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HOWARD SMITH

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On Subnormal Series with Factors of Finite Rank: The Join Problem.

HOWARD SMITH (*)

ABSTRACT - Let G be a group which is generated by subnormal subgroups H and K and let A and B be subnormal subgroups of H and K respectively. Some sufficient conditions are given for the subnormality of $J = \langle A, B \rangle$ in G. These conditions involve the nature of the embeddings of A and B in H and K.

1. Introduction.

Suppose the group G is generated by subnormal subgroups H and K and that A, B are subnormal subgroups of H, K respectively. In general, the subgroup $J = \langle A, B \rangle$ need not be subnormal in G, as may be seen by taking $H = K = G$ to be any group without the subnormal join property (see, in particular, P. Hall's example in [3]). Indeed, J need not be subnormal even in the case where the indices $|H:A|$ and $|K:B|$ are both finite [5], but, by imposing certain restrictions on G, it is possible to obtain some (reasonable) sufficient conditions for the finiteness of $|G:J|$ and hence for the subnormality of J. For these results the reader is referred to papers [5], [8] and [9]. Finiteness of index is replaced in this discussion by a much weaker hypothesis, which will be given following the introduction of a little notation.

Let Y be a group and X a subnormal subgroup of Y. This relationship will be indicated by «$X \operatorname{sn} Y$». If, further, there is a subnormal series from X to Y each of whose factors has finite (Prüfer) rank then we shall write «$X \operatorname{sn}(\mathrm{fr}) Y$».

(*) Indirizzo dell'A.: Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837, U.S.A.

The hypotheses which will recur throughout this discussion are summarised as follows.

($*$) $G = \langle H, K\rangle$, H and K subnormal in G. $A\,\mathrm{sn}\,(\mathrm{fr})\,H$, $B\,\mathrm{sn}\,(\mathrm{fr})\,K$ and $J = \langle A, B\rangle$.

The two obvious questions are «When is $J\,\mathrm{sn}\,G$?» and «When is $J\,\mathrm{sn}\,(\mathrm{fr})\,G$?». The subnormality of J does not imply $J\,\mathrm{sn}\,(\mathrm{fr})\,G$, and to see this we need look no further than the wreath product of an infinite elementary abelian p-group by a cyclic group of order p (see [5]). What will be shown in this paper is that, loosely speaking, most of the theorems that one might expect to hold, (upon examination of the hypotheses that are usually associated with the good behaviour of joins of subnormals—see, for instance, [9]), do indeed hold. There are, however, a couple of gaps, and these are indicated by means of a conjecture or two.

2. Some preliminary results and a theorem on permuting subnormals.

The main result of this section is the following.

THEOREM A. Suppose hypotheses ($*$) are satisfied and that $G = HK$. Then $J\,\mathrm{sn}\,(\mathrm{fr})\,G$.

Before giving the proof of the theorem, we need a number of lemmas which will also be of use during the proofs of the later results.

LEMMA 1. Suppose W, X, Y, Z are groups such that $X\,\mathrm{sn}\,(\mathrm{fr})\,Y$ and $X \leqslant Z\,\mathrm{sn}\,W \leqslant Y$. Then $Z\,\mathrm{sn}\,(\mathrm{fr})\,W$.

PROOF. By intersecting W with the terms of a suitable series from X to Y we easily obtain $X\,\mathrm{sn}\,(\mathrm{fr})\,W$. Similarly, $X\,\mathrm{sn}\,(\mathrm{fr})\,Z^W$, and a simple induction (on the defect of Z in W) allows us to assume $Z \lhd W$. Now adjoin Z to each term of a series from X to W with factors of finite rank. If $X_{i+1} \leqslant X_i$ are successive terms of this series, then clearly ZX_i/ZX_{i+1} has finite rank. The result follows.

COROLLARY 2. If $X\,\mathrm{sn}\,(\mathrm{fr})\,Y$ then each factor of the normal closure series of X in Y has finite rank.

LEMMA 3. Let X, Z be permuting subgroups of the group Y such that $X \cap Z\,\mathrm{sn}\,(\mathrm{fr})\,Z$. If $X\,\mathrm{sn}\,XZ$ then $X\,\mathrm{sn}\,(\mathrm{fr})\,XZ$.

