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On *-Modules Generating the Injectives.

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ABSTRACT - Relations between *-modules, quasi-progenerators and other generalizations of progenerators are studied. The *-modules generating all injective modules are shown to be finitely generated.

Introduction.

One of the examples showing that category theory is not only a language, but also a useful tool in algebra is the celebrated Morita theorem concerning equivalence of rings. For any ring \( R \), it implies e.g. the important fact that \( R \) and the full matrix ring \( M_n(R) \) share all the ring theoretic properties which are definable by means of categorical properties of modules.

More specifically, if \( R \) and \( S \) are rings, the Morita theorem ([AF, Corollary 22.4]) says that \( \text{mod-}R \) and \( \text{mod-}S \) are equivalent categories iff there exists a progenerator (= a finitely generated projective generator) \( P \) such that \( S = \text{End} \left( P_R \right) \).

In [F], Fuller generalized the theorem as follows: \( \text{mod-}S \) is equivalent to a full subcategory \( C \) of \( \text{mod-}R \) such that \( C \) is closed with respect to submodules, direct sums and quotients iff there exists a quasi-progenerator (= a finitely generated quasi-projective module generating all its submodules) \( P \) such that \( S = \text{End} \left( P_R \right) \) and \( C = \text{Gen} \left( P_R \right) \).

In both cases, the pair \((F, G)\) of functors realizing the equivalence is represented by \( P \), i.e. \( F \) and \( G \) are naturally equivalent to \(- \otimes_S P \) and \( \text{Hom}_R(P, -) \), respectively.


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In [MO, Theorem 3.1], Menini and Orsatti presented a further generalization: if \( B \) and \( C \) are equivalent categories, where \( B \subset \text{mod-}S \) is such that \( S \in B \) and \( B \) is closed with respect to submodules, and \( C \subset \text{mod-}R \) is closed with respect to direct sums and factors, then there is a module \( P \) such that \( S = \text{End}(P_R) \), \( C = \text{Gen}(P_R) \) and \( B = \text{Cog}(K_S) \), where \( K = \text{Hom}_R(P, Q) \) and \( Q \) is an injective cogenerator of \( \text{mod-}R \). Moreover, the equivalence is represented by \( P \).

In [MO, 3.5], a question was raised of characterizing the modules \( P \) that induce an equivalence between \( \text{Gen}(P_R) \) and \( \text{Cog}(\text{Hom}_R(P, Q)_S) \) with \( S = \text{End}(P_R) \). Since the question was denoted by (\( * \)), such modules are called \( * \)-modules ([C], [DH]).

For a ring \( R \), denote by \( PG \), \( QPG \) and \( STAR \) the class of all progenitors, quasi-progenitors and \( * \)-modules, respectively. Clearly, \( PG \subset QPG \subset STAR \). Surprisingly, there is another important class of \( * \)-modules, not connected with quasi-progenitors: a module \( P \) is a \( W \)-tilting module if \( P \) is finitely presented, \( \text{proj dim}(P) \leq 1 \), \( \text{Ext}_R(P, P) = 0 \), and there is an exact sequence \( 0 \to R \to P' \to P'' \to 0 \) such that \( P' \) and \( P'' \) are direct sums of summands of \( P \). For example, the \( W \)-tilting modules over finite dimensional algebras coincide with the tilting modules, introduced in [HR]. Denote by \( WTILT \) the class of all \( W \)-tilting modules and by \( ISTAR \) the class of all \( * \)-modules such that \( I(R) \in \text{Gen}(P_R) \), \( I(R) \) being the injective hull of \( R \). The surprise is that \( WTILT \subset ISTAR \) ([MO, Theorem 4.3]).

The class \( ISTAR \) was studied in more detail by Colpi and Menini in [C] and [CM]. By [CM, Proposition 1.5], if \( P \in ISTAR \) then \( \text{Gen}(P_R) = \{M \mid \text{Ext}_R(P, M) = 0\} \). Moreover, [CM, Proposition 1.5] implies that \( ISTAR \) coincides with the class of all \( * \)-modules \( P \) such that \( \text{Gen}(P_R) \supseteq \mathfrak{S} \), where \( \mathfrak{S} \) is the class of all injective modules. In [CM, Theorem 3.3], a complete characterization of the rings \( R \) for which there is a \( * \)-module \( P \) with \( \text{Gen}(P_R) = \mathfrak{S} \) was given.

