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Necessary and Sufficient Conditions for the Well Posedness of the Cauchy Problem for a Class of Hyperbolic Operators with High Variable Multiplicity.

V. SORDONI (*)

1. Introduction and main results.

Let X be an open set of $\mathbb{R}^{n+1} = \mathbb{R}_{x_0} \times \mathbb{R}^n_{x'}$, $x' = (x_1, \ldots, x_n)$, with $0 \in X$ and

$$P(x, D_x) = P_m(x, D_x) + P_{m-1}(x, D_x) + \ldots$$

be a differential operator of order m with C^∞ coefficients and let $P_{m-j}(x, D_x)$ denotes the homogeneous part of order $m-j$ of P, $j = 0, \ldots, m$.

Let us suppose that the principal symbol $p_m(x, \xi)$ is of the form

$$p_m(x, \xi) = q(x, \xi)^r$$

where

H$_1$) $q(x, \xi_0, \xi')$ is a real second order symbol, hyperbolic with respect to ξ_0.

In the following we will denote by $C = \{(x, \xi) \in T^*X \setminus 0 \mid q(x, \xi) = 0\}$ and by $\Sigma = \{(x, \xi) \in T^*X \setminus 0 \mid q(x, \xi) = dq(x, \xi) = 0\}$ the simple and the double characteristic set respectively. We will suppose that Σ is nowhere dense in C.

Our aim is to give necessary and sufficient conditions for the well posedness of the Cauchy problem for operators of the above

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type (for a definition of correctly posed Cauchy problem for P in $X_t = \{x \in X \mid x_0 < t\}$ we refer to [5]).

Let us observe that the standard Levi condition (cfr. [3] and [6]) implies that if the Cauchy problem for P is well posed in X_t, for t small, than $\partial^{\alpha}_{x,\xi} P_{m-j}(x, \xi) = 0$ $\forall (x, \xi) \in C$, $\forall \alpha$, $\forall j$ such that $|\alpha| + j < r$.

As a consequence, we can write $p_{m-j}(x, \xi) = a_j(x, \xi)\, q(x, \xi)^{r-j}$, $\forall j = 1, \dots, r$, for some $a_j \in S^j$. For this reason, denoting by $Q(x, D_x)$ a second order differential operator with principal symbol q, we can reduce ourselves to investigate necessary and sufficient conditions for the well posedness of the Cauchy problem for an operator P of the form

$$(1.1) \qquad P(x, D_x) = Q(x, D_x)^r + A_1(x, D_x)\, Q(x, D_x)^{r-1} + \dots + A_r(x, D_x)$$

where $A_j(x, D_x) \in L^j(X)$, $j = 1, \dots, r$.

If q_1^s denotes the subprincipal symbol of Q, F_ρ the Hamilton map corresponding to the Hessian of $q/2$ at a point $\rho \in \Sigma$ and $\mathrm{Tr}^+ F_\rho = \Sigma \mu$, with $\mu \geq 0$ and $i\mu \in \mathrm{sp}(F_\rho)$, the positive trace of q at ρ, we can associate to the operator (1.1), $\forall \rho \in \Sigma$, the polynomial in τ:

$$(1.2) \quad R_P(Q, \rho, \tau) = (\tau + q_1^s(\rho))^r + a_1(\rho)(\tau + q_1^s(\rho))^{r-1} + \dots + a_r(\rho).$$

Clearly, this polynomial is invariant by homogeneous canonical transformations.

The conditions for the well posedness of the Cauchy problem for $P(x, D_x)$ will be given in terms of the roots τ of polynomial $R_P(Q, \rho, \tau)$ and of the positive trace of q.

Our necessary result will be the following:

THEOREM 1.1. *Let $P(x, D_x)$ be a differential operators as in* (1.1). *Assume that the Cauchy problem for P is correctly posed in X_t, for small t, and that, for some $\rho = (0, \theta) \in T^*X \setminus 0$,*

$$q(\rho) = dq(\rho) = 0 \quad \textit{and} \quad \frac{\partial^2 q}{\partial \xi_0^2}(\rho) < 0.$$

If F_ρ has no non zero real eigenvalue then the roots $\lambda_j(\rho)$ of the polynomial $R_P(Q, \rho, \tau)$ are real and satisfy

$$(1.3) \qquad |\lambda_j(\rho)| \leq \mathrm{Tr}^+ F_\rho \qquad j = 1, \dots, r.$$

Our next result is concerned with sufficient conditions for P in order that the Cauchy problem will be well posed in X_0.

Let us introduce the following hypotheses:

H_2) the characteristic roots of $\xi_0 \to q(x, \xi_0, \xi')$ have multiplicity at most of order 2 and the double characteristic set of q, Σ, is a non empty C^∞ manifold such that the canonical 1-form $\omega = \sum_{j=0}^{n} \xi_j \, dx_j$ does not vanish identically on $T\Sigma$ and the canonical 2-form $\sigma = d\omega$ has constant rank on Σ.

If ρ is a point of Σ:

H_3) *a*) $\operatorname{Ker} F_\rho = T_\rho \Sigma$,

b) $\operatorname{Ker} F_\rho^2 \cap \operatorname{Im} F_\rho^2 = (0)$

c) $\operatorname{sp}(F_\rho) \subseteq i\mathbb{R}$,

d) $V^+ = \bigoplus_{\substack{i\mu \in \operatorname{sp}(F_\rho) \\ \mu > 0}} \operatorname{Ker}(F_\rho - i\mu I) \neq (0)$ and $\forall v \neq 0, v \in V^+, \frac{1}{i}\sigma(v, \bar{v}) > 0$.

Then, we can prove the following:

THEOREM 1.2. *Let $P(x, D_x)$ be a differential operators as in* (1.1) *satisfying* H_1), H_2) *and* $\mathrm{H}_3)_\rho$ $\forall \rho \in \Sigma$. *If, for each $\rho \in \Sigma$, the polynomial $R_P(Q, \rho, \tau)$ has r real simple roots $\lambda_j(\rho)$ such that*

$$|\lambda_j(\rho)| < \operatorname{Tr}^+ F_\rho \qquad j = 1, \ldots, r, \tag{1.4}$$

then the Cuachy problem for P is correctly posed in X_0.

Now few comments are in order:

(1) If $r = 1$ then $R_P(Q, \rho, \tau) = \tau + q_1^s(\rho)$ has the root $\lambda_1(\rho) = -q_1^s(\rho)$. Then the conditions in Theorem 1.1 (resp. Theorem 1.2) means that $\operatorname{Im} q_1^s(\rho) = 0$ and $|q_1^s(\rho)| \leqslant \operatorname{Tr}^+ F_\rho$ (resp. $\operatorname{Im} q_1^s(\rho) = 0$ and $|q_1^s(\rho)| < \operatorname{Tr}^+ F_\rho$) $\forall \rho \in \Sigma$. In this situation the above results are well-known (see [6], [5]).

