JAN TRLIFAJ

On $*$-modules generating the injectives

Rendiconti del Seminario Matematico della Università di Padova, tome 88 (1992), p. 211-220

<http://www.numdam.org/item?id=RSMUP_1992__88__211_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 88 (1992)

On $*$-Modules Generating the Injectives.

JAN TRLIFAJ (*)

ABSTRACT - Relations between $*$-modules, quasi-progenerators and other generalizations of progenerators are studied. The $*$-modules generating all injective modules are shown to be finitely generated.

Introduction.

One of the examples showing that category theory is not only a language, but also a useful tool in algebra is the celebrated Morita theorem concerning equivalence of rings. For any ring R, it implies e.g. the important fact that R and the full matrix ring $M_n(R)$ share all the ring theoretic properties which are definable by means of categorical properties of modules.

More specifically, if R and S are rings, the Morita theorem ([AF, Corollary 22.4]) says that mod-R and mod-S are equivalent categories iff there exists a *progenerator* ($=$ a finitely generated projective generator) P such that $S \simeq \mathrm{End}(P_R)$.

In [F], Fuller generalized the theorem as follows: mod-S is equivalent to a full subcategory C of mod-R such that C is closed with respect to submodules, direct sums and quotients iff there exists a *quasi-progenerator* ($=$ a finitely generated quasi-projective module generating all its submodules) P such that $S \simeq \mathrm{End}(P_R)$ and $C = \mathrm{Gen}(P_R)$.

In both cases, the pair (F, G) of functors realizing the equivalence is *represented* by P, i.e. F and G are naturally equivalent to $- \otimes_S P$ and $\mathrm{Hom}_R(P, -)$, respectively.

(*) Indirizzo dell'A.: Department of Algebra, Fac. Math. Phys., Charles University, Prague, Czechoslovakia.

This research was done in part during the author's stay at the University of Padua. The author would like to thank Professor Luigi Salce for his hospitality and support.

In [MO, Theorem 3.1], Menini and Orsatti presented a further generalization: if B and C are equivalent categories, where $B \subseteq$ mod-S is such that $S \in B$ and B is closed with respect to submodules, and $C \subseteq$ mod-R is closed with respect to direct sums and factors, then there is a module P such that $S \simeq \mathrm{End}(P_R)$, $C = \mathrm{Gen}(P_R)$ and $B = \mathrm{Cog}(K_S)$, where $K = \mathrm{Hom}_R(P, Q)$ and Q is an injective cogenerator of mod-R. Moreover, the equivalence is represented by P.

In [MO, 3.5], a question was raised of characterizing the modules P that induce an equivalence between $\mathrm{Gen}(P_R)$ and $\mathrm{Cog}((\mathrm{Hom}_R(P, Q))_S)$ with $S \simeq \mathrm{End}(P_R)$. Since the question was denoted by ($*$), such modules are called $*$*-modules* ([C], [DH]).

For a ring R, denote by PG, QPG and $STAR$ the class of all progenerators, quasi-progenerators and $*$-modules, respectively. Clearly, $PG \subseteq QPG \subseteq STAR$. Surprisingly, there is another important class of $*$-modules, not connected with quasi-progenerators: a module P is a *W-tilting* module if P is finitely presented, proj dim $(P) \leqslant 1$, $\mathrm{Ext}_R(P, P) = 0$, and there is an exact sequence $0 \to R \to P' \to P'' \to 0$ such that P' and P'' are direct sums of summands of P. For example, the W-tilting modules over finite dimensional algebras coincide with the tilting modules, introduced in [HR]. Denote by $WTILT$ the class of all W-tilting modules and by $ISTAR$ the class of all $*$-modules such that $I(R) \in \mathrm{Gen}(P_R)$, $I(R)$ being the injective hull of R. The surprise is that $WTILT \subseteq ISTAR$ ([MO, Theorem 4.3]).

The class $ISTAR$ was studied in more detail by Colpi and Menini in [C] and [CM]. By [CM, Proposition 1.5], if $P \in ISTAR$ then $\mathrm{Gen}(P_R) = \{M \mid \mathrm{Ext}_R(P, M) = 0\}$. Moreover, [CM, Proposition 1.5] implies that $ISTAR$ coincides with the class of all $*$-modules P such that $\mathrm{Gen}(P_R) \supseteq \mathfrak{J}$, where $\mathfrak{J}$ is the class of all injective modules. In [CM, Theorem 3.3], a complete characterization of the rings R for which there is a $*$-module P with $\mathrm{Gen}(P_R) = \mathfrak{J}$ was given.

