible. For $f \in R$, this is obvious. Thus we may suppose that $f \in R[\langle X \rangle]_{\mathbb{K}} \setminus R$ and that the assertion is proved for all large polynomials of smaller degree. Clearly, $f = p_1 \cdots p_r f^*$, where $p_i \in R$ are primes of R and $f^* \in R[\langle X \rangle]_{\mathbb{K}} \setminus R$ is primitive. If f^* is irreducible, we are done; otherwise $f^* = f_1^* f_2^*$ where $f_i^* \in R[\langle X \rangle]_{\mathbb{K}} \setminus R$ and hence $\deg(f_i^*) < \deg(f^*) = \deg(f)$ ($i = 1, 2$). Applying the induction hypothesis for f_i^* , the assertion follows. ■

For the next result, we need a Lemma.

LEMMA 1. Let $(R_\alpha)_{\alpha \in \Lambda}$ be a family of Krull domains contained in a field K , and set

$$R = \bigcap_{\alpha \in \Lambda} R_\alpha.$$

Suppose that for every $0 \neq x \in R$ the set $\{\alpha \in \Lambda \mid x \notin R_\alpha^\times\}$ is finite. Then R is Krull domain.

PROOF. [9], Lemma 1.2. ■

THEOREM 2. If R is a Krull domain, then $R[\langle X \rangle]_\kappa$ is also a Krull domain.

PROOF. Let K be a quotient field of R and $(V_\alpha)_{\alpha \in \Lambda}$ a family of discrete valuation rings of K such that $R = \bigcap_{\alpha \in \Lambda} V_\alpha$ and, for each $0 \neq x \in R$, the set $\{\alpha \in \Lambda \mid x \notin V_\alpha^\times\}$ is finite. For $\alpha \in \Lambda$, set

$$N_\alpha = \{f \in V_\alpha[\langle X \rangle] \mid f \text{ is primitive}\}.$$

By Proposition 7, N_α is a multiplicatively closed subset of $V_\alpha[\langle X \rangle]$. By Theorem 1, the domains $K[\langle X \rangle]_\kappa$ and $V_\alpha[\langle X \rangle]$ are factorial and hence the localisations $V_\alpha[\langle X \rangle]_{N_\alpha}$ are also factorial. If $0 \neq f \in R[\langle X \rangle]$ then the set $\{\alpha \in \Lambda \mid x \notin N_\alpha^\times\}$ is finite; this implies $f \in (V_\alpha[\langle X \rangle]_{N_\alpha})^\times$ for all but finitely many $\alpha \in \Lambda$. By Lemma 1 it is sufficient to prove that

$$R[\langle X \rangle]_\kappa = K[\langle X \rangle]_\kappa \cap \bigcap_{\alpha \in \Lambda} V_\alpha[\langle X \rangle]_{N_\alpha}.$$

Obviously $R[\langle X \rangle]_\kappa$ is contained in $K[\langle X \rangle]_\kappa$ and in each $V_\alpha[\langle X \rangle]_{N_\alpha}$. If $\alpha \in \Lambda$ and $f \in K[\langle X \rangle]_\kappa \cap V_\alpha[\langle X \rangle]_{N_\alpha}$, then there exists some $g_\alpha \in N_\alpha$ such that $fg_\alpha \in V_\alpha[\langle X \rangle]$. Since $g_\alpha \in V_\alpha[\langle X \rangle]$ is primitive, Proposition 7, ii) implies $f \in V_\alpha[\langle X \rangle]$. Thus we obtain

$$K[\langle X \rangle]_\kappa \cap \bigcap_{\alpha \in \Lambda} V_\alpha[\langle X \rangle]_{N_\alpha} \subset K[\langle X \rangle]_\kappa \cap \bigcap_{\alpha \in \Lambda} V_\alpha[\langle X \rangle] = R[\langle X \rangle]_\kappa. \quad \blacksquare$$