The main result of our paper is Theorem 1.3 showing that \( ISTAR \) is very close to the class \( WTILT \). In particular, all elements of \( ISTAR \) are finitely presented. Thus, for this case, we confirm the conjecture of Colpi and Menini (cp. [CM, Propositions 1.8 and 1.9]), D'Este and Happel ([DH, Remark 4]), and Zanardo ([Z, Remark 4]), which claims that every \( * \)-module is finitely generated. In Propositions 1.6 and 1.7, the structure if \( ISTAR \) over semiperfect rings is described in greater detail.

Then we turn to applications to particular classes of rings. We show that \( ISTAR = PG \) provided \( R \) is either a commutative or a local or a von Neumann regular ring (Theorem 1.9). Finally, Theorem 1.10 shows that properties concerning \( P^\perp \) which are slightly weaker than the ones induced by \( * \)-modules, can be independent of \( ZFC \).
In the following, all rings are associative with unit. Let $R$ be a ring. The category of (unitary right $R$-) modules is denoted by $\text{mod} \cdot R$. Homomorphisms in $\text{mod} \cdot R$ are written as acting on the left. If $M \in \text{mod} \cdot R$, then $\text{Rad}(M)$ denotes the Jacobson radical of $M$. Further, $R$ is completely reducible provided $R$ is a finite ring direct sum of full matrix rings over skew fields. $R$ is semiperfect provided $R/\text{Rad}(R)$ is completely reducible and idempotents lift modulo $\text{Rad}(R)$.

Let $M$ be a module. Then $\text{gen}(M)$ denotes the minimal cardinality of an $R$-generating subset of $M$ and $I(M)$ the injective hull of $M$. The category of all modules generated by $P$ is denoted by $\text{Gen}(P_R)$, and $\overline{\text{Gen}}(P_R)$ is the category consisting of all submodules of elements of $\text{Gen}(P_R)$. $M$ is said to be small provided for every sequence of modules $(N_x | x \in A)$ and every homomorphism $h \in \text{Hom}_{\mathcal{R}}(M, \bigoplus_{x \in A} N_x)$ there is a finite set $F \subset A$ such that $\text{Im}(h) \subset \bigoplus_{x \in F} N_x$. The module $M$ is finitely presented provided there is an exact sequence $0 \to G \to F \to M \to 0$ in $\text{mod} \cdot R$ such that $F$ is projective, and $F$ and $G$ are finitely generated. Further, $\text{proj dim}(M)$ denotes the projective dimension of $M$, and $M^\perp$ the cotorsion class generated by $M$, i.e. $M^\perp = \{ N \in \text{mod} \cdot R | \text{Ext}_R(M, N) = 0 \}$ (see [S] or [T, § 1]). For further concepts and notation, the reader is referred to [AF] and [EM].

1. The structure of $\text{ISTAR}$.

**Lemma 1.1.** Let $R$ be a ring and $P$ a small module. Then either $P$ is finitely generated or $\text{gen}(P) \geq \aleph_1$.

**Proof.** An easy modification of the proof of [CM, Proposition 1.9].

**Lemma 1.2.** Let $R$ be a ring and $P$ a module. Then the following conditions are equivalent:

1. $P$ is small and $P^\perp$ is closed with respect to direct sums and factors,

2. $P$ is finitely presented and $\text{proj dim}(P) \leq 1$.