The main result of our paper is Theorem 1.3 showing that $ISTAR$ is very close to the class $WTILT$. In particular, all elements of $ISTAR$ are finitely presented. Thus, for this case, we confirm the conjecture of Colpi and Menini (cp. [CM, Propositions 1.8 and 1.9]), D'Este and Happel ([DH, Remark 4]), and Zanardo ([Z, Remark 4]), which claims that every $*$-module is finitely generated. In Propositions 1.6 and 1.7, the structure if $ISTAR$ over semiperfect rings is described in greater detail.

Then we turn to applications to particular classes of rings. We show that $ISTAR = PG$ provided R is either a commutative or a local or a von Neumann regular ring (Theorem 1.9). Finally, Theorem 1.10 shows that properties concerning $P^{\perp}$ which are slightly weaker than the ones induced by $*$-modules, can be independent of ZFC.

In the following, all rings are associative with unit. Let R be a ring. The category of (unitary right R-) modules is denoted by mod-R. Homomorphisms in mod-R are written as acting on the left. If $M \in$ mod-R, then $\mathrm{Rad}(M)$ denotes the Jacobson radical of M. Further, R is *completely reducible* provided R is a finite ring direct sum of full matrix rings over skew fields. R is *semiperfect* provided $R/\mathrm{Rad}(R)$ is completely reducible and idempotents lift modulo $\mathrm{Rad}(R)$.

Let M be a module. Then $\mathrm{gen}(M)$ denotes the minimal cardinality of an R-generating subset of M and $I(M)$ the injective hull of M. The category of all modules generated by P is denoted by $\mathrm{Gen}(P_R)$, and $\overline{\mathrm{Gen}}(P_R)$ is the category consisting of all submodules of elements of $\mathrm{Gen}(P_R)$. M is said to be *small* provided for every sequence of modules $(N_\alpha \mid \alpha \in A)$ and every homomorphism $h \in \mathrm{Hom}_R(M, \oplus \sum_{\alpha \in A} N_\alpha)$ there is a finite set $F \subseteq A$ such that $\mathrm{Im}(h) \subseteq \oplus \sum_{\alpha \in F} N_\alpha$. The module M is *finitely presented* provided there is an exact sequence $0 \to G \to F \to M \to 0$ in mod-R such that F is projective, and F and G are finitely generated. Further, $\mathrm{proj\,dim}(M)$ denotes the projective dimension of M, and $M^\perp$ the cotorsion class generated by M, i.e. $M^\perp = \{N \in \text{mod-}R \mid \mathrm{Ext}_R(M, N) = 0\}$ (see [S] or [T, § 1]). For further concepts and notation, the reader is referred to [AF] and [EM].

1. The structure of *ISTAR*.

LEMMA 1.1. *Let R be a ring and P a small module. Then either P is finitely generated or* $\mathrm{gen}(P) \geq \aleph_1$.

PROOF. An easy modification of the proof of [CM, Proposition 1.9].

LEMMA 1.2. *Let R be a ring and P a module. Then the following conditions are equivalent:*

(1) *P is small and $P^\perp$ is closed with respect to direct sums and factors,*

(2) *P is finitely presented and* $\mathrm{proj\,dim}(P) \leq 1$.

PROOF. Assume (1). Clearly, $P = R^{(\kappa)}/Q$ for a cardinal κ and a submodule Q of $R^{(\kappa)}$. First we observe that Q is projective. Take an arbitrary $N \in$ mod-R. Since the sequence $0 \to Q \to R^{(\kappa)} \to P \to 0$ is exact and $0 = \mathrm{Ext}_R(R^{(\kappa)}, N) = \mathrm{Ext}^2_R(R^{(\kappa)}, N)$, the abelian groups $\mathrm{Ext}_R(Q, N)$ and $\mathrm{Ext}^2_R(P, N)$ are isomorphic. Since the sequence $0 \to N \to I(N) \to$ $\to I(N)/N \to 0$ is exact and $0 = \mathrm{Ext}_R(P, I(N)) = \mathrm{Ext}^2_R(P, I(N))$, the abelian groups $\mathrm{Ext}_R(P, I(N)/N)$ and $\mathrm{Ext}^2_R(P, N)$ are isomorphic. Now,