3. Polynomial functions on modules.

Throughout this section, let F be a commutative ring, R a commutative F -algebra and V an F -module. A mapping $p: V \rightarrow R$ is called a *homogeneous F -polynomial function of degree $d \in \mathbb{N}$* , if there exists an F -multilinear mapping $p^*: V^d \rightarrow R$ such that $p(x) = p^*(x, \dots, x)$ for all $x \in V$. We denote by $\mathcal{P}_F(V, R)_d$ the set of all homogeneous F -polynomial functions $p: V \rightarrow R$ of degree $d \in \mathbb{N}$; $\mathcal{P}_F(V, R)_0$ denotes the set of all con-

stant functions $p: V \rightarrow R$ which we call *homogeneous F -polynomial functions of degree 0*. For any $d \in \mathbb{N}_0$, $\mathcal{P}_F(V, R)_d$ is an R -module under pointwise addition and scalar multiplication. If $p \in \mathcal{P}_F(V, R)_d$, $x \in V$ and $t \in F$, then $p(tx) = t^d p(x)$.

It is usual to define polynomial functions with values in F -modules, see e.g. [3; Ch. IV, §5, no. 9]. In this paper however, we are mainly interested in the polynomial algebra (with a pointwise multiplication), and therefore we restrict ourselves to polynomial functions taking values in an F -algebra.

PROPOSITION 9. *Let $d, e \in \mathbb{N}_0$, $p \in \mathcal{P}_F(V, R)_d$ and $q \in \mathcal{P}_F(V, R)_e$ be given. If $pq: V \rightarrow R$ is defined pointwise, i.e. $(pq)(x) = p(x)q(x)$, then $pq \in \mathcal{P}_F(V, R)_{d+e}$.*

PROOF. Let $p^*: V^d \rightarrow R$ and $q^*: V^e \rightarrow R$ be F -multilinear mappings such that $p(x) = p^*(x, \dots, x)$ and $q(x) = q^*(x, \dots, x)$ for all $x \in V$. If $r: V^{d+e} = V^d \times V^e \rightarrow R$ is defined by $r(x_1, \dots, x_d, y_1, \dots, y_e) = p(x_1, \dots, x_d)q(y_1, \dots, y_e)$, then r is F -multilinear, and $(pq)(x) = r(x, \dots, x)$ for all $x \in V$. ■

A mapping $p: V \rightarrow R$ is called an *F -polynomial function*, if there exists some $d \in \mathbb{N}_0$ and homogeneous F -polynomial functions $p_0, \dots, p_d: V \rightarrow R$ such that $p(x) = p_0(x) + \dots + p_d(x)$ for all $x \in V$; p is called a *local F -polynomial function*, if $p|_M: M \rightarrow R$ is an F -polynomial function for every finitely generated R -submodule M of V . We denote by $\mathcal{P}_F(V, R)$ the set of all F -polynomial functions and by $\overline{\mathcal{P}}_F(V, R)$ the set of all local F -polynomial functions $p: V \rightarrow R$. Obviously,

$$\mathcal{P}_F(V, R) \subset \overline{\mathcal{P}}_F(V, R) \subset R^V$$

are R -subalgebras if R^V is viewed as the R -algebra of all functions $f: V \rightarrow R$ under pointwise addition, multiplication and scalar multiplication.

On the algebra R^V we introduce a topology as follows. Denote by $\mathfrak{E}(V)$ the set of all finitely generated F -submodules of V . For $M \in \mathfrak{E}(V)$, define $\pi_M: R^V \rightarrow R^M$ by $\pi_M(f) = f|_M$, and for $M, M' \in \mathfrak{E}(V)$, $M \supset M'$, define $\pi_{M, M'}: R^M \rightarrow R^{M'}$ by $\pi_{M, M'}(g) = g|_{M'}$. With the mappings $\pi_{M, M'}$, the system $(R^M)_{M \in \mathfrak{E}(V)}$ becomes a projective system of R -algebras, and

$$\pi = \varprojlim_{M \in \mathfrak{E}(V)} \pi_M: R^V \rightarrow \varprojlim_{M \in \mathfrak{E}(V)} R^M$$

is an R -algebra isomorphism. If $f \in R^V$, then $\pi(f) = (f|_M)_{M \in \mathfrak{E}(V)}$. For

every $M \in \mathfrak{E}(V)$, we give R^M the discrete topology, and we shift the topology of the projective limit to R^V by means of the isomorphism π . This topology on R^V (obviously different from the product topology) will be called the *limit topology*.