(2) Results of above type have been announced by O. V. Zaitseva and V. Ia. Ivrii in [8] but, as far as we know, no proofs have yet appeared.

Moreover T. Okaji gives in [7] some necessary conditions for differential operator whose principal symbol is a product of second order operators with commutative Hamilton maps but unfortunately he has sufficient conditions only for very special type of operators.

2. Necessary conditions.

Since the proof of Theorem 1.1 can be obtained using exactly the same and of argument of Theorem 1.5.1 in Hormander [5], we will be very short and give only the main lines of the argument.

PROOF OF THEOREM 1.1. Let $\rho = (0, e_n) \in \Sigma$. Using the symplectic dilatations $y \to \rho^{-s} y$, $D \to \rho^s D$, $s = (s_0, s_1, \ldots, s_n)$, of Section 1.3 of [5], we get, with the same notation used there:

$$P_\rho = \rho^{-rs_n} P(\rho^{-s} y, \rho^s D) =$$

$$= \rho^{-rs_n}(Q(\rho^s y, \rho^s D)^r + A_1(\rho^{-s} y, \rho^s D) Q(\rho^{-s} y, \rho^s D)^{r-1} + \ldots + A_r(\rho^{-s} y, \rho^s D)) =$$

$$= (\rho^{-s_n} Q(\rho^{-s} y, \rho^s D))^r +$$

$$+ \rho^{-s_n} A_1(\rho^{-s} y, \rho^s D)(\rho^{-s_n} Q(\rho^{-s} y, \rho^s D))^{r-1} + \ldots + \rho^{-rs_n} A_r(\rho^{-s} y, \rho^s D).$$

With a suitable choice of s, we have:

$$P_\rho = (Q_\infty(D_n y, D) + q_1(0, e_n) D_n)^r +$$

$$+ A_1(0, e_n) D_n (Q_\infty(D_n y, D) + q_1(0, e_n) D_n)^{r-1} + \ldots + A_r(0, e_n) D_n^r + O(\rho^{-N}) =$$

$$= \prod_{j=1}^{r} (Q_\infty(D_n y, D) + (q_1(0, e_n) - q_1^s(0, e_n) - \lambda_j(0, e_n)) D_n) + O(\rho^{-N}) =$$

$$= \prod_{j=1}^{r} L^{(j)} + O(\rho^{-N})$$

where

$$L^{(j)} = L - \lambda_j(0, e_n) D_n = Q_\infty(D_n y, D) + (q_1(0, e_n) - q_1^s(0, e_n) - \lambda_j(0, e_n)) D_n.$$

Let us choose a matrix E as in [5] (pag. 141) and put $E_\rho = e^{-i\rho^2(y_n + \langle Ey, y\rangle/2)}$. Then

$$E_\rho^{-1} P_\rho E_\rho = \prod_{j=1}^{r} L_\rho^{(j)} + O(\rho^{-N})$$

where $L_\rho^{(j)} = \rho^2(2\langle My, D\rangle - 2\langle My, Ey\rangle D_n - i \operatorname{Tr} M - \lambda_j(0, e_n)) + L^{(j)}$ (here M is the matrix defined in [5], pag. 140).

Finally, with a function φ to be determined, let us consider the operator

$$\tilde{P}_\rho = e^{-i\rho\varphi} E_\rho^{-1} P_\rho E_\rho e^{i\rho\varphi} + O(\rho^{-N}).$$

If some $\lambda_j(0, e_n)$ is not real or $\mathrm{Tr}^+ F_\rho - \lambda_j(0, e_n) < 0$ for at least one j, arguing as in [5], it is possible to prove the existence of a phase function φ and of a formal power series v_ρ in y and $1/\rho$ such that $v_\infty(0) = 1$ and $\tilde{P}_\rho v_\rho = 0$ and this contradicts the assumption of the well posedness of the Cauchy problem.

As a partial result, we obtain that $\lambda_j(0, e_n) \in \mathbb{R}$ and $\lambda_j(0, e_n) \leqslant \leqslant \mathrm{Tr}^+ F_\rho$.

Now, we observe that

$$
\begin{aligned}
{}^tP &= ({}^tQ)^r + ({}^tQ)^{r-1}\, {}^tA_1 + \ldots + {}^tA_r = \\
&= ({}^tQ)^r + {}^tA_1 ({}^tQ)^{r-1} + \ldots + {}^tA_r + \sum_{j=0}^{r-2} S_{r-j} ({}^tQ)^j
\end{aligned}
$$

where $S_j \in L^j(X)$, $j = 2, \ldots, r$ have principal symbols vanishing at ρ.

Since ${}^tP$ is of type (1.1) the polynomial associated to ${}^tP$ will be:

$$
\begin{aligned}
&R_{{}^tP}({}^tQ, \rho, \tau) = \\
&\quad = (\tau + \sigma_1^s({}^tQ)(\rho))^r + \sigma_1(A_1)(\rho)(\tau + \sigma_1^s({}^tQ)(\rho))^{r-1} + \ldots + \sigma_r(A_r)(\rho) = \\
&\quad = (\tau - q_1^s(\rho))^r - a_1(\rho)(\tau - q_1^s(\rho))^{r-1} + \ldots + (-1)^r a_r(\rho) = \\
&\quad = (-1)^r [(-\tau + q_1^s(\rho))^r + a_1(\rho)(-\tau + q_1^s(\rho))^{r-1} + \ldots + a_r(\rho)] = \\
&\quad = (-1)^r R_P(Q, \rho, -\tau)
\end{aligned}
$$

with roots $\tilde{\lambda}_j = -\lambda_j$.

Applying the same argument as above to ${}^tP$, we conclude that $\lambda_j(0, e_n)$ must be real $\forall j = 1, \ldots, r$, and satisfy $|\lambda_j(0, e_n)| \leqslant \mathrm{Tr}^+ F_\rho$. ∎

3. Sufficent conditions.

We will prove Theorem 1.2 using the method of energy estimates (cfr. [5] and also [1], [2]). Such estimate will be obtained associating to the operator P a system of second order pdo's.

First of all, let us observe that we can always assume that the operator Q in (1.1) has subprincipal symbol identically zero on Σ.