$I(N) \in P^{\perp}$ and $P^{\perp}$ is closed with respect to factors, and $\mathrm{Ext}_R(Q, N) \simeq \mathrm{Ext}_R(P, I(N)/N) = 0$, whence Q is projective. Thus, proj dim $(P) \leqslant 1$. By [AF, Corollary 26.2], the projective module Q is a direct sum of countably generated modules, $Q = \oplus \sum_{\alpha < \lambda} Q_\alpha$. Put $D = \oplus \sum_{\alpha < \lambda} I(Q_\alpha)$. Since $P^{\perp}$ is closed with respect to direct sums, we have $\mathrm{Ext}_R(P, D) = 0$. In particular, the inclusion $i \in \mathrm{Hom}_R(Q, D)$ has a prolongation $g \in \mathrm{Hom}_R(R^{(\kappa)}, D)$, $g|Q = i$. For $\alpha < \lambda$, denote by π_α and by ρ_α the α-th projection of D onto $I(Q_\alpha)$ and of $I(Q_\alpha)$ onto $I(Q_\alpha)/Q_\alpha$, respectively. For $\alpha < \lambda$, put $g_\alpha = \rho_\alpha \pi_\alpha g$.

If $h = \oplus \sum_{\alpha < \lambda} g_\alpha \in \mathrm{Hom}_R(R^{(\kappa)}, \oplus \sum_{\alpha < \lambda} I(Q_\alpha)/Q_\alpha)$ then $Q \subseteq \mathrm{Ker}(h)$ and h induces a homomorphism $\bar{h} \in \mathrm{Hom}_R(P, \oplus \sum_{\alpha < \lambda} I(Q_\alpha)/Q_\alpha)$. Since P is small, there is a finite subset $F \subseteq \lambda$ such that $\mathrm{Im}(\bar{h}) \subseteq \oplus \sum_{\alpha \in F} I(Q_\alpha)/Q_\alpha$. Thus $\mathrm{Im}(g) \subseteq \oplus \sum_{\alpha \in F} I(Q_\alpha) + \oplus \sum_{\alpha < \lambda} Q_\alpha$. Denote by π the projection od D onto $\oplus \sum_{\alpha < \lambda, \alpha \notin F} I(Q_\alpha)$. Put $\bar{g} = \pi g$. Then $\bar{g} \in \mathrm{Hom}_R(R^{(\kappa)}, \bar{Q})$, where $\bar{Q} = \oplus \sum_{\alpha < \lambda, \alpha \notin F} Q_\alpha$. Since $\bar{g}|\bar{Q} = \mathrm{id}$, we have $R^{(\kappa)} = \mathrm{Ker}(\bar{g}) \oplus \bar{Q}$. Put $A = \mathrm{Ker}(\bar{g}) \cap Q = \oplus \sum_{\alpha \in F} Q_\alpha$. Then $P = R^{(\kappa)}/Q = (\mathrm{Ker}(\bar{g}) + Q)/Q \simeq \mathrm{Ker}(\bar{g})/A$. Since $\mathrm{Ker}(\bar{g})$ is projective, [AF, Corollary 26.2], implies it is a direct sum of countably generated projective modules. Since A is countably generated, we infer that P is a direct sum of a countably genrated module C and a projective module B. Since P is small, B is countably generated. Hence, P is a countably generated small module, and 1.1 implies P is finitely generated.

Now, if P is finitely generated and proj dim $(P) \leqslant 1$, there is an exact sequence $0 \to L \to R^{(n)} \to P \to 0$ with L projective, i.e. L a summand of some $R^{(X)}$. Since $P^{\perp}$ is closed with respect to direct sums, we have $I(R)^{(X)} \in P^{\perp}$ and the same argument as in the second part of the proof of [CM, Proposition 1.7] shows that L is finitely generated. Hence, P is finitely presented.

Assume (2). Clearly, P is a small module. Since proj dim $(P) \leqslant 1$, $P^{\perp}$ is closed with respect to factors. Moreover, $P = X/Y$, where X is a projective module and Y is a finitely generated module. Hence every homomorphism of Y into a direct sum of modules actually maps into a finite direct sub-sum. Therefore, as $P^{\perp}$ is closed with respect to finite direct sums, it is closed with respect to the arbitrary ones.