PROOF. Suppose $X = X_m \lhd X_{m-1} \lhd \ldots X_1 \lhd X_0 = XZ$. By Lemma 1 $X_i \cap Z \operatorname{sn}(\mathrm{fr}) Z$, for each i, and then $X_i \cap Z \operatorname{sn}(\mathrm{fr}) X_{i-1} \cap Z$ $(i \geqslant 1)$. But $X_i \cap Z \lhd X_{i-1} \cap Z$ and so X_{i-1}/X_i, which equals $X_i(X_{i-1} \cap Z)/X_i$ and is therefore isomorphic to $X_{i-1} \cap Z/X_i \cap Z$, has finite rank, as required.

The bulk of the proof of Theorem 2.2 of [7] was concerned with establishing that which constitutes our next result, except that the «(fr)» part of the conclusion is derived from the previous lemma.

LEMMA 4. Let G be a group and let D be a normal abelian subgroup of G such that $G = DJ$, where J is the join of two subnormal subgroups A and B of G. If D has finite rank, then $J \operatorname{sn}(\mathrm{fr}) G$.

The final two lemmas are special cases of Theorem A.

LEMMA 5. Suppose the hypotheses of Theorem A hold and that G is soluble. Then $J \operatorname{sn}(\mathrm{fr}) G$.

PROOF. (i) J is subnormal in G. If G is abelian this is trivial. Otherwise, by induction, we may assume $JD \operatorname{sn} G$, where D is abelian and normal in G. Suppose $H \lhd^m G$ and $K \lhd^n G$. Then $[D, {}_mH] \leqslant H \cap D$ and $[D, {}_nK] \leqslant K \cap D$ and so, by Proposition C of [2] (applied to the group ring $\mathbb{Z}G$ and its module D) there is an integer t such that $D^* = [D, {}_tG] \leqslant (H \cap D)(K \cap D)$. Then, modulo D^*, we have $D \leqslant Z_t(G)$ and then of course $JD^* \lhd^t JD$. It remains to show that $J \operatorname{sn} JD^*$. Since $H \cap D/A \cap D$ and $K \cap D/B \cap D$ have finite rank it follows that D^* has finite rank modulo $J \cap D^*$. Applying Lemma 4 to the group $JD^*/J \cap D^*$ we obtain the result.

(ii) $J \operatorname{sn}(\mathrm{fr}) G$. To establish this we use part (i) of the proof and proceed by induction on the defect m of H in G. If $m = 0$ then $A \operatorname{sn}(\mathrm{fr}) G$ and $J \operatorname{sn} G$ and Lemma 1 applies. Otherwise, writing $H_1 = H^G = H(H_1 \cap K)$, by the inductive hypothesis it follows that $J^* = \langle A, H_1 \cap B \rangle \operatorname{sn}(\mathrm{fr}) H_1$ and so, again using the subnormality of J and applying Lemma 1, $J \cap H_1 \operatorname{sn}(\mathrm{fr}) H_1$. Further, it is clear that $JH_1 = BH_1 \operatorname{sn}(\mathrm{fr}) KH_1 = G$. The result now follows from Lemma 3.

LEMMA 6. Suppose $G = HB$, where H is normal and B is subnormal in G, and suppose also that $A \operatorname{sn}(\mathrm{fr}) H$. Let $J = \langle A, B \rangle$. Then $J \operatorname{sn}(\mathrm{fr}) G$.

PROOF. We argue by induction on the defect n of A in H. If $n = 0$ then $J = G$ and there is nothing to prove. Assuming $n > 0$, there is a subgroup L of defect $n - 1$ in H such that $A \lhd (\mathrm{fr}) L \operatorname{sn}(\mathrm{fr}) H$ and, by induction, we may suppose $M = \langle L, B\rangle \operatorname{sn}(\mathrm{fr}) G$. The next part of the argument is a modified version of the proof of Lemma 2.1 of [7].