With the limit topology, R^V is a separated complete topological R -algebra. For $f \in R^V$ and $M \in \mathfrak{E}(V)$, we set

$$\mathcal{U}_M(f) = \{g \in R^V \mid g|M = f|M\}.$$

Then $\{\mathcal{U}_M(f) \mid M \in \mathfrak{E}(V)\}$ is a fundamental system of neighbourhoods of f , and therefore the limit topology on R^V coincides with the topology of $\mathfrak{E}(V)$ -convergence; see [4], Ch. X, § 1.

For the next result, let $\mathfrak{E}^+(V)$ be the set of all finitely generated F -submodules of V which are F -direct summands.

PROPOSITION 10. i) $\overline{\mathcal{P}_F(V, R)}$ is closed in R^V , and

$$\pi(\overline{\mathcal{P}_F(V, R)}) = \lim_{\leftarrow M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R) \subset \lim_{\leftarrow M \in \mathfrak{E}(V)} R^M.$$

ii) Let $M \in \mathfrak{E}(V)$ be given and suppose that either $M \in \mathfrak{E}^+(V)$ or R is an injective F -module. Then the restriction map

$$\rho: \begin{cases} \mathcal{P}_F(V, R) & \rightarrow & \mathcal{P}_F(M, R), \\ f & \mapsto & f|M, \end{cases}$$

is surjective.

iii) Suppose that either $\mathfrak{E}^+(V)$ is cofinal in $\mathfrak{E}(V)$ or R is an injective F -module. Then

$$\overline{\mathcal{P}_F(V, R)} = \overline{\mathcal{P}_F(V, R)} \subset R^V.$$

PROOF. i) A function $f \in R^V$ lies in $\overline{\mathcal{P}_F(V, R)}$ if and only if $f|M \in \mathcal{P}_F(M, R)$ for all $M \in \mathfrak{E}(V)$, i.e.,

$$\pi(f) = (f|M)_{M \in \mathfrak{E}(V)} \in \lim_{\leftarrow M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R).$$

This implies $\pi(\overline{\mathcal{P}_F(V, R)}) = \lim_{\leftarrow} \mathcal{P}_F(M, R)$. If $f \in \overline{\mathcal{P}_F(V, R)}$, then

$$\pi(f) \in \overline{\pi(\mathcal{P}_F(V, R))} = \lim_{\leftarrow M \in \mathfrak{E}(V)} \pi_M \overline{\mathcal{P}_F(V, R)} \subset \lim_{\leftarrow M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R),$$

and consequently $f \in \overline{\mathcal{P}_F(V, R)}$. Hence $\overline{\mathcal{P}_F(V, R)}$ is closed in R^V .

ii) It is sufficient to prove that every homogeneous F -polynomial function $q: M \rightarrow R$ of degree $d \geq 1$ can be extended to an F -polynomial function $\tilde{q}: V \rightarrow R$. Let $q^*: M^d \rightarrow R$ be F -multilinear such that $q(x) = q^*(x, \dots, x)$ for all $x \in M$. If either $M \in \mathfrak{E}^+(V)$ or R is F -injective, then there exists an F -multilinear mapping $\tilde{q}^*: V^d \rightarrow R$ such that $\tilde{q}^*|M^d = q^*$, and $\tilde{q}: V \rightarrow R$, defined by $\tilde{q}(x) = \tilde{q}^*(x, \dots, x)$, is an F -polynomial function extending q .

iii) If $M \in \mathfrak{E}^+(V)$ or R is F -injective, ii) implies

$$\mathcal{P}_F(M, R) = \pi_M \mathcal{P}_F(V, R) \subset \pi_M \overline{\mathcal{P}_F}(V, R) \subset \overline{\mathcal{P}_F}(M, R),$$

whence equality holds. This implies

$$\overline{\pi \mathcal{P}_F(V, R)} = \lim_{M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R) = \pi \overline{\mathcal{P}_F}(V, R),$$

and consequently $\overline{\mathcal{P}_F(V, R)} = \overline{\mathcal{P}_F}(V, R)$. ■

Next we investigate the connection between F -polynomial functions and large polynomials; we start with the case of polynomials in a finite number of indeterminates.