THEOREM 1.3. *Let R be a ring and P a module.*

(i) *If $P \in ISTAR$, then P is finitely presented,* proj dim $(P) \leqslant 1$,

$\mathrm{Ext}_R(P, P) = 0$, *and there is an exact sequence* $0 \to R \to P' \to P'' \to 0$ *such that* P' *is a finite direct power of* P.

(ii) $P \in ISTAR$ *iff* P *is finitely generated and* $\mathrm{Gen}(P_R) = P^{\perp}$.

PROOF. (i) By [CM, Propositions 1.5 and 1.8], P is a small module with $\mathrm{Gen}(P_R) = P^{\perp}$. In particular, $\mathrm{Ext}_R(P, P) = 0$. By 1.2, P is finitely presented and $\mathrm{proj\,dim}(P) \leqslant 1$. Finally, [CM, Proposition 1.5] implies R embeds into a finite direct power of P.

(ii) By 1.2 and [CM, Proposition 1.5].

By 1.3(i), the classes $ISTAR$ and $WTILT$ are quite close to each other. Moreover,

PROPOSITION 1.4. *Let* R *be a finite dimensional algebra over a field. Then* $ISTAR = WTILT$.

PROOF. By [MO, Theorem 4.3], $WTILT \subseteq ISTAR$. On the other hand, every $P \in ISTAR$ is finitely generated by 1.3(ii) and it is faithful by [CM, Proposition 1.5]. Thus [DH, Theorem 1] implies $P \in WTILT$.

Now, 1.4 and [CM, Theorem 3.3] suggest the following.

PROBLEM 1.5. *Characterize the rings* R *such that* $WTILT = ISTAR$.

PROPOSITION 1.6. *Let* R *be a semiperfect ring and* B *a basic set of idempotents of* R. *Let* $P \in ISTAR$. *Then there exist a non-empty subset* C *of* B, *a positive integer* n, *and, for each* $i < n$, *modules* F_i *and* G_i *such that*

(1) F_i *is a non-zero direct sum of direct powers of the modules* eR, $e \in C$,

(2) G_i *is a superfluous submodule of* F_i,

(3) G_i *is isomorphic to a direct sum of direct powers of the modules* eR, $e \in B \setminus C$,

(4) *for every* $e \in C$ *and* $e \in B \setminus C$, *the module* eR *appears as a summand of* F_i *and* G_i, *respectively, for some* $i < n$,

(5) *the module* F_i/G_i *is indecomposable*,

(6) $P \simeq \bigoplus \sum_{i<n} F_i/G_i$.

PROOF. By ([AF, Corollary 15.8]), we have $\mathrm{Rad}(P) = P.\ \mathrm{Rad}(R)$ and, by 1.3(ii), $P/\mathrm{Rad}(P)$ is a non-zero finitely generated completely reducible module. Hence P is a direct sum of indecomposable modules, $P = \oplus \sum_{i<n} P_i$, for a positive integer n. Of course, each P_i has a projective cover, and $\operatorname{proj\,dim}(P_i) \leq 1$ by 1.3(i). By [AF, Theorem 27.11], there exist modules F_i and G_i such that $P_i \simeq F_i/G_i$, where F_i is a non-zero direct sum of non-zero direct powers of the modules eR, $e \in C_i$, for some $C_i \subseteq B$, G_i is a superfluous submodule of F_i, and either $G_i = 0$ and $D_i = \emptyset$, or G_i is isomorphic to a non-zero direct sum of nonzero direct powers of the modules eR, $e \in D_i$, for some $D_i \subseteq B$. Put $C = \bigcup_{i<n} C_i$ and $D = \bigcup_{i<n} D_i$. It remains to prove that $C \cap D = \emptyset$ and $C \cup D = B$. Assume $e \in C_i \cap D_j$. By [AF, Proposition 27.10], there is maximal submodule H of G_j such that $G_j/H \simeq eR/\mathrm{Rad}(eR)$ and G_j/H is isomorphic to a summand of the completely reducible module $F_i/\mathrm{Rad}(F_i)$.