**Proof.** Assume (1). Clearly, $P = R^{(\kappa)}/Q$ for a cardinal $\kappa$ and a submodule $Q$ of $R^{(\kappa)}$. First we observe that $Q$ is projective. Take an arbitrary $N \in \text{mod} \cdot R$. Since the sequence $0 \to Q \to R^{(\kappa)} \to P \to 0$ is exact and $0 = \text{Ext}_R(R^{(\kappa)}, N) = \text{Ext}_R^2(R^{(\kappa)}, N)$, the abelian groups $\text{Ext}_R(Q, N)$ and $\text{Ext}_R^2(P, N)$ are isomorphic. Since the sequence $0 \to N \to I(N) \to I(N)/N \to 0$ is exact and $0 = \text{Ext}_R(P, I(N)) = \text{Ext}_R^2(P, I(N))$, the abelian groups $\text{Ext}_R(P, I(N)/N)$ and $\text{Ext}_R^2(P, N)$ are isomorphic. Now,
I(N) ∈ P ⊥ and P ⊥ is closed with respect to factors, and Ext_R(Q, N) = Ext_R(P, I(N)/N) = 0, whence Q is projective. Thus, proj dim (P) ≤ 1. By [AF, Corollary 26.2], the projective module Q is a direct sum of countably generated modules, Q = \( \bigoplus \sum \limits_{\alpha < \lambda} Q_{\alpha} \). Put \( D = \bigoplus \sum \limits_{\alpha < \lambda} I(Q_{\alpha}) \).

Since P ⊥ is closed with respect to direct sums, we have Ext_R(P, D) = 0. In particular, the inclusion \( i \in \text{Hom}_R(Q, D) \) has a prolongation \( g \in \text{Hom}_R(R^{(\alpha)}, D) \), \( g\big|_Q = i \). For \( \alpha < \lambda \), denote by \( \pi_{\alpha} \) and by \( \rho_{\alpha} \) the \( \alpha \)-th projection of \( D \) onto \( I(Q_{\alpha}) \) and of \( I(Q_{\alpha})/Q_{\alpha} \), respectively. For \( \alpha < \lambda \), put \( g_{\alpha} = \rho_{\alpha} \pi_{\alpha} g \).

If \( h = \bigoplus \sum \limits_{\alpha < \lambda} g_{\alpha} \in \text{Hom}_R(R^{(\alpha)}, \bigoplus \sum \limits_{\alpha < \lambda} I(Q_{\alpha})/Q_{\alpha}) \) then \( Q \subset \text{Ker} (h) \) and \( h \) induces a homomorphism \( \tilde{h} \in \text{Hom}_R(P, \bigoplus \sum \limits_{\alpha < \lambda} I(Q_{\alpha})/Q_{\alpha}) \). Since \( P \) is small, there is a finite subset \( F \subset \lambda \) such that \( \text{Im} (\tilde{h}) \subset \bigoplus \sum \limits_{\alpha \in F} I(Q_{\alpha})/Q_{\alpha} \).

Thus \( \text{Im} (g) \subset \bigoplus \sum \limits_{\alpha \in F} I(Q_{\alpha}) + \bigoplus \sum \limits_{\alpha < \lambda} Q_{\alpha} \). Denote by \( \pi \) the projection of \( D \) onto \( \bigoplus \sum \limits_{\alpha < \lambda} I(Q_{\alpha}) \). Put \( \tilde{g} = \pi g \). Then \( \tilde{g} \in \text{Hom}_R(R^{(\alpha)}, \overline{Q}) \), where \( \overline{Q} = \bigoplus \sum \limits_{\alpha \in F} Q_{\alpha} \). Since \( \tilde{g}\big|_\overline{Q} = \text{id} \), we have \( R^{(\alpha)} = \text{Ker} (\tilde{g}) + \overline{Q} \). Put \( A = \text{Ker} (\tilde{g}) \cap Q = \bigoplus \sum \limits_{\alpha \in F} Q_{\alpha} \). Then \( P = R^{(\alpha)}/Q = (\text{Ker} (\tilde{g}) + Q)/Q = \text{Ker} (\tilde{g})/A \). Since \( \text{Ker} (\tilde{g}) \) is projective, [AF, Corollary 26.2], implies it is a direct sum of countably generated projective modules. Since \( A \) is countably generated, we infer that \( P \) is a direct sum of a countably generated module \( C \) and a projective module \( B \). Since \( P \) is small, \( B \) is countably generated. Hence, \( P \) is a countably generated small module, and 1.1 implies \( P \) is finitely generated.