Otherwise, if it is not, we rewrite Q as $Q = \tilde{Q} + B$ with $\tilde{q}_1^s(\rho) = 0$

and $\sigma_1(B) = q_1^s(\rho)$. Then

$$P(x, D_x) = \tilde{Q}(x, D_x)^r + \tilde{A}_1(x, D_x)\tilde{Q}(x, D_x)^{r-1} + \ldots + \tilde{A}_r(x, D_x)$$

with

$$\tilde{A}_{r-k} = \binom{r}{k} B^{r-k} + \sum_{j=k}^{r-1} \binom{j}{k} A_{r-j} B^{j-k} + T_{r-k}, \qquad k = 0, \ldots, r-1,$$

where $T_1 = 0$ and $T_j \in L^j$, $j = 2, \ldots, r$, have principal symbol vanishing on Σ.

Then

$$R_{\tilde{Q}}(P, \rho, \tau) = \tau^r + \tilde{a}_1(\rho)\tau^{r-1} + \ldots + \tilde{a}_1(\rho) =$$

$$= (\tau + q_1^s(\rho))^r + a_1(\rho)(\tau + q_1^s(\rho))^{r-1} + \ldots + a_r(\rho) = R_Q(P, \rho, \tau)$$

since

$$\sigma_{r-k}(\tilde{A}_{r-k}) = \binom{r}{k}(q_1^s(\rho))^r + \sum_{j=k}^{r-1}\binom{j}{k} a_{r-j}(\rho)(q_1^s(\rho))^{r-k}, \qquad \forall \rho \in \Sigma.$$

Consider now a point $\rho_0 \in \Sigma$. Without loss of generality, we can suppose that $\rho_0 = (y_0 = 0, y_0'; \eta_0 = 0, \eta_0')$ and that

$$\partial_{\xi_0}^j q(\rho_0) = 0, \qquad j = 0, 1 \qquad \text{and} \qquad \partial_{\xi_0}^2 q(\rho_0) \neq 0.$$

We will use the following

LEMMA 3.1. *There exist:*

i) *a neighborhood $I \times U \subset X$ of $(y_0 = 0, y_0')$ and a conic neighborhood $\Gamma \subset T^* U \setminus 0$ of $(y_0'; \eta_0')$;*

ii) *pseudodifferential operators*

$$E(y, D_y) \in L^0(I \times U),$$

$$B_j(y, D_{y'}) \in C^\infty(I, L^j(U)), \qquad j = 1, 2,$$

$$C_k^{(1)}(y, D_{y'}) \in C^\infty(I, L^{k-1}(U)), \; C_k^{(0)}(y, D_{y'}) \in C^\infty(I, L^k(U)), \; k = 1, \ldots, r$$

such that, if $\tilde{\Gamma} = \{(y, \eta) \mid (y_0, (y'; \eta')) \in I \times \Gamma, \eta_0 \in \mathbb{R}\}$, we have:

(*a*) *the principal symbol e_0 of E is real and does not vanish on $\tilde{\Gamma}$;*

(*b*) $Q'(y, D_y) = -D_0^2 + B_1(y, D_{y'})D_0 + B_2(y, D_{y'})$ *satisfies the same hypotheses of* Q;

(*c*) the principal symbols $c_k^{(0)}$ of $C_k^{(0)}(y, D_{y'})$ are given by

$$c_k^{(0)}(\rho_0) = e_0(\rho_0)^k a_k(\rho_0);$$

(*d*) $E(y, D_y)P(y, D_y) \sim Q'(y, D_y)^r +$

$$+(C_1^{(1)}(y, D_{y'})D_0 + C_1^{(0)}(y, D_{y'}))Q'(y, D_y)^{r-1} + \ldots +$$

$$+(C_r^{(1)}(y, D_{y'})D_0 + C_r^{(0)}(y, D_{y'})) \text{ on } \tilde{\Gamma}.$$

PROOF. We will prove the Lemma only in the case $r = 2$ (the proof of the general case is analogous).

By Malgrange Preparation Theorem there exist a neighborhood $I \times U \subset X$ of $(y_0 = 0, y_0')$, a conic neighborhood $\Gamma \subset T^*U \setminus 0$ of $(y_0'; \eta_0')$ and operators $\tilde{E}(y, D_y) \in L^0(I \times U)$, $B_j(y, D_{y'}) \in C^\infty(I, L^j(U))$, $j = 1,2$, such that

$$\tilde{E}(y, D_y)Q(y, D_y) \sim Q'(y, D_y) = -D_0^2 + B_1(y, D_{y'})D_0 + B_2(y, D_{y'}) \text{ on } \tilde{\Gamma}$$

where $\tilde{e}_0(y, \eta) \neq 0$ and Q' has principal symbol $q' = -\eta_0^2 + b_1(y, \eta')\eta_0 + b_2(y, \eta')$, $b_j(\rho_0) = 0$ for $j = 1, 2$.

Then

$$\tilde{E}^2(Q^2 + A_1Q + A_2) - Q'^2 \sim ([\tilde{E}, Q] + \tilde{E}A_1)Q' +$$

$$+([\tilde{E},[\tilde{E}, Q]]Q + \tilde{E}[\tilde{E}, A_1]Q + \tilde{E}^2A_2) \sim F_1Q' + F_2 = S_3$$

with F_j of order j on $\tilde{\Gamma}$.

By Mather Division Theorem

$$\sigma_1(F_1) = \beta_{-1,1}q' + g_{1,1} \quad \text{on } \tilde{\Gamma}$$

where $\beta_{-1,1}(y, \eta)$ is a symbol of order -1 and $g_{1,1} = c_{1,1}^{(1)}(y, \eta')\eta_0 + c_{1,1}^{(0)}(y, \eta')$ with $c_{1,1}^{(j)}$ symbol of order $1-j$, $j = 0, 1$.

Let us notice that $\sigma_1(F_1)(\rho_0) = \tilde{e}_0(\rho_0)a_1(\rho_0) = g_{1,1}(\rho_0) = c_{1,1}^{(0)}(\rho_0)$.

Let $\mathcal{B}_{-1,1}$ and $G_{1,1} = C_{1,1}^{(1)}D_0 + C_{1,1}^{(0)}$ be pdo's with principal symbol $\beta_{-1,1}$ and $g_{1,1}$ respectively such that $T_2 = S_3 - \mathcal{B}_{-1,1}Q'^2 - G_{1,1}Q'$ is a second order pdo on $\tilde{\Gamma}$. Then

$$\tilde{E}^2(Q^2 + A_1Q + A_2) - ((1 + \mathcal{B}_{-1,1})Q'^2 + G_{1,1}Q') = T_2$$

and

$$(1+\mathcal{B}_{-1,1})^{-1}\tilde{E}^2(Q^2+A_1Q+A_2)-(Q'^2+G_{1,1}Q')\sim$$

$$\sim(1+\mathcal{B}_{-1,1})^{-1}T_2+((1+\mathcal{B}_{-1,1})^{-1}-1)G_{1,1}Q'=S_2 \quad \text{on } \tilde{\Gamma},$$

where S_2 is a second order operator on $\tilde{\Gamma}$ since $((1+\mathcal{B}_{-1,1})^{-1}-1)$ is of order -1. Now, we write

$$\sigma_2(S_2)=\beta_{0,2}q'+g_{2,2} \qquad \text{on } \tilde{\Gamma}$$

where $\beta_{0,2}(y,\eta)$ is a symbol of order 0 and $g_{2,2}=c_{2,2}^{(1)}(y,\eta')\eta_0+c_{2,2}^{(0)}(y,\eta')$ with $c_{2,2}^{(j)}$ symbol of order $2-j$, $j=0,1$.