We say that F has no zero divisors on R if

$$t \in F, x \in R, tx = 0 \text{ implies } t = 0 \text{ or } x = 0;$$

notice that this condition implies that F itself is a domain.

A polynomial $f \in R[X_1, \dots, X_n]$ (in $n \in \mathbb{N}$ indeterminates) is called q -reduced (for some $q \in \mathbb{N}$), if $\deg_{X_j}(f) < q$ for all $j \in \{1, \dots, n\}$.

LEMMA 2. Suppose that F has no zero divisors on R , and let $f \in R[X_1, \dots, X_n]$ be a polynomial.

i) If F is infinite and $f(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in F^n$, then $f = 0$.

ii) If $\#F = q \in \mathbb{N}$, then there exists a unique q -reduced polynomial $f_0 \in R[X_1, \dots, X_n]$ such that $f(x_1, \dots, x_n) = f_0(x_1, \dots, x_n)$ for all $(x_1, \dots, x_n) \in F^n$.

PROOF. Exactly as in the classical case; cf.[14], Ch. V, § 4. ■

Now let again $X = (X_i)_{i \in I}$ be a family of indeterminates, and adopt all notations of section 1.

THEOREM 3. *For*

$$f = \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \dots \cdot X_{i_n} \in \overline{R[\langle X \rangle]}$$

we define $f^F: F^{(I)} \rightarrow R$ *by*

$$f^F((x_i)_{i \in I}) = \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \lambda_{i_1, \dots, i_n} x_{i_1} \cdot \dots \cdot x_{i_n}.$$

i) $f \in \overline{R[\langle X \rangle]}$ *implies* $f^F \in \overline{\mathcal{P}_F}(F^{(I)}, R)$, *and* $f \in R[\langle X \rangle]$ *implies* $f^F \in \mathcal{P}_F(F^{(I)}, R)$.

ii) *The mapping*

$$\phi^F: \begin{cases} \overline{R[\langle X \rangle]} & \rightarrow \overline{\mathcal{P}_F}(F^{(I)}, R), \\ f & \mapsto f^F, \end{cases}$$

is a homomorphism of R -*algebras satisfying*

$$\phi^F(R[\langle X \rangle]) = \mathcal{P}_F(F^{(I)}, R).$$

iii) *Suppose that* F *has no zero divisors on* R ; *then*

$$\phi^F(\overline{R[\langle X \rangle]}) = \overline{\mathcal{P}_F}(F^{(I)}, R).$$

If moreover F *is infinite, then* ϕ^F *is a topological isomorphism.*

PROOF. We set $V = F^{(I)}$, and we denote by $(e_i)_{i \in I}$ the canonical basis of V , i.e.,

$$\mathbf{x} = (x_i)_{i \in I} = \sum_{i \in I} x_i e_i \quad \text{for all} \quad \mathbf{x} \in V.$$

For $J \subset I$, we set

$$V_J = \bigoplus_{i \in J} R e_i \subset V.$$

If $J \in \mathcal{E}(I)$, then $V_J \in \mathfrak{E}^+(V)$, and the system $\{V_J \mid J \in \mathcal{E}(I)\}$ is cofinal in $\mathfrak{E}(V)$. Identifying V_J with $F^{(J)}$ we obtain, for any $f \in \overline{R[\langle X \rangle]}$,

$$f^F|_{V_J} = (f_J)^F.$$

i) We show first that $f \in R[\langle X \rangle]$ implies $f^F \in \mathcal{P}_F(V, R)$ and it suffices to do this for large forms $f \in R[\mathbf{X}]_n$, where $n \in \mathbb{N}$. Suppose that

$$f = \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \dots \cdot X_{i_n} \in R[\mathbf{X}]_n,$$

and let $p^*: V^n \rightarrow R$ be the unique F -multilinear mapping satisfying

$$p^*(e_{i_1}, \dots, e_{i_n}) = \begin{cases} \lambda_{i_1, \dots, i_n} & \text{if } (i_1, \dots, i_n) \in I_{\leq}^n, \\ 0 & \text{otherwise.} \end{cases}$$