Now, if \( P \) is finitely generated and proj dim (P) ≤ 1, there is an exact sequence \( 0 \rightarrow L \rightarrow R^{(X)} \rightarrow P \rightarrow 0 \) with \( L \) projective, i.e. \( L \) a summand of some \( R^{(X)} \). Since \( P^\perp \) is closed with respect to direct sums, we have \( I(R^{(X)}) \subset P^\perp \) and the same argument as in the second part of the proof of [CM, Proposition 1.7] shows that \( L \) is finitely generated. Hence, \( P \) is finitely presented.

Assume (2). Clearly, \( P \) is a small module. Since proj dim (P) ≤ 1, \( P^\perp \) is closed with respect to factors. Moreover, \( P = X/Y \), where \( X \) is a projective module and \( Y \) is a finitely generated module. Hence every homomorphism of \( Y \) into a direct sum of modules actually maps into a finite direct sub-sum. Therefore, as \( P^\perp \) is closed with respect to finite direct sums, it is closed with respect to the arbitrary ones.

**Theorem 1.3.** Let \( R \) be a ring and \( P \) a module.

(i) If \( P \in \text{ISTAR} \), then \( P \) is finitely presented, proj dim (P) ≤ 1,
Ext$_R$(P, P) = 0, and there is an exact sequence $0 \rightarrow R \rightarrow P' \rightarrow P'' \rightarrow 0$ such that $P'$ is a finite direct power of $P$.

(ii) $P \in$ ISTAR iff $P$ is finitely generated and $\text{Gen}(P_R) = P^\perp$.

PROOF. (i) By [CM, Propositions 1.5 and 1.8], $P$ is a small module with $\text{Gen}(P_R) = P^\perp$. In particular, Ext$_R$(P, P) = 0. By 1.2, $P$ is finitely presented and proj dim $(P) \leq 1$. Finally, [CM, Proposition 1.5] implies $R$ embeds into a finite direct power of $P$.

(ii) By 1.2 and [CM, Proposition 1.5].

By 1.3(i), the classes ISTAR and WTILT are quite close to each other. Moreover,

PROPOSITION 1.4. Let $R$ be a finite dimensional algebra over a field. Then ISTAR = WTILT.

PROOF. By [MO, Theorem 4.3], WTILT $\subseteq$ ISTAR. On the other hand, every $P \in$ ISTAR is finitely generated by 1.3(ii) and it is faithful by [CM, Proposition 1.5]. Thus [DH, Theorem 1] implies $P \in$ WTILT.

Now, 1.4 and [CM, Theorem 3.3] suggest the following.

PROBLEM 1.5. Characterize the rings $R$ such that WTILT = ISTAR.

PROPOSITION 1.6. Let $R$ be a semiperfect ring and $B$ a basic set of idempotents of $R$. Let $P \in$ ISTAR. Then there exist a non-empty subset $C$ of $B$, a positive integer $n$, and, for each $i < n$, modules $F_i$ and $G_i$ such that

1. $F_i$ is a non-zero direct sum of direct powers of the modules $eR$, $e \in C$,

2. $G_i$ is a superfluous submodule of $F_i$,

3. $G_i$ is isomorphic to a direct sum of direct powers of the modules $eR$, $e \in B \setminus C$,

4. for every $e \in C$ and $e \in B \setminus C$, the module $eR$ appears as a summand of $F_i$ and $G_i$, respectively, for some $i < n$,

5. the module $F_i/G_i$ is indecomposable,

6. $P = \bigoplus_{i < n} F_i/G_i$. 