Likewise

$$\beta_{0,2}=\beta_{-2,2}q'+g_{0,2} \qquad \text{on } \tilde{\Gamma}$$

where $\beta_{-2,2}(y,\eta)$ is a symbol of order -2 and $g_{0,2}=c_{0,2}^{(1)}(y,\eta')\eta_0+c_{0,2}^{(0)}(y,\eta')$ with $c_{0,2}^{(j)}$ symbol of order $-j, j=0,1$.

Therefore

$$\sigma_2(S_2)=\beta_{-2,2}q'^2+g_{0,2}q'+g_{2,2} \qquad \text{on } \tilde{\Gamma}.$$

Note that $\sigma_2(S_2)(\rho_0)=\tilde{e}_0^2(\rho_0)a_2(\rho_0)=g_{2,2}(\rho_0)=c_{2,2}^{(0)}(\rho_0)$.

Let $\mathcal{B}_{-2,2}$ and $G_{0,2}=C_{0,2}^{(1)}D_0+C_{0,2}^{(0)}$ e $G_{2,2}=C_{2,2}^{(1)}D_0+C_{2,2}^{(0)}$ be operators with principal symbols $\beta_{-2,2}$, $g_{0,2}$ and $g_{2,2}$ respectively such that $T_1=S_2-\mathcal{B}_{-2,2}Q'^2-G_{0,2}Q'-G_{2,2}$ is a first order pdo $\tilde{\Gamma}$. Then, on $\tilde{\Gamma}$, we have

$$(1+\mathcal{B}_{-2,2})^{-1}(1+\mathcal{B}_{-1,1})^{-1}\tilde{E}^2(Q^2+A_1Q+A_2)-$$

$$-(Q'^2+(G_{1,1}+G_{0,2})Q'+G_{2,2})=$$

$$=(1+\mathcal{B}_{-2,2})^{-1}T_1+((1+\mathcal{B}_{-2,2})^{-1}-1)((G_{1,1}+G_{0,2})Q'+G_{2,2}))=S_1$$

where S_1 is a first order pdo on $\tilde{\Gamma}$ since $((1+\mathcal{B}_{-2,2})^{-1}-1)$ is of order -2 on $\tilde{\Gamma}$. Continuing in the same way, we finally obtain

$$\prod_{j=1}^{\infty}(1+\mathcal{B}_{-j,j})^{-1}\tilde{E}^2(Q^2+A_1Q+A_2)\sim Q'^2+G_1Q'+G_2$$

where $G_k=C_k^{(1)}D_0+C_k^{(0)}$, $k=1,2$, and $C_k^{(j)}(y,D_{y'})\in C^\infty(I,L^{k-j}(U))$, $j=1,2$. ■

By Lemma 3.1, disregarding the elliptic factor E and possibly after making a canonical transformation which preserves the planes

$x_0 = \text{const}$, we can suppose that $P(x, D_x)$ is of the form

$$(3.2)\qquad P(x, D_x) = Q^r + (C_1^{(1)}(x, D_{x'})D_0 + C_1^{(0)}(x, D_{x'}))Q^{r-1} + \\ + \ldots + (C_r^{(1)}(x, D_{x'})D_0 + C_r^{(0)}(x, D_{x'}))$$

where $Q = (-D_0^2 + A(x, D_{x'}))$ and A is a second order pseudodifferential operator in the x' variable depending smoothly on x_0 as a parameter such that $\sigma_2(A) = a_2 \geq 0$ and

$$\Sigma = \{(x, \xi) \in T^*X\setminus 0 \mid \xi_0 = 0,\ a_2(x, \xi') = da_2(x, \xi') = 0\}.$$

The polynomial associated to P will be, $\forall \rho \in \Sigma$

$$R_P(Q, \rho, \tau) = \tau^r + c_1^{(0)}(\rho)\tau^{r-1} + \ldots + c_r^{(0)}(\rho).$$

It is easy to verify that the roots of this polynomial satisfy the hypotheses of Theorem 1.2 at ρ_0.

Let $u \in C_0^\infty(K)$ with $K \subset\subset X$ and Λ_s be a selfadjoint operator with principal symbol $|\xi'|^s$. We put:

$$u_j = \Lambda_{r-j}(-D_0^2 + A(x, D_{x'}))^{j-1}u, \qquad j = 1, \ldots, r.$$

Then

$$(-D_0^2 + A(x, D_{x'}))u_j = \Lambda u_{j+1} + iT_j u_j, \qquad j = 1, \ldots, r-1$$

where $T_j = T_j(x, D_{x'})$ are selfadjoint first order operators with principal symbol vanishing on Σ and

$$(-D_0^2 + A(x, D_{x'}))u_r = Pu - \tilde{G}_1 u_r - \tilde{G}u_{r-1} - \ldots - \tilde{G}_r u_1$$

where $\tilde{G}_j = \tilde{C}_j^{(1)}D_0 + \tilde{C}_j^{(0)} = (C_j^{(1)}\Lambda_{-j+1})D_0 + (C_j^{(0)}\Lambda_{-j+1})$ and $\tilde{C}_j^{(1)}$, $\tilde{C}_j^{(0)}$ are pdo's in the x' variable, depending on x_0 as a parameter, of order 0 and 1 respectively. If $v = (u_1, u_2, \ldots, u_r)$ we have the $r \times r$ system:

$$(3.3)\qquad (-D_0^2 I - \mathcal{K}D_0 + \mathcal{A} - \mathcal{G})v = \mathcal{N}.$$

Here we have set

$$\mathcal{A} = \mathcal{A}' + i\mathcal{A}'' = \begin{bmatrix} A' + iA_1'' & 0 & \ldots & 0 & 0 \\ 0 & A' + iA_2'' & \ldots & 0 & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & \ldots & A' + iA_{r-1}'' & 0 \\ 0 & 0 & \ldots & 0 & A' + iA_r'' \end{bmatrix}$$

where A' (resp. A_j'') is second order (resp. first order) selfadjoint operator in the x' variable, depending on x_0 as a parameter such that $\sigma_2(A') = \sigma_2(A)$, $\sigma_1(A') = \mathrm{Re}\, q_1^s = 0$ on Σ, $\sigma_1(A_j'') = \mathrm{Im}\, q_1^s + \mathrm{Re}\, \sigma_1(T_j) = 0$ on Σ $(T_r \equiv 0)$ $\forall j = 1, \ldots, r$.