If $p \in \mathcal{P}_F(V, R)$ is defined by $p(\mathbf{x}) = p^*(\mathbf{x}, \dots, \mathbf{x})$, then

$$\begin{aligned} p((x_i)_{i \in I}) &= p^*\left(\sum_{i \in I} x_i e_i, \dots, \sum_{i \in I} x_i e_i\right) = \\ &= \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} x_{i_1} \cdot \dots \cdot x_{i_n} \lambda_{i_1, \dots, i_n} = f^F((x_i)_{i \in I}), \end{aligned}$$

whence $f^F = p \in \mathcal{P}_F(V, R)$.

If $f \in \overline{R[\langle X \rangle]}$, then $f_J \in R[X_J]$ for all $J \in \mathcal{E}(I)$ and consequently $(f_J)^F = f^F|_{V_J} \in \mathcal{P}_F(V_J, R)$, which implies $f^F \in \overline{\mathcal{P}_F(V, R)}$.

ii) Clearly, ϕ^F is a homomorphism of R -algebras. In order to prove the equality $\phi^F(R[\langle X \rangle]) = \mathcal{P}_F(V, R)$, it is sufficient to show that every homogeneous F -polynomial function $p: V \rightarrow R$ of degree $n \geq 1$ is of the form $p = f^F$ for some $f \in R[\langle X \rangle]_n$.

Let $p: V \rightarrow R$ be a homogeneous F -polynomial function, and let $p^*: V^n \rightarrow R$ be F -multilinear such that $p(\mathbf{x}) = p^*(\mathbf{x}, \dots, \mathbf{x})$ for all $\mathbf{x} \in V$. For $(i_1, \dots, i_n) \in I^n$, we set

$$[i_1, \dots, i_n] = \{(i_{\sigma(1)}, \dots, i_{\sigma(n)}) \mid \sigma \in S_n\},$$

and we define $f \in R[\langle X \rangle]_n$ by

$$f = \sum_{(i_1, \dots, i_n) \in I_{\leq}^n} \left(\sum_{(j_1, \dots, j_n) \in [i_1, \dots, i_n]} p^*(e_{j_1}, \dots, e_{j_n}) \right) X_{i_1} \cdot \dots \cdot X_{i_n}.$$

Then we obtain

$$\begin{aligned} f^F((x_i)_{i \in I}) &= \sum_{(i_1, \dots, i_n) \in I^n} p^*(e_{i_1}, \dots, e_{i_n}) x_{i_1} \cdot \dots \cdot x_{i_n} = \\ &= p^*\left(\sum_{i=1}^n x_i e_i, \dots, \sum_{i=1}^n x_i e_i\right) = p((x_i)_{i \in I}), \end{aligned}$$

whence $p = f^F$.

iii) For a large polynomial $f \in R[\langle X \rangle]$, we have $f^F = 0$ if and only if $0 = f^F|_{V_J} = (f_J)^F$ for all $J \in \mathcal{E}(I)$. If F is infinite, this implies $f_J = 0$ for all $J \in \mathcal{E}(I)$ (by Lemma 2) and hence $f = 0$; therefore

$$\phi^F|_{R[\langle X \rangle]}: R[\langle X \rangle] \xrightarrow{\sim} \mathcal{P}_F(V, R)$$

is an isomorphism and

$$(\phi^F | R[X_J]: R[X_J] \xrightarrow{\sim} \mathcal{P}_F(V_J, R))_{J \in \mathcal{E}(I)}$$

is a family of isomorphisms, compatible with the mappings $\pi_{J, J'}$ of the projective systems on either side. Taking projective limits and observing the commutative diagram

$$\begin{array}{ccc} \overline{R[\langle X \rangle]} & \xrightarrow{\phi^F} & \overline{\mathcal{P}_F(V, R)} \\ \pi \downarrow & & \downarrow \pi \\ \varprojlim R[X_J] & \xrightarrow{\sim} & \varprojlim \mathcal{P}_F(V_J, R), \end{array}$$

it follows from Proposition 9, iii) that ϕ^F is an isomorphism if F is infinite.