PROOF. By ([AF, Corollary 15.8]), we have \( \text{Rad}(P) = P \). \( \text{Rad}(R) \) and, by 1.3(ii), \( P/\text{Rad}(P) \) is a non-zero finitely generated completely reducible module. Hence \( P \) is a direct sum of indecomposable modules, \( P = \bigoplus \sum P_i \), for a positive integer \( n \). Of course, each \( P_i \) has a projective cover, \( \text{and proj dim}(P_i) \leq 1 \) by 1.3(i). By [AF, Theorem 27.11], there exist modules \( F_i \) and \( G_i \) such that \( P_i = F_i / G_i \), where \( F_i \) is a non-zero direct sum of non-zero direct powers of the modules \( eR, e \in C_i \), for some \( C_i \subseteq B \), \( G_i \) is a superfluous submodule of \( F_i \), and either \( G_i = 0 \) and \( D_i = \emptyset \), or \( G_i \) is isomorphic to a non-zero direct sum of nonzero direct powers of the modules \( eR, e \in D_i \), for some \( D_i \subseteq B \). Put \( C = \bigcup_{i < n} C_i \) and \( D = \bigcup_{i < n} D_i \). It remains to prove that \( C \cap D = \emptyset \) and \( C \cup D = B \). Assume \( e \in C_i \cap D_j \). By [AF, Proposition 27.10], there is maximal submodule \( H \) of \( G_j \) such that \( G_j / H \cong eR / \text{Rad}(eR) \) and \( G_j / H \) is isomorphic to a summand of the completely reducible module \( F_i / \text{Rad}(F_i) \).

Let \( \phi \in \text{Hom}_R(G_j, F_i / \text{Rad}(F_i)) \) be the composition of these isomorphisms and of the projection of \( G_j \) onto \( G_j / H \). Assume there is some \( \varphi \in \text{Hom}_R(F_j, F_i / \text{Rad}(F_i)) \) such that \( \phi = \varphi \circ \nu \), where \( \nu \) is the inclusion of \( G_j \) into \( F_j \). Then \( \text{Ker}(\phi) \) is a maximal submodule of \( F_j \), whence \( G_j \subseteq \text{Rad}(F_j) \subseteq \text{Ker}(\phi) \) and \( \varphi \circ \nu = 0 \), a contradiction. Therefore, \( \text{Ext}_R(P_j, F_i / \text{Rad}(F_i)) \neq 0 \). But \( F_i / \text{Rad}(F_i) \) is a factor-module of \( P_i \) and \( \text{proj dim}(P_i) \leq 1 \), whence \( \text{Ext}_R(P_i, P_j) \neq 0 \), a contradiction with [CM, Proposition 1.5].

Assume there is some \( e \in B \setminus (C \cup D) \). Then [AF, Proposition 27.10] implies \( \text{Hom}_R(\bigoplus_{i < n} G_i, M) = 0 \), where \( M = eR / \text{Rad}(eR) \) is a simple module. Hence \( \text{Ext}_R(P, M) = 0 \). By [CM, Proposition 1.5], \( M \in \text{Gen}(P_R) \subseteq \text{Gen}(\bigoplus_{i < n} F_i) \), in contradiction with [AF, 27.13].

PROPOSITION 1.7. Let \( R \) be a semiperfect ring and \( P \in \text{ISTAR} \).

(i) Put \( G = \bigoplus_{i < n} G_i \) and \( F = \bigoplus_{i < n} F_i \) (see 1.6 for the notation). Consider the following two conditions:

1. \( N \in \text{Gen}(P_R) \),
2. The completely reducible modules \( N/\text{Rad}(N) \) and \( G/\text{Rad}(G) \) have no isomorphic direct summands.

Then (1) implies (2) for any \( N \in \text{mod-R} \). If \( N \) is completely reducible, then (1) is equivalent to (2). Moreover, (1) is equivalent to (2) for all finitely generated modules \( N \) iff every homomorphism of \( G \) into \( \text{Rad}(F) \) can be prolonged into an endomorphism of \( F \) iff \( \text{Gen}(P_R) = \text{Gen}(F_R) \).