Moreover

$$\mathcal{G} = \begin{bmatrix} 0 & \Lambda & 0 & \ldots & 0 \\ 0 & 0 & \Lambda & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & \Lambda \\ -\tilde{C}_r^{(0)} & -\tilde{C}_{r-1}^{(0)} & -\tilde{C}_{r-2}^{(0)} & \ldots & -\tilde{C}_1^{(0)} \end{bmatrix}$$

with $\det(\tau I - \sigma_1(\mathcal{G})) = R_P(Q, \rho, \tau)$, while

$$\mathcal{H} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & 0 \\ -\tilde{C}_r^{(1)} & -\tilde{C}_{r-1}^{(1)} & -\tilde{C}_{r-2}^{(1)} & \ldots & -\tilde{C}_1^{(1)} \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ Pu \end{bmatrix}.$$

We can prove the following theorem containing the crucial energy estimates.

THEOREM 3.1. *Let $P(x, D_x)$ be as in* (3.2) *satisfying* $\mathrm{H}_1)$, $\mathrm{H}_2)$ $\mathrm{H}_3)_\rho$, $\forall \rho \in \Sigma$.

Let us assume that, $\forall \rho \in \Sigma$, the polynomial $R_P(Q, \rho, \tau)$ has r real simple roots $\lambda_j(\rho)$, $j = 1, \ldots, r$, such that:

$$\lambda_j(\rho) < \mathrm{Tr}^+ F_\rho \quad \forall j = 1, \ldots, r, \ \forall \rho \in \Sigma. \tag{3.4}$$

Let S_j, $j = 1, \ldots, r$, be first order operators which are differential in x_0 and pseudodifferential in x' with principal symbols vanishing on Σ.

Then, if $K \subset\subset X$, there exist a constant $C = C_K > 0$ and $\tau_K > 0$ such that $\forall u \in C_0^\infty(K)$ and $\forall \tau > \tau_K$ the following inequality holds:

$$C \int_{x_0 < 0} \|Pu(x_0, \cdot)\|_0^2 e^{-2\tau x_0} dx_0 \geqslant \tau^3 \sum_{j=1}^r \|(Q^{j-1}u)(0, \cdot)\|_{r-j}^2 + \tag{3.5}$$

$$+ \tau \sum_{j=1}^r \|(Q^{j-1}u)(0, \cdot)\|_{r-j+1/2}^2 + \tau^4 \sum_{j=1}^r \int_{x_0<0} \|(Q^{j-1}u)(x_0, \cdot)\|_{r-j}^2 e^{-2\tau x_0} dx_0 +$$

$$+\tau^2 \sum_{j=1}^{r} \int_{x_0<0} \|(Q^{j-1}u)(x_0, \cdot)\|^2_{r-j+1/2} e^{-2\tau x_0} dx_0 +$$

$$+\tau^2 \sum_{j=1}^{r} \int_{x_0<0} \|(S_j Q^{j-1}u)(x_0, \cdot)\|^2_{r-j} e^{-2\tau x_0} dx_0 +$$

PROOF. Define $\mathcal{P}v = (-D_0^2 I - \mathcal{H} D_0 + \mathcal{A}' + i\mathcal{A}'' - \mathcal{G})v$. Arguing as in [4], we can find first order symbols $X_j(x, \xi')$, $j = 1, \ldots, 2k+h$, vanishing on Σ and such that, in a conic neighborhood of a fixed point $\rho_0 \in S$, Σ is given locally as

$$\{\xi_0 = 0, X_j(x, \xi') = 0, j = 1, \ldots, 2k+h\}.$$

We can suppose that $\{X_{k+i}(x, \xi'), X_j(x, \xi')\} = \delta_{i,j}$ for $i, j = 1, \ldots, k$ and $\{X_{2k+i}(x, \xi'), X_s(x, \xi')\} = 0$ for $i = 1, \ldots, h$ and $s = 1, \ldots, 2k+h$.

By the geometric hypotheses H_2), H_3), using the arguments of [5] (§ 4.3), it is possible to choose a first order selfadjoint operator $B = B(x, D')$ such that, if $Y_j(x, \xi') = X_{k+j}(x, \xi') - iX_j(x, \xi')$, $j = 1, \ldots, k$ and $Y_{j+k}(x, \xi') = X_{2k+j}(x, \xi')$, $j = 1, \ldots, h$,

$$A' - B^2 = \sum_{j=1}^{k+h} Y_j^* Y_j + \tilde{F} \tag{3.6}$$

where $\tilde{F} = \tilde{F}(x, D_{x'})$ is a first order selfadjoint pdo with $\sigma_1(\tilde{F})|_\Sigma = \mathrm{Tr}^+ F$.

Moreover, if $M = -D_0 + B$, the principal symbols of $[Y_j, M]$, $[Y_j^*, M]$ and $[D_0, B]$ vanish on Σ.

Putting $\mathcal{Y}_j = Y_j I$, $\mathcal{F} = \tilde{F} I$, $\mathcal{M} = MI$ and $\mathcal{B} = BI$ we can rewrite $\mathcal{P}v$ as

$$\mathcal{P}v = D_0 \mathcal{M} v - D_0 \mathcal{B} v - \mathcal{H} D_0 v + \sum_{j=1}^{k+h} \mathcal{Y}_j^* \mathcal{Y}_j + (\mathcal{F} - \mathcal{G})v + i\mathcal{A}''v + \mathcal{B}^2 v. \tag{3.7}$$

Let $\sigma_1(\mathcal{F} - \mathcal{G}) = \alpha$ be the principal symbol of the matrix $\mathcal{F} - \mathcal{G}$. It is immediate to verify that for $\rho \in \Sigma$ the eigenvalues of $\alpha(\rho)$ are exactly $-\lambda_j(\rho) + \mathrm{Tr}^+ F_\rho$, $j = 1, \ldots, r$.