Now consider the case $\#F = q \in \mathbb{N}$. If $g \in \mathcal{P}_F(V, R)$ and $J \in \mathcal{E}(I)$, then $g|V_J \in \mathcal{P}_F(V_J, R)$, and by ii), there exists a polynomial $f^{(J)} \in R[X_J]$ such that $(f^{(J)})^F = g|V_J$. By Lemma 2, there exists a unique q -reduced polynomial $f_0^{(J)} \in R[X_J]$ such that $(f_0^{(J)})^F = (f^{(J)})^F = g|V_J$. If $J, J' \in \mathcal{E}(I)$ and $J \supset J'$, then $\pi_{J, J'}(f_0^{(J)})$ is q -reduced, and $\pi_{J, J'}(f_0^{(J)})^F = (f_0^{(J)})^F|V_{J'} = g|V_{J'}$, whence $\pi_{J, J'}(f_0^{(J)}) = f_0^{(J')}$. This implies $(f_0^{(J)})_{J \in \mathcal{E}(I)} \in \varprojlim R[X_J]$, $f = \pi^{-1}((f_0^{(J)})_{J \in \mathcal{E}(I)}) \in R[\langle X \rangle]$ and $f^F|V_J = (f_J)^F = (f_0^{(J)})^F = g|V_J$, whence $f^F = g$. ■

4. Polynomial functions on groups.

In this section we study $(\mathbb{Z}-)$ polynomial functions and local $(\mathbb{Z}-)$ polynomial functions $q: G \rightarrow R$, where G is an abelian group and R is a commutative ring containing a prime field F .

Let G be an (additively written) abelian group, F a prime field (i.e. $F = \mathbb{Q}$ or $F = \mathbb{F}_p$ for some prime number p) and R a commutative F -algebra. We shall be concerned with the R -algebras $\mathcal{P}(G, R) = \mathcal{P}_{\mathbb{Z}}(G, R)$ and $\overline{\mathcal{P}}(G, R) = \overline{\mathcal{P}}_{\mathbb{Z}}(G, R)$; we always write $\otimes$ instead of $\otimes_{\mathbb{Z}}$.

$F \otimes G$ is a vector space over F , and $F \otimes G = \{\lambda \otimes g | \lambda \in F, g \in G\}$. Let $\omega: G \rightarrow F \otimes G$ be the group homomorphism defined by

$$\omega(g) = 1 \otimes g.$$

Let $\mathcal{L}^n(G, R)$ resp. $\mathcal{L}_F^n(F \otimes G, R)$ be the R -module of all multiadditive functions $G^n \rightarrow R$ resp. F -multilinear functions $(F \otimes G)^n \rightarrow R$. For $p^* \in$

$\in \mathcal{L}_F^n(F \otimes G, R)$ we define $\omega^n(p^*) \in \mathcal{L}^n(G, R)$ by

$$\omega^n(p^*)(g_1, \dots, g_n) = p^*(1 \otimes g_1, \dots, 1 \otimes g_n).$$

Then we obtain the following Lemma.