Let $\phi \in \mathrm{Hom}_R(G_j, F_i/\mathrm{Rad}(F_i))$ be the composition of these isomorphisms and of the projection of G_j onto G_j/H. Assume there is some $\varphi \in \mathrm{Hom}_R(F_j, F_i/\mathrm{Rad}(F_i))$ such that $\phi = \varphi\nu$, ν being the inclusion of G_j into F_j. Then $\mathrm{Ker}(\varphi)$ is a maximal submodule of F_j, whence $G_j \subseteq \mathrm{Rad}(F_j) \subseteq \mathrm{Ker}(\varphi)$ and $\varphi\nu = 0$, a contradiction. Therefore, $\mathrm{Ext}_R(P_j, F_i/\mathrm{Rad}(F_i)) \neq 0$. But $F_i/\mathrm{Rad}(F_i)$ is a factor-module of P_i and $\operatorname{proj\,dim}(P_i) \leq 1$, whence $\mathrm{Ext}_R(P_i, P_j) \neq 0$, a contradiction with [CM, Proposition 1.5].

Assume there is some $e \in B \setminus (C \cup D)$. Then [AF, Proposition 27.10] implies $\mathrm{Hom}_R(\oplus \sum_{i<n} G_i, M) = 0$, where $M = eR/\mathrm{Rad}(eR)$ is a simple module. Hence $\mathrm{Ext}_R(P, M) = 0$. By [CM, Proposition 1.5], $M \in \mathrm{Gen}(P_R) \subseteq \mathrm{Gen}((\oplus \sum_{i<n} F_i)_R)$, in contradiction with [AF, 27.13].

PROPOSITION 1.7. *Let R be a semiperfect ring and $P \in ISTAR$.*

(i) *Put $G = \oplus \sum_{i<n} G_i$ and $F = \oplus \sum_{i<n} F_i$ (see* 1.6 *for the notation). Consider the following two conditions:*

(1) $N \in \mathrm{Gen}(P_R)$,

(2) *The completely reducible modules $N/\mathrm{Rad}(N)$ and $G/\mathrm{Rad}(G)$ have no isomorphic direct summands.*

Then (1) *implies* (2) *for any $N \in$ mod-R. If N is completely reducible, then* (1) *is equivalent to* (2). *Moreover,* (1) *is equivalent to* (2) *for all finitely generated modules N iff every homomorphism of G into* $\mathrm{Rad}(F)$ *can be prolonged into an endomorphism of F iff* $\mathrm{Gen}(P_R) = \mathrm{Gen}(F_R)$.

(ii) *$P \in PG$ iff* $\mathrm{Gen}(P_R)$ *contains all simple modules.*

PROOF. (i) Assume (1). Then $\mathrm{Ext}_R(P, N) = 0$, by [CM, Proposition 1.5]. Suppose (2) does not hold. Then there exist a homomorphism $\xi \in \mathrm{Hom}_R(G/\mathrm{Rad}(G), N/\mathrm{Rad}(N))$ such that $\mathrm{Im}(\xi)$ is a simple module. Put $\phi = \xi\pi$, where $\pi\colon G \to G/\mathrm{Rad}(G)$ is the projection. Then $\phi \in \mathrm{Hom}_R(G, N/\mathrm{Rad}(N))$ and by 1.3 (ii), there is $\varphi \in \mathrm{Hom}_R(F, N/\mathrm{Rad}(N))$ such that $\phi = \varphi\nu$, ν being the inclusion of G into F. In particular, $\mathrm{Ker}(\varphi)$ is a maximal submodule of F, $G \subseteq \mathrm{Rad}(F) \subseteq \mathrm{Ker}(\varphi)$, a contradiction.

If N is completely reducible and (2) holds, then $\mathrm{Hom}_R(G/\mathrm{Rad}(G), N) = 0$. Hence $\mathrm{Hom}_R(G, N) = 0$, $\mathrm{Ext}_R(P, N) = 0$, and [CM, Proposition 1.5] implies $N \in \mathrm{Gen}(P_R)$.

Assume (2) implies (1) for all finitely generated modules N. By 1.6, $\mathrm{Hom}_R(F/\mathrm{Rad}(F), G/\mathrm{Rad}(G)) = 0$. For $N = F$, we get $\mathrm{Gen}(P_R) = \mathrm{Gen}(F_R)$.