It follows that on a conic neighborhood ω of ρ_0 the matrix $\alpha(\rho)$ has smooth distinct eigenvalues $\tilde{\lambda}_j(\rho)$, $j = 1, \ldots, r$ (possibly complex for $\rho \in \omega \setminus \Sigma$) with $\mathrm{Re}\,\tilde{\lambda}_j > 0$ near ρ_0. Denote by $\pi_j(\rho)$ the projector on $\mathrm{Ker}(\tilde{\lambda}_j(\rho) I - \alpha(\rho))$, $j = 1, \ldots, r$, we can suppose that the $\pi_j' s$ are symbol homogeneous of degree 0 in ξ. Let Π_j the pdo's with principal symbol π_j and put $\mathcal{R} = \sum_{j=1}^{r} \Pi_j^* \Pi_j$. It is easy to verify that $\mathcal{R} = \mathcal{R}^*$, $\mathcal{R} \geq cI$ with $c > 0$, $\mathcal{R}(\mathcal{F} - \mathcal{G}) - (\mathcal{F} - \mathcal{G})^* \mathcal{R} = \mathcal{T}$ is a first order matrix with principal symbol vanishing on Σ (near ρ_0) and $\Pi_j(\mathcal{F} - \mathcal{G}) = \tilde{\Lambda}_j \Pi_j$,

$j = 1, \dots, r$, where the $\tilde{\Lambda}_j$'s are zero order pdo's with principal symbol $\tilde{\lambda}_j$.

Denoting by $\langle , \rangle$ the scalar product in $L^2(\mathbb{R}^n_{x'})$ we have:

(3.8) $$2i \operatorname{Im} \langle \mathcal{P}v, \mathcal{R}\mathcal{M}v \rangle =$$

$$= D_0 \left\{ \operatorname{Re} \langle \mathcal{R}\mathcal{M}v, \mathcal{M}v \rangle + \sum_{j=1}^{k+h} \operatorname{Re} \langle \mathcal{R}\mathcal{Y}_j v, \mathcal{Y}_j v \rangle + \operatorname{Re} \langle \mathcal{R}(\mathcal{F} - \mathcal{G}) v, v \rangle \right\} +$$

$$- i \operatorname{Im} \langle ([D_0 I + \mathcal{B}, \mathcal{R}] + 2\mathcal{R}\mathcal{H}) \mathcal{M}v, \mathcal{M}v \rangle +$$

$$- i \operatorname{Im} \left\langle \left(2\mathcal{R}[D_0 I, \mathcal{B}] + 2 \sum_{j=1}^{k+h} [\mathcal{Y}_j^*, \mathcal{R}] \mathcal{Y}_j - \mathcal{C} - 2i\mathcal{R}\mathcal{A}'' \right) v, \mathcal{M}v \right\rangle +$$

$$- 2i \sum_{j=1}^{k+h} \operatorname{Im} \langle \mathcal{R}\mathcal{Y}_j v, [\mathcal{M}, \mathcal{Y}_j] v \rangle - i \sum_{j=1}^{k+h} \operatorname{Im} \langle [\mathcal{R}, \mathcal{M}] \mathcal{Y}_j v, \mathcal{Y}_j v \rangle +$$

$$- i \operatorname{Im} \langle [\mathcal{R}(\mathcal{F} - \mathcal{G}), \mathcal{M}] v, v \rangle .$$

Moltiplying (3.8) by $ie^{-2\tau x_0}$ and integrating for $x_0 < 0$ we get, $\forall \varepsilon > 0$:

$$-2 \int_{x_0 < 0} \operatorname{Im} \langle \mathcal{P}v, \mathcal{R}\mathcal{M}v \rangle e^{-2\tau x_0} dx_0 \leq$$

$$\leq \frac{1}{\varepsilon\tau} \int_{x_0 < 0} \| \mathcal{P}v(x_0, \cdot) \|_0^2 e^{-2\tau x_0} dx_0 + C\varepsilon\tau \int_{x_0 < 0} \| \mathcal{M}v(x_0, \cdot) \|_0^2 e^{-2\tau x_0} dx_0 .$$

Estimating from above the terms in the right hand side of (3.8) except the first one, we obtain:

$$\int_{x_0 < 0} \operatorname{Im} \langle ([D_0 I + \mathcal{B}, \mathcal{R}] + 2\mathcal{R}\mathcal{H}) \mathcal{M}v(x_0, \cdot), \mathcal{M}v(x_0, \cdot) \rangle e^{-2\tau x_0} dx_0 \leq$$

$$\leq C_1 \int_{x_0 < 0} \| \mathcal{M}v(x_0, \cdot) \|_0^2 e^{-2\tau x_0} dx_0 ;$$

$$\int_{x_0 < 0} \operatorname{Im} \Big\langle (2\mathcal{R}[D_0 I, \mathcal{B}] + 2 \sum_{j=1}^{k+h} [\mathcal{Y}_j^*, \mathcal{R}] \mathcal{Y}_j +$$

$$- \mathcal{C} - 2i\mathcal{R}\mathcal{A}'') v(x_0, \cdot), \mathcal{M}v(x_0, \cdot) \Big\rangle e^{-2\tau x_0} dx_0 \leq$$

$$\leqslant C_2 \int_{x_0<0} \|\mathcal{M}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 + C_3 \int_{x_0<0} \|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 +$$

$$+ C_4 \int_{x_0<0} (\|[D_0 I, \mathcal{B}]\, v(x_0,\cdot)\|_0^2 + \|\mathcal{A}'' v(x_0,\cdot)\|_0^2 +$$

$$+ \|\mathcal{C}v(x_0,\cdot)\|_0^2 + \sum_{j=1}^{k+h} \|\mathcal{Y}_j v(x_0,\cdot)\|_0^2)\, e^{2\tau x_0}\,dx_0 ;$$

$$2 \sum_{j=1}^{k+h} \int_{x_0<0} \operatorname{Im} \langle \mathcal{R}\mathcal{Y}_j v(x_0,\cdot), [\mathcal{M}, \mathcal{Y}_j]\, v(x_0,\cdot)\rangle\, e^{-2\tau x_0}\,dx_0 \leqslant$$

$$\leqslant C_5 \sum_{j=1}^{k+h} \int_{x_0<0} (\|\mathcal{Y}_j v(x_0,\cdot)\|_0^2 + \|[\mathcal{M}, \mathcal{Y}_j]\, v(x_0,\cdot)\|_0^2)\, e^{-2\tau x_0}\,dx_0 ;$$

$$\sum_{j=1}^{k+h} \int_{x_0<0} \operatorname{Im} \langle [\mathcal{R}, \mathcal{M}]\, \mathcal{Y}_j v(x_0,\cdot), \mathcal{Y}_j v(x_0,\cdot)\rangle\, e^{-2\tau x_0}\,dx_0 \leqslant$$

$$\leqslant C_6 \sum_{j=1}^{k+h} \int_{x_0<0} \|\mathcal{Y}_j v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 ;$$

$$\int_{x_0<0} \operatorname{Im} \langle [\mathcal{R}(\mathcal{F}-\mathcal{G}), \mathcal{M}]\, v(x_0,\cdot), v(x_0,\cdot)\rangle\, e^{-2\tau x_0}\,dx_0 \leqslant C_7 \int_{x_0<0} \|v(x_0,\cdot)\|_{1/2}^2 e^{-2\tau x_0}\,dx_0 .$$