LEMMA 3. *The mapping $\omega^n: \mathcal{L}_F^n(F \otimes G, R) \rightarrow \mathcal{L}^n(G, R)$ is an isomorphism of R -modules.*

PROOF. ω^n is R -linear by definition. Now we consider the canonical isomorphism

$$\mathcal{L}^n(G, R) \xrightarrow{\sim} \text{Hom}(G \otimes \dots \otimes G, R),$$

$$\mathcal{L}_F^n(F \otimes G, R) \xrightarrow{\sim} \text{Hom}_F((F \otimes G) \bigotimes_F \dots \bigotimes_F (F \otimes G), R)$$

and

$$(F \otimes G) \bigotimes_F \dots \bigotimes_F (F \otimes G) \xrightarrow{\sim} F \otimes (G \otimes \dots \otimes G);$$

they induce a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_F^n(F \otimes G, R) & \xrightarrow{\sim} & \text{Hom}_F(F \otimes G^{\otimes n}, R) \\ \omega^n \downarrow & & \downarrow \omega^* \\ \mathcal{L}^n(G, R) & \xrightarrow{\sim} & \text{Hom}(G^{\otimes n}, R) \end{array}$$

where $\omega^*(\phi)(g_1 \otimes \dots \otimes g_n) = \phi(1 \otimes g_1 \otimes \dots \otimes g_n)$. By [11], Lemma 2, ω^* is an isomorphism (there R is assumed to be a field, but this is immaterial). Hence ω^* is also an isomorphism. ■

THEOREM 4. *The mapping*

$$\omega^*: \begin{cases} \overline{\mathcal{P}}_F(F \otimes G, R) & \rightarrow & \overline{\mathcal{P}}(G, R), \\ p & \mapsto & p \circ \omega, \end{cases}$$

is an isomorphism of K -algebras satisfying

$$\omega^*(\mathcal{P}_F(F \otimes G, R)) = \mathcal{P}(G, R).$$

PROOF. We prove first that $p \in \mathcal{P}_F(F \otimes G, R)$ implies $\omega^*(p) \in \mathcal{P}(G, R)$, and it is sufficient to do this in the case where $p: F \otimes G \rightarrow R$ is a homogeneous F -polynomial function of degree $n \geq 1$. In this case, let

$p^* \in \mathcal{L}_F^n(F \otimes G, R)$ be such that $p(z) = p^*(z, \dots, z)$ for all $z \in F \otimes G$. Then we obtain, for $g \in G$,

$$\omega^*(p)(g) = p(1 \otimes g) = p^*(1 \otimes g, \dots, 1 \otimes g) = (\omega^n p)(g, \dots, g),$$

which implies $\omega^*(p) \in \mathcal{P}(G, R)$. Now we set

$$\tilde{\omega} = \omega^* \mid \mathcal{P}_F(F \otimes G, R): \mathcal{P}_F(F \otimes G, R) \rightarrow \mathcal{P}(G, R),$$

and we prove that $\tilde{\omega}$ is an isomorphism of R -algebras.

In order to prove that $\tilde{\omega}$ is surjective it suffices to show that every homogeneous polynomial function $q: G \rightarrow R$ of degree $n \geq 1$ lies in the image of $\tilde{\omega}$. Let $q: G \rightarrow R$ be a homogeneous polynomial function of degree $n \geq 1$, and let $q^* \in \mathcal{L}^n(G, R)$ be such that $q(g) = q^*(g, \dots, g)$ for all $g \in G$. By Lemma 3, $q^* = \omega^n(p^*)$ for some $p^* \in \mathcal{L}_F^n(F \otimes G, R)$. If $p: F \otimes G \rightarrow R$ is defined by $p(z) = p^*(z, \dots, z)$ then

$$\omega^*(p)(g) = p^*(1 \otimes g, \dots, 1 \otimes g) = q^*(g, \dots, g) = q(g)$$

for all $g \in G$, whence $\omega^*(p) = q$.

In order to prove that $\tilde{\omega}$ is injective, let $p \in \mathcal{P}_F(F \otimes G, R)$ be in the kernel of $\tilde{\omega}$, i.e., $p(1 \otimes g) = 0$ for all $g \in G$.

CASE 1. $\text{char}(R) = p > 0$, $F = \mathbb{F}_p$. In this case, all elements of $F \otimes G$ are of the form $z = \bar{m} \otimes g = 1 \otimes mg$ for some $m \in \mathbb{Z}$, which implies $p = 0$.