Assume $\mathrm{Gen}(P_R) = \mathrm{Gen}(F_R)$. Then [CM, Proposition 1.5] implies $\mathrm{Ext}_R(P, F) = 0$. Thus, even every homomorphism of G into F has the desired prolongation.

Assume the prolongations exist and let N be a finitely generated module satisfying (2). By [AF, Theorem 27.6], there are a finitely generated projective module A and a superfluous submodule B of A such that $N = A/B$. In particular, $\mathrm{Rad}(N) = \mathrm{Rad}(A)/B$. By (2), 1.6 and [AF, Theorem 27.11], there exist positive integers p and q such that $A^{(p)}$ is a summand of $F^{(q)}$. Let $\phi \in \mathrm{Hom}_R(G, N^{(p)})$. By (2), $\rho\phi = 0$, where $\rho\colon N^{(p)} \to N^{(p)}/\mathrm{Rad}(N^{(p)})$ is the projection. Hence, $\mathrm{Im}(\phi) \subseteq \mathrm{Rad}(N^{(p)}) = \mathrm{Rad}(A^{(p)})/B^{(p)}$. Since G is projective, there exists $\theta \in \mathrm{Hom}_R(G, \mathrm{Rad}(A^{(p)}))$ such that $\sigma\theta = \phi$, where σ is the projection of $A^{(p)}$ onto $A^{(p)}/B^{(p)}$. Using the premise, it is easy to see that θ has a prolongation into a $\varphi \in \mathrm{Hom}_R(F, A^{(p)})$. Thus, $\sigma\varphi\nu = \sigma\theta = \phi$, where ν is the inclusion of G into F. This implies $\mathrm{Ext}_R(P, N^{(p)}) = 0$, and [CM, Proposition 1.5] gives (1).

(ii) If $\mathrm{Gen}(P_R)$ contains all simple modules, then (i) implies $G = 0$ and $P \in PG$.

Clearly, for any ring R, we have $PG \subseteq QPG \subseteq STAR$ and $PG \subseteq ISTAR \subseteq STAR$. Moreover, [C, Proposition 4.5 and Theorem 4.7] imply $QPG \cap ISTAR = PG$.

PROPOSITION 1.8. *$QPG = PG$ iff R is a simple completely reducible ring.*

PROOF. Assume $QPG = PG$. Denote by $\mathcal{S}$ the class of all simple modules. Clearly $\mathcal{S} \subseteq QPG$, whence every simple module is projective and R is completely reducible. Moreover, since every element of $\mathcal{S}$ is a generator, R is simple. The opposite implication is obvious.

We turn to applications to particular classes of rings:

THEOREM 1.9. *$ISTAR = PG$ provided one of the following conditions is true:*

(i) *R is a commutative ring,*

(ii) *R is a local ring,*

(iii) *R is a von Neumann regular ring.*

PROOF. (i) Let $P \in ISTAR$. By [CM, Proposition 1.5], P is a faithful $*$-module. By 1.3(ii), P is finitely generated and [CM, Theorem 2.3] shows that $P \in PG$.

(ii) By 1.6, since $\operatorname{card}(B) = 1$.

(iii) By 1.3(i), since every finitely presented module is projective.

In view of 1.3 and 1.9(iii), it is surprising that even for von Neumann regular hereditary rings, the question whether the class $M^{\perp}$ is closed with respect to countable direct powers for a non-projective module M, can be quite difficult to answer.

THEOREM 1.10. *Let R be a simple right hereditary non-completely reducible von Neumann regular ring with $\operatorname{card}(R) \leq \aleph_1$ (e.g. R can be any simple countable non-completely reducible von Neumann regular ring). Then, for every module M, the class $M^{\perp}$ is closed with respect to factors, but the assertion*

«$N^{(\aleph_0)} \notin M^{\perp}$ whenever $M, N \in \text{mod-}R$ are such that M is non-projective and $0 \neq N \in M^{\perp}$»

is independent of $ZFC + GCH$.

PROOF. Since R is right hereditary, $M^{\perp}$ is closed with respect to factors for any module M.