Let P be the permutizer of B in L. Then $A^B \lhd \langle A, B, P\rangle = J^*$, say, and, modulo A^B, P has finite rank and $J^* = BP$. It follows easily from Lemma 3 that $J \operatorname{sn}(\mathrm{fr}) J^*$. By [4], some term $L^{(\lambda)}$ of the derived series of L lies in P and, since $J \operatorname{sn}(\mathrm{fr})\langle AL^{(\lambda)}, B\rangle$, we may replace A by $AL^{(\lambda)}$ and thus assume L/A is soluble. Again from [4], we have $M^{(\mu)} \leqslant J$, for some μ. Noting that $A \operatorname{sn}(\mathrm{fr}) M \cap H$ and $M = (M \cap H) B$, we may apply Lemma 5 to the appropriate (soluble) image of M to obtain the result.

Proof of Theorem A. Assume the hypotheses are satisfied. The conclusion is established by induction on m, the defect of H in G. If $m = 0$ then $A \operatorname{sn}(\mathrm{fr}) G$ and Lemma 6 may be used. If m is greater than zero then, inductively, we may assume $A^* = \langle A, H_1 \cap B\rangle \operatorname{sn}(\mathrm{fr}) H_1$, where $H_1 = H^G = H(H_1 \cap K)$. Replacing A by A^* and H by H_1, we may suppose that H is normal in G. But then $HB \operatorname{sn}(\mathrm{fr}) KH = G$ and, by Lemma 6, $J \operatorname{sn}(\mathrm{fr}) HB$. This concludes the proof.

3. Commutator subgroups and tensor products.

We have seen that, with hypotheses ($*$), the join J is subnormal provided H and K permute. Here we turn our attention to other conditions which ensure the subnormality of J. These are as follows.

(1) $[H, K]$ has finite rank modulo every term of the lower central series of G.

(2) $H/H' \otimes K/K'$ (viewed as an additive group) is a direct sum of a group having finite rank and a periodic divisible group.

Hypotheses (1) and (2) are considered in papers [6] and [12] respectively with regard to the traditional join problem. It is not surprising that these hypotheses are once more decisive (see also [9]), but note the example following the proof of Theorem C below. Another property considered by J. P. Williams in [11] and which does not appear above is the finiteness of rank of $G'/\gamma_3(G)$. This condition implies (1) and is more significant for us here when considering joins of several subgroups (Theorem E).

For the proofs of Theorems B and D below we require two further preliminary results. These are incorporated into the statement of Proposition 8, which appears later (together with its proof) for the sake of convenience.

THEOREM B. Suppose hypotheses ($*$) are satisfied and that A and B are normal in H and K respectively. If either of conditions (1) and (2) hold then J is subnormal in G.

PROOF. By considering the permutizer of B in H and arguing as in the proof of Lemma 6, we can (with either hypothesis) reduce to the case where H/A is soluble. A similar argument then allows us to assume K/B is soluble and, using results from [4], we may suppose G is soluble. Induction on the derived length gives $JD \operatorname{sn} G$, where D is abelian and normal in G. We may now argue as in the proof of Lemma 5, but appealing at the appropriate stage to Proposition 8 in place of [2, Proposition C], to obtain the result.

THEOREM C. With hypotheses ($*$) and (1) satisfied, if $J \operatorname{sn} G$ then $J \operatorname{sn}(\mathrm{fr}) G$.

There is of course the following consequence of Theorems B and C.

COROLLARY 7. If hypotheses ($*$) and (1) hold and if A and B are normal in H and K then $J \operatorname{sn}(\mathrm{fr}) G$.

PROOF OF THEOREM C. Let P be the permutizer of K in H. Then $A \cap P \operatorname{sn}(\mathrm{fr}) P$ and $B \operatorname{sn}(\mathrm{fr}) K$ and thus, by Theorem A, $\langle A \cap P, B\rangle \operatorname{sn}(\mathrm{fr}) PK$. Now $\langle A \cap P, B\rangle$ is contained in $J \cap PK$, which is subnormal in PK by hypothesis. By Lemma 1, therefore, $J \cap PK \operatorname{sn}(\mathrm{fr}) PK$. Further, from the Proposition in [6], we can deduce that there is an integer a such that $\gamma_a(H) \leqslant P$ and then an integer c such that $\gamma_c(G) \leqslant PK$. It follows that $J \cap \gamma_c(G) \operatorname{sn}(\mathrm{fr}) \gamma_c(G)$. Since J is subnormal in $J\gamma_c(G)$, Lemma 3 gives $J \operatorname{sn}(\mathrm{fr}) J\gamma_c(G)$ and it only remains to show that $J\gamma_c(G) \operatorname{sn}(\mathrm{fr}) G$. Factoring, we may suppose G to be nilpotent. This gives $[H, K]$ of finite rank and hence $J \operatorname{sn}(\mathrm{fr}) J[H, K]$. Again factoring, we assume that $[H, K] = 1$. Thus $G = HK$ and Theorem A applies. This concludes the proof.