Now, integrating by parts and using Garding inequality, we obtain:

$$\int_{x_0<0} iD_0 \{\operatorname{Re} \langle \mathcal{R}\mathcal{M}v(x_0,\cdot), \mathcal{M}v(x_0,\cdot)\rangle +$$

$$+ \sum_{j=1}^{k+h} \operatorname{Re} \langle \mathcal{R}\mathcal{Y}_j v(x_0,\cdot), \mathcal{Y}_j v((x_0,\cdot)\rangle + \operatorname{Re} \langle \mathcal{R}(\mathcal{F}-\mathcal{G})\, v, v\rangle\}\, e^{-2\tau x_0}\,dx_0 =$$

$$= \operatorname{Re} \langle \mathcal{R}\mathcal{M}v(0,\cdot), \mathcal{M}v(0,\cdot)\rangle + \sum_{j=1}^{k+h} \operatorname{Re} \langle \mathcal{R}\mathcal{Y}_j v(0,\cdot), \mathcal{Y}_j v(0,\cdot)\rangle +$$

$$+ \sum_{i=1}^{r} \operatorname{Re} \langle \Pi_i (\mathcal{F}-\mathcal{G})\, v(0,\cdot), \Pi_i v(0,\cdot)\rangle +$$

$$+ 2\tau \int_{x_0<0} \operatorname{Re} \langle \mathcal{R}\mathcal{M}v(x_0,\cdot), \mathcal{M}v(x_0,\cdot)\rangle\, e^{-2\tau x_0}\,dx_0 +$$

$$+2\tau \sum_{j=1}^{k+h} \int_{x_0<0} \operatorname{Re}\langle \mathcal{R}\mathcal{Y}_j v(x_0,\cdot), \mathcal{Y}_j v(x_0,\cdot)\rangle e^{-2\tau x_0} dx_0 +$$

$$+2\tau \sum_{i=1}^{r} \int_{x_0<0} \operatorname{Re}\langle \Pi_i(\mathcal{F}-\mathcal{G}) v(x_0,\cdot), \Pi_i v(x_0,\cdot)\rangle e^{-2\tau x_0} dx_0 \geqslant$$

$$\geqslant c\|\mathcal{M}v(0,\cdot)\|_0^2 + c\sum_{j=1}^{k+h} \|\mathcal{Y}_j v(0,\cdot)\|_0^2 +$$

$$+\sum_{i=1}^{r} \operatorname{Re}\langle \tilde{\Lambda}_i \Pi_i v(0,\cdot), \Pi_i v(0,\cdot)\rangle - c'\|v(0,\cdot)\|_0^2 +$$

$$+c\tau \int_{x_0<0} \|\mathcal{M}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 + c\tau \sum_{j=1}^{k+h} \int_{x_0<0} \|\mathcal{Y}_j v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$+2\tau \sum_{i=1}^{r} \int_{x_0<0} \operatorname{Re}\langle \tilde{\Lambda}_i \Pi_i v(x_0,\cdot), \Pi_i v(x_0,\cdot)\rangle e^{-2\tau x_0} dx_0 +$$

$$-c'\tau \int_{x_0<0} \|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 \geqslant$$

$$\geqslant c\|\mathcal{M}v(0,\cdot)\|_0^2 + c\sum_{j=1}^{k+h} \|\mathcal{Y}_j v(0,\cdot)\|_0^2 + c''\|v(0,\cdot)\|_{1/2}^2 - c'''\|v(0,\cdot)\|_0^2 +$$

$$+c\tau \int_{x_0<0} \|\mathcal{M}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 + c\tau \sum_{j=1}^{k+h} \int_{x_0<0} \|\mathcal{Y}_j v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$+c''' \int_{x_0<0} \|v(x_0,\cdot)\|_{1/2}^2 e^{-2\tau x_0} dx_0 - c'''\tau \int_{x_0<0} \|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 .$$

Summing up, if ε is chosen small enough and τ suitably large, we have, with a new constant $C > 0$:

$$\text{(3.9)} \quad C \int_{x_0<0} \|\mathcal{P}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 \geqslant$$

$$\geqslant \tau\|\mathcal{M}v(0,\cdot)\|_0^2 + \tau \sum_{j=1}^{k+h} \|\mathcal{Y}_j v(0,\cdot)\|_0^2 + \tau\|v(0,\cdot)\|_{1/2}^2 - \tau\|v(0,\cdot)\|_0^2 +$$

$$+\tau^2 \int_{x_0<0} \|\mathcal{M}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 + \tau^2 \sum_{j=1}^{k+h} \int_{x_0<0} \|\mathcal{Y}_j v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$+\tau^2 \int_{x_0<0} \|v(x_0,\cdot)\|_{1/2}^2 e^{-2\tau x_0} dx_0 - \tau^2 \int_{x_0<0} \|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$+\tau \int_{x_0<0} (\|[D_0 I, \mathcal{B}]\, v(x_0,\cdot)\|_0^2 + \|\mathcal{A}'' v(x_0,\cdot)\|_0^2 +$$

$$+\|\mathcal{C} v(x_0,\cdot)\|_0^2 + \sum_{j=1}^{k+h} \|[\mathcal{M}, \mathcal{Y}_j]\, v(x_0,\cdot)\|_0^2)\, e^{-2\tau x_0} dx_0 .$$

Now, let

$$\mathcal{S} = \begin{bmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & S_r \end{bmatrix},$$

a diagonal matrix of first order operators which are differential in x_0 and pseudodifferential in x' having principal symbol vanishing on Σ.

We can write $\mathcal{S} = \mathcal{S}'\mathcal{M} + \mathcal{S}''$ where $\mathcal{S}'$ and $\mathcal{S}''$ are diagonal matrices of pdo's in the x' variable, depending on x_0 as a parameter, of order 0 and 1 respectively.