(ii) \( P \in \text{PG} \) iff \( \text{Gen}(P_R) \) contains all simple modules.
PROOF. (i) Assume (1). Then Ext$_R$(P, N) = 0, by [CM, Proposition 1.5]. Suppose (2) does not hold. Then there exist a homomorphism $\xi \in \text{Hom}_R(G/\text{Rad}(G), N/\text{Rad}(N))$ such that $\text{Im}(\xi)$ is a simple module. Put $\phi = \xi\pi$, where $\pi: G \to G/\text{Rad}(G)$ is the projection. Then $\phi \in \text{Hom}_R(G, N/\text{Rad}(N))$ and by 1.3 (ii), there is $\varphi \in \text{Hom}_R(F, N/\text{Rad}(N))$ such that $\phi = \varphi\nu$, $\nu$ being the inclusion of $G$ into $F$. In particular, $\text{Ker}(\varphi)$ is a maximal submodule of $F$, $G \subset \text{Rad}(F) \subset \text{Ker}(\varphi)$, a contradiction.

If $N$ is completely reducible and (2) holds, then $\text{Hom}_R(G/\text{Rad}(G), N) = 0$. Hence $\text{Hom}_R(G, N) = 0$, $\text{Ext}_R(P, N) = 0$, and [CM, Proposition 1.5] implies $N \in \text{Gen}(P_R)$.

Assume (2) implies (1) for all finitely generated modules $N$. By 1.6, $\text{Hom}_R(F/\text{Rad}(F), G/\text{Rad}(G)) = 0$. For $N = F$, we get $\text{Gen}(P_R) = \text{Gen}(F_R)$.

Assume $\text{Gen}(P_R) = \text{Gen}(F_R)$. Then [CM, Proposition 1.5] implies $\text{Ext}_R(P, F) = 0$. Thus, even every homomorphism of $G$ into $F$ has the desired prolongation.

Assume the prolongations exist and let $N$ be a finitely generated module satisfying (2). By [AF, Theorem 27.6], there are a finitely generated projective module $A$ and a superfluous submodule $B$ of $A$ such that $N = A/B$. In particular, $\text{Rad}(N) = \text{Rad}(A)/B$. By (2), 1.6 and [AF, Theorem 27.11], there exist positive integers $p$ and $q$ such that $A^{(p)}$ is a summand of $F^{(q)}$. Let $\xi \in \text{Hom}_R(G, N^{(p)})$. By (2), $\rho\phi = 0$, where $\varphi: N^{(p)} \to N^{(p)}/\text{Rad}(N^{(p)})$ is the projection. Hence, $\text{Im}(\xi) \subset \text{Rad}(N^{(p)}) = \text{Rad}(A^{(p)})/B^{(p)}$. Since $G$ is projective, there exists $\theta \in \text{Hom}_R(G, \text{Rad}(A^{(p)}))$ such that $\sigma\theta = \phi$, where $\sigma$ is the projection of $A^{(p)}$ onto $A^{(p)}/B^{(p)}$. Using the premise, it is easy to see that $\theta$ has a prolongation into a $\varphi \in \text{Hom}_R(F, A^{(p)})$. Thus, $\sigma\varphi\nu = \sigma\theta = \phi$, where $\nu$ is the inclusion of $G$ into $F$. This implies $\text{Ext}_R(P, N^{(p)}) = 0$, and [CM, Proposition 1.5] gives (1).

(ii) If $\text{Gen}(P_R)$ contains all simple modules, then (i) implies $G = 0$ and $P \in PG$.

Clearly, for any ring $R$, we have $PG \subset QPG \subset \text{STAR}$ and $PG \subset \text{ISTAR} \subset \text{STAR}$. Moreover, [C, Proposition 4.5 and Theorem 4.7] imply $QPG \cap \text{ISTAR} = PG$.

PROPOSITION 1.8. $QPG = PG$ iff $R$ is a simple completely reducible ring.

PROOF. Assume $QPG = PG$. Denote by $S$ the class of all simple modules. Clearly $S \subset QPG$, whence every simple module is projective
and $R$ is completely reducible. Moreover, since every element of $s$ is a generator, $R$ is simple. The opposite implication is obvious.

We turn to applications to particular classes of rings:

**Theorem 1.9.** $\text{ISTAR} = \text{PG}$ provided one of the following conditions is true:

(i) $R$ is a commutative ring,

(ii) $R$ is a local ring,

(iii) $R$ is a von Neumann regular ring.