If hypothesis (1) is replaced by (2) in the above theorem, then the resulting statement is false, even with the additional hypotheses that A and B are normal. This is perhaps not surprising, as the divisible (or radicable) component of the tensor product is of course allowed to have infinite rank. The following easy example shows just what can go wrong.

EXAMPLE. Let K be a direct product of infinitely many copies of a Prüfer group C_{p^∞} (for a fixed prime p) and let H be an infinite cyclic group. Define G to be second nilpotent product of H and K. It is wellknown that $[H, K]$, which is central, is isomorphic to $H \otimes K$, which clearly has infinite rank. Setting $A = 1$, $B = K$ and noting that $K \cap [H, K]$ is trivial, it is certainly not the case that $\langle A, B\rangle \operatorname{sn}(\mathrm{fr}) G$, although H has finite rank and $\langle A, B\rangle = K \lhd^2 G$.

Returning now to hypotheses ($*$) without the extra condition of normality, we note that the permutizer argument referred to in the proof of Theorem B does not go through. It seems reasonable to hope that this argument can be suitably amended, given the hypotheses involving factors of finite rank. This optimism is the basis of the conjecture which appears following the next theorem.

THEOREM D. Suppose that hypoyteses ($*$) and either (1) or (2) hold. If G is soluble then $J \operatorname{sn} G$.

PROOF. By induction on the derived length of G we may suppose that $G = JD$, where D is abelian and normal in G. Imitating the proof of Lemma 5 once more, but relying on Proposition 8 as in the proof of Theorem B, we arrive at the desired conclusion.

CONJECTURE 1. Theorem D remains valid without the hypothesis of solubility.

The final result is one concerning the join of several subgroups. The basic hypotheses this time are as follows.

($*$ $*$) G = $\langle H_1, \dots, H_n\rangle$, where each H_i is subnormal in G. For each i, $A_i \operatorname{sn}(\mathrm{fr}) H_i$. $J = \langle A_1, \dots, A_n\rangle$.

THEOREM E. Suppose hypotheses ($*$ $*$) hold and assume further that $G'/\gamma_3(G)$ has finite rank.

(i) If G is soluble then $J \operatorname{sn} G$.

(ii) If $J \operatorname{sn} G$ then $J \operatorname{sn}(\mathrm{fr}) G$.

PROOF. (i) By induction, $JD \operatorname{sn} G$ for some abelian normal subgroup D. Write $D_1 = \langle D \cap H_i : i = 1, \dots, n\rangle$, $D_2 = \langle D \cap A_i : i = 1, \dots n\rangle$. Then D_1/D_2 has finite rank and, by Proposition 8, there is an integer t such that $D^* = [D, {}_tG] \leqslant D_1$. So $JD^* \operatorname{sn} G$ and $D^*/D^* \cap J$ has finite rank. Noting that Lemma 4 extends easily to the case of several subgroups, we obtain the result.

(ii) By considering a sequence of permutizers and proceeding as in the proof of Theorem 4(a) of [9] (an idea taken from [4]) we can reduce to the case where G is soluble—the details are omitted, except for the remark that the prerequisites are Theorem A and Lemmas 1 and 3 above. (Recall that J is subnormal by hypothesis). The proof proceeds by induction on the nilpotent length of G. If G is nilpotent then G' has finite rank and $J \operatorname{sn}(\mathrm{fr}) JG' \lhd (\mathrm{fr}) G$. Otherwise, arguing as in step (ii) of the proof of Theorem 4 of [9] (and noting that step (i) was unnecessary!), and appealing again to Theorem A and Lemmas 1 and 3 we reach our conclusion.

CONJECTURE 2. The hypothesis of solubility is not required in Theorem E.