Summing and subtracting the term $\varepsilon\tau^2 \int_{x_0<0} \|\mathcal{S} v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0$ in the right hand side of (3.9), we have:

$$(3.10)\qquad C' \int_{x_0<0} \|\mathcal{P} v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 \geq$$

$$\geq \tau\|\mathcal{M} v(0,\cdot)\|_0^2 + \tau \sum_{j=1}^{k+h} \|\mathcal{Y}_j v(0,\cdot)\|_0^2 + \tau\|v(0,\cdot)\|_{1/2}^2 - \tau\|v(0,\cdot)\|_0^2 +$$

$$+\tau^2 \int_{x_0<0} \|\mathcal{M} v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 - \varepsilon\tau^2 \int_{x_0<0} \|\mathcal{S}'\mathcal{M} v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$+\tau^2 \sum_{j=1}^{k+h} \int_{x_0<0} \|\mathcal{Y}_j (x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 + \varepsilon\tau^2 \int_{x_0<0} \|\mathcal{S} v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$+\tau^2 \int_{x_0<0} \|v(x_0,\cdot)\|_{1/2}^2 e^{-2\tau x_0} dx_0 - \tau^2 \int_{x_0<0} \|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0} dx_0 +$$

$$-\tau \int_{x_0<0} \Big(\|[D_0 I, \mathcal{B}]\, v(x_0, \cdot)\|_0^2 + \|\mathcal{A}'' v(x_0, \cdot)\|_0^2 +$$

$$+ \sum_{j=1}^{k+h} \|[\mathcal{M}, \mathcal{Y}_j]\, v(x_0, \cdot)\|_0^2 \Big) e^{-2\tau x_0}\, dx_0 - \varepsilon\tau^2 \int_{x_0<0} \|S'' v(x_0, \cdot)\|_0^2 e^{-2\tau x_0}\, dx_0 .$$

Choosing ε small enough, the term $\varepsilon\tau^2 \int_{x_0<0} \|S' \mathcal{M} v(x_0, \cdot)\|_0^2 e^{-2\tau x_0}\, dx_0$ can be controlled by the term $\tau^2 \int_{x_0<0} \|\mathcal{M} v(x_0, \cdot)\|_0^2 e^{-2\tau x_0}\, dx_0$ on the right hand side of (3.10).

Since S'' has principal symbol vanishing on Σ, there exist diagonal $r \times r$ matrices $\alpha_j = \alpha_j(x, D')$, $\beta_j = \beta_j(x, D')$ and $\gamma = \gamma(x, D')$ of order 0 such that $S'' = \sum_{j=1}^{k+h} \alpha_j \mathcal{Y}_j + \mathcal{Y}_j^* \beta_j + \gamma$.

Then

$$\|S'' v(x_0, \cdot)\|_0^2 \leqslant$$

$$\leqslant C \Big(\sum_{j=1}^{k+h} \|\mathcal{Y}_j v(x_0, \cdot)\|_0^2 + \sum_{j=1}^{k+h} \mathrm{Re}\, \langle [\mathcal{Y}_j, \mathcal{Y}_j^*] \beta_j v(x_0, \cdot), \beta_j v(x_0, \cdot) \rangle +$$

$$+ \sum_{j=1}^{k+h} \|\mathcal{Y}_j \beta_j v(x_0, \cdot)\|_0^2 + \|v(x_0, \cdot)\|_0^2 \Big) \leqslant C' \Big(\sum_{j=1}^{k+h} \|\mathcal{Y}_j v(x_0, \cdot)\|_0^2 + \|v(x_0, \cdot)\|_{1/2}^2 \Big).$$

Again, choosing ε small enough, the term $-\varepsilon\tau^2 \int_{x_0<0} \|S'' v(x_0, \cdot)\|_0^2 e^{-2\tau x_0}\, dx_0$ can be controlled by the remaning terms on the right side of (3.10).

In the same way, if τ is large enough, the term

$$-\tau \int_{x_0<0} \big(\|[D_0 I, \mathcal{B}]\, v(x_0, \cdot)\|_0^2 + \|\mathcal{A}'' v(x_0, \cdot)\|_0^2 +$$

$$+ \|\mathcal{C} v(x_0, \cdot)\|_0^2 + \sum_{j=1}^{k+h} \|[\mathcal{M}, \mathcal{Y}_j]\, v(x_0, \cdot)\|_0^2 \big) e^{-2\tau x_0}\, dx_0 ,$$

can be controlled by the remaining terms on the right side of (3.10).

In conclusion we obtain:

$$(3.11)\qquad C'' \int_{x_0<0} \|\mathcal{P}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 \geqslant$$

$$\geqslant \tau\|\mathcal{M}v(0,\cdot)\|_0^2 + \tau\sum_{j=1}^{k+h}\|\mathcal{Y}_j v(0,\cdot)\|_0^2 + \tau\|v(0,\cdot)\|_{1/2}^2 - \tau\|v(0,\cdot)\|_0^2 +$$

$$+\tau^2\int_{x_0<0}\|\mathcal{M}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 + \tau^2\sum_{j=1}^{k+h}\int_{x_0<0}\|\mathcal{Y}_j v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 +$$

$$+\tau^2\int_{x_0<0}\|Sv(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 + \tau^2\int_{x_0<0}\|v(x_0,\cdot)\|_{1/2}^2 e^{-2\tau x_0}\,dx_0 +$$

$$-\tau^2\int_{x_0<0}\|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0.$$

On the other hand, the following estimates holds (see also [5]):

$$(3.12)\qquad C\int_{x_0<0}\|\mathcal{M}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 \geqslant$$

$$\geqslant \tau\|v(0,\cdot)\|_0^2 + \tau^2\int_{x_0<0}\|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0.$$

From (3.11) and (3.12) we obtain:

$$(3.13)\qquad \tilde{C}\int_{x_0<0}\|\mathcal{P}v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0 \geqslant$$

$$\geqslant \tau\|v(0,\cdot)\|_{1/2}^2 + \tau^3\|v(0,\cdot)\|_0^2 + r^2\int_{x_0<0}\|Sv(x_0,\cdot)\|_0^2\, e^{-2\tau x_0}\,dx_0 +$$

$$+\tau^2\int_{x_0<0}\|v(x_0,\cdot)\|_{1/2}^2 e^{-2\tau x_0}\,dx_0 + \tau^4\int_{x_0<0}\|v(x_0,\cdot)\|_0^2 e^{-2\tau x_0}\,dx_0.$$

Since $v = (u_1, \dots, u_r)$ with $u_j = \Lambda_{r-j}Q^{j-1}u$, $j = 1, \dots, r$, and $u \in C_0^\infty(K)$, from (3.13) we can easily deduce the energy inequality (3.6) for the operator P. ■

Now, we can sketch the proof of Theorem 1.2.

Let us point out that, since for a differential operator P,

$R_{{}^tP}({}^tQ, \rho, \tau) = (-1)^r R_P(Q, \rho, -\tau)$, if (1.4) holds we can prove for ${}^tP$ an estimates analogous to (3.5).

Having obtained the estimates near Σ, we observe that, in a neighborhood of a point $\rho_0 \in \mathrm{Char}(P) \setminus \Sigma$, P is an hyperbolic operator with characteristics of constant multiplicity that satisfies Levi condition which are known to be sufficient for the well posedness of the Cauchy problem (cfr. Theorem 2.10 in [3]).

Using these informations, the proof will be finished arguing as in [5], § 4. ∎

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