CASE 2. $\text{char}(R) = 0$, $F = \mathbb{Q}$. We write p in the form $p = p_1 + \dots + p_d$, where $p_i: F \otimes G \rightarrow R$ is a homogeneous F -polynomial function of degree i . For $t \in \mathbb{Q}$ and $g \in G$, we obtain

$$p(t \otimes g) = \sum_{i=0}^d t^i p_i(1 \otimes g) \in R,$$

and if $t \in \mathbb{Z}$, then $p(t \otimes g) = p(1 \otimes tg) = 0$. Hence the polynomial

$$\sum_{i=0}^d p_i(1 \otimes g) T^i \in R[T]$$

vanishes on $\mathbb{Z}$ which, by Lemma 2, implies $p_i(1 \otimes g) = 0$ for all $i \in \{0, \dots, d\}$ and $g \in G$. Therefore we obtain $p(t \otimes g) = 0$ for all $t \in \mathbb{Q}$ and $g \in G$, i.e., $p = 0$.

Now we consider local polynomial functions. Let $\mathfrak{E}(G)$ be the set of all finitely generated subgroups of G and $\mathfrak{E}(F \otimes G)$ the set of all finitely generated F -submodules of $F \otimes G$. Obviously, the set

$$\mathfrak{E}_0(F \otimes G) = \{F \otimes C \mid C \in \mathfrak{E}(G)\}$$

is cofinal in $\mathfrak{C}(F \otimes G)$ and therefore a function $p: F \otimes G \rightarrow R$ lies in $\overline{\mathcal{P}}_F(F \otimes G, R)$ if and only if $p|F \otimes C \in \mathcal{P}_F(F \otimes G, R)$ for all $C \in \mathfrak{C}(G)$. If $p \in \overline{\mathcal{P}}_F(F \otimes G, R)$, then $(p \circ \omega)|C = (p|F \otimes C) \circ (\omega|C) \in \mathcal{P}(C, R)$ for all $C \in \mathfrak{C}(G)$ which implies $\omega^*(p) = p \circ \omega \in \overline{\mathcal{P}}(G, R)$. For $C \in \mathfrak{C}(G)$, we have established an isomorphism

$$\tilde{\omega}_C: \mathcal{P}_F(F \otimes C, R) \xrightarrow{\sim} \mathcal{P}(C, R)$$

satisfying $\tilde{\omega}_C(p) = p \circ \omega$; the family $(\tilde{\omega}_C)_{C \in \mathfrak{C}(G)}$ is compatible with the morphisms of the projective system, and therefore we get a commutative diagram.

$$\begin{array}{ccc} \overline{\mathcal{P}}_F(F \otimes G, R) & \xrightarrow{\omega^*} & \overline{\mathcal{P}}(G, R) \\ \pi \downarrow & & \downarrow \pi \\ \varprojlim_{C \in \mathfrak{C}(G)} \mathcal{P}_F(F \otimes C, R) & \xrightarrow[\varprojlim \tilde{\omega}_C]{\sim} & \varprojlim_{C \in \mathfrak{C}(G)} \mathcal{P}(C, R). \end{array}$$

The left vertical arrow is an isomorphism by Proposition 9. Hence the right vertical arrow is surjective, and since it clearly is injective it is also an isomorphism. Therefore ω^* is an isomorphism. ■

COROLLARY. *Let K be a field of characteristic zero. Then $\overline{\mathcal{P}}(G, K)$ is a factorial domain.*

PROOF. By Theorem 4, $\mathcal{P}(G, K) \simeq \mathcal{P}_F(F \otimes G, K)$, where $F \simeq \mathbb{Q}$ is the prime field of K . If $F \otimes G = \{0\}$ then $\mathcal{P}_F(F \otimes G, K) \simeq K$; thus we suppose that $F \otimes G \simeq F^{(I)}$ for some set $I \neq \emptyset$. Then we obtain $\mathcal{P}_F(F \otimes G, K) \simeq \mathcal{P}_F(F^{(I)}, K) \simeq K[\langle X \rangle]$ by Theorem 3, and the latter ring is a factorial domain by Theorem 1. ■

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