For the proof of Lemma 5, we were able to quote Proposition C of [2], a result about augmentation ideals of group rings in the situation where two subnormal subgroups permute. Somewhat similar results were required for the proofs of Theorems B, D and E, but the relevant theorems from [6], [11] and [12] do not quite suffice (especially in the case of [6]). Exactly what is needed to complete the proofs is the following.

PROPOSITION 8. Let D be a normal abelian subgroup of the group G.

(i) Suppose G is generated by subnormal subgroups H and K and that either (1) or (2) holds. Then there is an integer t such that $[D, {}_tG] \leqslant (D \cap H)(D \cap K)$.

(ii) Suppose G is generated by subnormal subgroups $H_1, \dots, H_l$ and that $G'/\gamma_3(G)$ has finite rank. Then there is an integer t such that $[D, {}_tG] \leqslant \langle D \cap H_i : i = 1, \dots, l\rangle$.

PROOF. (i) Let G and D be as stated, where $H \lhd^m G$, $K \lhd^n G$, and suppose (1) or (2) holds. Then $H \cap D \lhd HD$ and, using bars to denote factor groups modulo $H \cap D$, we have $[\overline{D}, {}_m\overline{H}] \leqslant \overline{D} \cap \overline{H} = 1$. Letting C denote the centralizer of $\overline{D}$ in $\overline{H}$, it follows that $\overline{H}/C$ embeds in Aut $\overline{D}$ and acts nilpotently on $\overline{D}$. So $\overline{H}/C$ is nilpotent [1]. Consequently $[D, \gamma_a(H)] \leqslant H \cap D$, for some integer a. Similarly $[D, \gamma_b(K)] \leqslant K \cap D$, for some b. Now, by the Proposition in [6] or by Theorem B of [12], there is an integer c such that $\gamma_c(G) \leqslant \gamma_a(H)\gamma_b(K)$. Then $[D, \gamma_c(G)] \leqslant (H \cap D)(K \cap D)$ and, factoring, we may assume $\gamma_c(G)$ centralises D. The action of G on D is now the same as that of the nilpotent group $G/\gamma_c(G) = \Gamma$, say. Viewing D as a $\mathbb{Z}\Gamma$-module in the natural way we may, in the case where hypothesis (2) holds, apply Theorem C (i) of [12]. In group-theoretic terms, this gives an integer t such that $[D, {}_tG] \leqslant [D, {}_mH][D, {}_nK] \leqslant (D \cap H)(D \cap K)$, as required. For hypothesis (1) a little more remains to be done—this is accomplished by Lemma 9 below.

(ii) A straightforward adaptation of the above argument, using Theorems C (i) and B of [11], gives the result here. The details are omitted.

The following result on augmentation ideals completes the proof of Proposition 8. The notation is standard (see [2]).

LEMMA 9. Suppose that X is a nilpotent group, generated by subgroups Y and Z, and that $[Y, Z]$ has finite rank r. Then, given positive integers a and b, there is an integer t such that $\mathfrak{X}^t \leq \mathbb{Z}X(\mathfrak{Y}^a) + \mathbb{Z}X(\mathfrak{Z}^b)$.

PROOF. We may (and shall) suppose that X is finitely generated, provided it is shown that the integer t can be bounded in terms of a, b, r and c, the nilpotency class of X. By Lemma 2.1 of [10], $[Y, Y]$ can be generated by (at most) $2r$ elements of the form $[y, z]$, where $y \in Y$, $z \in Z$. Writing W for the subgroup $\langle Y_0, Z_0 \rangle$ generated by the $4r$ elements of $Y \cup Z$ appearing in these commutators, we can apply Theorem B of [2] to deduce that (with the obvious notation) there is an integer t_1 such that $\mathfrak{W}^{t_1} \leq \mathbb{Z}W(\mathfrak{Y}_0^a) + \mathbb{Z}W(\mathfrak{Z}_0^b) \leq \mathbb{Z}X(\mathfrak{Y}^a) + \mathbb{Z}X(\mathfrak{Z}^b)$ and such that, as indicated in [2], t_1 can be suitably bounded. Certainly $[Y, Z]$ is contained in W. Further, $Y^Z = Y[Y, Z]$ and $X = Y^Z Z$. Two succesive applications of Proposition C of [2] now give the result. (Note that the subnormal indices of Y and Z in X are of course at most c and that the proof of Proposition C of [2] is easily adapted to include the relevant bounds).

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