**Proof.** (i) Let $P \in \text{ISTAR}$. By [CM, Proposition 1.5], $P$ is a faithful $\ast$-module. By 1.3(ii), $P$ is finitely generated and [CM, Theorem 2.3] shows that $P \in \text{PG}$.

(ii) By 1.6, since $\text{card} (B) = 1$.

(iii) By 1.3(i), since every finitely presented module is projective.

In view of 1.3 and 1.9(iii), it is surprising that even for von Neumann regular hereditary rings, the question whether the class $M \perp$ is closed with respect to countable direct powers for a non-projective module $M$, can be quite difficult to answer.

**Theorem 1.10.** Let $R$ be a simple right hereditary non-completely reducible von Neumann regular ring with $\text{card} (R) \leq \kappa_1$ (e.g. $R$ can be any simple countable non-completely reducible von Neumann regular ring). Then, for every module $M$, the class $M \perp$ is closed with respect to factors, but the assertion

\[ N^{(\kappa_0)} \notin M \perp \text{ whenever } M, N \in \text{mod-}R \text{ are such that } M \text{ is non-projective and } 0 \neq N \in M \perp \]

is independent of $\text{ZFC + GCH}$.

**Proof.** Since $R$ is right hereditary, $M \perp$ is closed with respect to factors for any module $M$.

Since $R$ is not right perfect, [ES, Corollary 2.2] implies that it is consistent with $\text{ZFC + GCH}$ that for every uncountable cardinal $\kappa$ such that $\text{card} (R) \leq \kappa$ and $\text{cf} (\kappa) = \kappa_0$ there is a non-projective module $M$ such that $\text{card} (M) = \kappa^+$ and $\text{Ext}_R (M, N) = 0$ for all modules $N$ with $\text{card} (N) < \kappa$. In particular, the negation of our assertion is consistent. On the other hand, assume the axiom of constructibility ($V = L$). We prove the assertion by induction on $\text{gen} (M) = \lambda$. 
If $\lambda < \aleph_0$, then $M = R^{(\lambda)}/I$ for an infinitely generated module $I$. Since $I$ is projective, there exists a countable $R$-independent set $\{x_n \mid n < \aleph_0\}$ generating a summand of $I$. Let $e_n$ be the non-zero idempotent of $R$ such that $\text{Ann}(x_n) = (1 - e_n)R$. Since $R$ is simple, there is some $0 \neq y_n \in Ne_n$ for all $n < \aleph_0$. Define $h \in \text{Hom}_R(I, N^{(\kappa_0)})$ by $hx_n = \pi_n y_n$, where $\pi_n$ is the $n$-th inclusion of $N$ into $N^{(\kappa_0)}$. Then $h$ does not extend into an element of $\text{Hom}_R(R^{(\lambda)}, N^{(\kappa_0)})$, whence $\text{Ext}_R(M, N^{(\kappa_0)}) \neq 0$.

If $\lambda = \aleph_0$, [T, Lemma 10.3] shows there exists a non-projective finitely generated submodule $F$ of $M$ such that $\text{Ext}_R(F, N) = 0$ and the induction works.

If $\lambda$ is a regular uncountable cardinal, then [T, Lemma 10.7] shows there is a $\lambda$-filtration $(C_\alpha \mid \alpha < \lambda)$ of $M$ such that $\text{Ext}_R(C_{\alpha+1}/C_\alpha, N) = 0$ for all $\alpha < \lambda$, and the set $E = \{\alpha < \lambda \mid C_{\alpha+1}/C_\alpha$ is non-projective$\}$ is stationary in $\lambda$. By the induction premise, $\text{Ext}_R(C_{\alpha+1}/C_\alpha, N^{(\kappa_0)}) \neq 0$ for all $\alpha \in E$. By [T, Lemma 10.6], this implies $\text{Ext}_R(M, N^{(\kappa_0)}) \neq 0$.

If $\lambda$ is singular, then the general compactness theorem [EM, Theorem IV.3.7] implies there is a non-projective submodule $U$ of $M$ such that $\text{gen}(U) < \lambda$ and $\text{Ext}_R(U, N) = 0$, and the induction works.

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