Since R is not right perfect, [ES, Corollary 2.2] implies that it is consistent with $ZFC + GCH$ that for every uncountable cardinal κ such that $\operatorname{card}(R) \leq \kappa$ and $cf(\kappa) = \aleph_0$ there is a non-projective module M such that $\operatorname{card}(M) = \kappa^{+}$ and $\operatorname{Ext}_R(M, N) = 0$ for all modules N with $\operatorname{card}(N) < \kappa$. In particular, the negation of our assertion is consistent. On the other hand, assume the axiom of constructibility ($V = L$). We prove the assertion by induction on $\operatorname{gen}(M) = \lambda$.

If $\lambda < \aleph_0$, then $M \simeq R^{(\lambda)}/I$ for an infinitely generated module I. Since I is projective, there exists a countable R-independent set $\{x_n \mid n < \aleph_0\}$ generating a summand of I. Let e_n be the non-zero idempotent of R such that $\mathrm{Ann}(x_n) = (1 - e_n)R$. Since R is simple, there is some $0 \neq y_n \in Ne_n$ for all $n < \aleph_0$. Define $h \in \mathrm{Hom}_R(I, N^{(\aleph_0)})$ by $hx_n = \pi_n y_n$, where π_n is the n-th inclusion of N into $N^{(\aleph_0)}$. Then h does not extend into an element of $\mathrm{Hom}_R(R^{(\lambda)}, N^{(\aleph_0)})$, whence $\mathrm{Ext}_R(M, N^{(\aleph_0)}) \neq 0$.

If $\lambda = \aleph_0$, [T, Lemma 10.3] shows there exists a non-projective finitely generated submodule F of M such that $\mathrm{Ext}_R(F, N) = 0$ and the induction works.

If λ is a regular uncountable cardinal, then [T, Lemma 10.7] shows there is a λ-filtration $(C_\alpha \mid \alpha < \lambda)$ of M such that $\mathrm{Ext}_R(C_{\alpha+1}/C_\alpha, N) = 0$ for all $\alpha < \lambda$, and the set $E = \{\alpha < \lambda \mid C_{\alpha+1}/C_\alpha$ is non-projective$\}$ is stationary in λ. By the induction premise, $\mathrm{Ext}_R(C_{\alpha+1}/C_\alpha, N^{(\aleph_0)}) \neq 0$ for all $\alpha \in E$. By [T, Lemma 10.6], this implies $\mathrm{Ext}_R(M, N^{(\aleph_0)}) \neq 0$.

If λ is singular, then the general compactness theorem [EM, Theorem IV.3.7] implies there is a non-projective submodule U of M such that $\mathrm{gen}(U) < \lambda$ and $\mathrm{Ext}_R(U, N) = 0$, and the induction works.

REFERENCES

[AF] F. D. ANDERSON - K. R. FULLER, *Rings and Categories of Modules*, Springer, New York (1974).

[C] R. COLPI, *Some remarks on equivalences between categories of modules*, Comm. Alg., 18 (1990), pp. 1935-1951.

[CM] R. COLPI - C. MENINI, *On the structure of $*$-modules*, J. Alg. (as appear).

[DH] G. D'ESTE - D. HAPPEL, *Representable equivalences are represented by tilting modules*, Rend. Sem. Mat. Univ. Padova, 83 (1990), pp. 77-80.

[EM] P. C. EKLOF - A. H. MEKLER, *Almost Free Modules*, North-Holland, Amsterdam (1990).

[ES] P. C. EKLOF - S. SHELAH, *On Whitehead modules*, J. Alg., **142** (1991), pp. 492-510.

[F] K. R. FULLER, *Density and equivalence*, J. Alg., **29** (1974), pp. 528-550.

[HR] D. HAPPEL - C. M. RINGEL, *Tilted algebras*, Trans. Amer. Math. Soc., **274** (1982), pp. 399-443.

[MO] C. MENINI - A. ORSATTI, *Representable equivalences between categories of modules and applications*, Rend. Sem. Mat. Univ. Padova, 82 (1989), pp. 203-231.

[S] L. SALCE, *Cotorsion theories for abelian groups*, Symp. Math., **23** (1979), pp. 11-32.

[T] J. TRLIFAJ, *Associative Rings and the Whitehead Property of Modules*, R. Fischer, Munich (1990).

[Z] P. ZANARDO, *On $*$-modules over valuation rings*, Rend. Sem. Mat. Univ. Padova, **83** (1990), pp. 193-199.

Manoscritto pervenuto in redazione il 22 agosto 1991.