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Andrea Lucchini

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Generating the Augmentation Ideal.

ANDREA LUCCHINI (*)

Introduction.

Suppose that a finite group G can be generated by s subgroups, $H_1, \dots, H_s$, each of those being generatable by r elements. L. G. Kovács and Hyo-Seob Sim have recently proved [1] that if the orders of the subgroups H_i are pairwise coprime and the group G is solvable, then G can be generated by $r + s - 1$ elements. A similar result is also proved for the case when the indices are coprime. Their proofs make use of the theory, due to Gaschütz, of the chief factors in soluble groups.

In this paper we exploit a different technique, using properties of the group ring $\mathbb{Z}G$, and in particular the relation between the minimal number of generators for the group G and the minimal number of module generators $d_G(I_G)$ of the augmentation ideal I_G of $\mathbb{Z}G$. In this way we get more general results; in addition to that, our proofs are simpler.

Precisely we will prove the following results:

THEOREM 1. *If a finite group G is generated by s subgroups $H_1, \dots, H_s$ of pairwise coprime orders, and if for each H_i, $1 \leq i \leq s$, the augmentation ideal of $\mathbb{Z}H_i$ can be generated as H_i-module by r elements, then the augmentation ideal of $\mathbb{Z}G$ can be generated as G module by $r + s - 1$ elements.*

THEOREM 2. *If a finite group G has a family of subgroups whose indices have no common divisors, and if for each subgroup H in this family the augmentation ideal of $\mathbb{Z}H$ can be generated as H-module by d elements, then the augmentation ideal of $\mathbb{Z}G$ can be generated as G-module by $d + 1$ elements.*

(*) Indirizzo dell'A.: Dipartimento di Matematica Pura e Applicata, Via Belzoni 7, 35131 Padova, Italy.

The connection between the minimal number of generators $d(G)$ of a finite group G and $d_G(I_G)$ is given by the following result due to Roggenkamp ([2]):

$$d(G) = d_G(I_G) + pr(G),$$

where $pr(G)$ is a non negative integer, called the presentation rank of the group G, whose definition is introduced in the study of relation modules ([2], [3]).

The class of group with zero presentation rank is known to be large: for instance it contains all soluble groups ([2], Prop. 5.9), Frobenius groups ([6], Prop. 3.3), 2-generator groups ([2], Lemma 5.5). For all these groups one can get immediately:

COROLLARY. *Assume G is a finite group with zero presentation rank.*

i) *If G satisfies the hypotheses of Theorem 1, then $d(G) \leq r+s-1$.*

ii) *If G satisfies the hypotheses of Theorem 2, then $d(G) \leq d+1$.*

In the particular case when G is a finite soluble group, we find in this way alternative proofs for the results in [1].

In [1] the authors mentioned a connection between the study of these bounds for the number of generators and a question about pro-$\mathcal{C}$-groups, proposed in [4] by Ribes and Wong.

Precisely in [4] the equivalence of the following two facts is proved, for every class $\mathcal{C}$ of finite groups closed under the operations of taking subgroups, quotients and extensions:

i) For every pair H_1 and H_2 of pro-$\mathcal{C}$-groups, $d(H_1 \coprod H_2) = d(H_1) + d(H_2)$, where $H_1 \coprod H_2$ indicates the free pro-$\mathcal{C}$-product of H_1 and H_2 (i.e. an analogue of the Grushko-Neumann theorem for free products of abstract groups holds for free pro-$\mathcal{C}$-products of pro-$\mathcal{C}$-groups).

ii) For every pair H_1 and H_2 of finite groups in $\mathcal{C}$ there is a finite group H in $\mathcal{C}$ such that H_1 and H_2 are subgroups of H, the group H is generated by H_1 and H_2, and $d(H) = d(H_1) + d(H_2)$.

Whether (ii) is true or not for the class of finite groups is still open.

However in [1] it is remarked that if H_1, H_2 are soluble finite groups with coprime orders and $d(H_1) = d(H_2) = r$ then $d(H) \leq r+1$ for any soluble group H with $H = \langle H_1, H_2 \rangle$: so (ii) does not hold for $\mathcal{C}$ the class of soluble groups.

With the same argument, if H_1, H_2 are finite groups with coprime orders and with $d(H_1) = d(H_2) = r$, and if a finite group H exists such that $H = \langle H_1, H_2 \rangle$ and $d(H) = d(H_1) + d(H_2) = 2r$, then by Theorem 1, $d_H(I_H) \leqslant r + 1$, and consequently, $pr(H) \geqslant r - 1$. In particular this implies that if H_1, H_2 are not cyclic, then $pr(H) \neq 0$, and so $|H|$ is very large even if $|H_1|$ and $|H_2|$ are small (the smallest known group with non zero presentation rank has order 60^{20}); furthermore from ([5], Cor. 2.9) it follows that the Sylow 2-subgroup of H cannot be generated by less than $2r - 1$ elements.

Proof of Theorem 1.

To prove Theorems 1 and 2 we use that augmentation ideal is a Swan module, i.e. the following holds ([2], p. 188, Prop. 5.3):

$$d_G(I_G) = \max_{p/|G|} d_G(I_G/pI_G).$$

Therefore to prove Theorem 1 it is enough to prove:

PROPOSITION 1. *Suppose that G is a finite group generated by s subgroups $H_1, \ldots, H_s$ with pairwise coprime orders and that for each H_i, $1 \leqslant i \leqslant s$, the augmentation ideal of $\mathbb{Z}H_i$ can be generated as H_i-module by r elements. If p is a prime divisor of $|G|$, then $d_G(I_G/pI_G) \leqslant \leqslant r + s - 1$.*

PROOF. For each i, $1 \leqslant i \leqslant s$, define $e_i = \sum_{h \in H_i} (1 - h)$. We claim:

$$(*) \qquad e_i(1 - g) = |H_i|(1 - g) \quad \text{for any g in } H_i.$$

In fact

$$e_i(1 - g) = \sum_{h \in H_i} (1 - h)(1 - g) = \sum_{h \in H_i} 1 - \sum_{h \in H_i} h - \sum_{h \in H_i} g + \sum_{h \in H_i} hg =$$

$$= (\text{since } \sum_{h \in H_i} h = \sum_{h \in H_i} hg) = \sum_{h \in H_i} (1 - g) = |H_i|(1 - g).$$

Since the orders of the subgroups $H_1, \ldots, H_s$ are pairwise coprime, we may assume, without loss of generality, that p does not divide $|H_i|$, if $i \geqslant 2$.

By hypothesis the augmentation ideal of H_1 can be generated by r elements, say $f_1, \ldots, f_r$.

Now let M be the G-submodule of I_G/pI_G generated by the $r + s - 1$ elements $f_1 + pI_G, \ldots, f_r + pI_G$, $e_2 + pI_G, \ldots, e_s + pI_G$.

If x an element of H_1, then $1 - x + pI_G$ is contained in the G-submodule of I_G/pI_G generated by $f_1 + pI_G, \ldots, f_r + pI_G$, and so it is contained also in M. If x is an element of H_i $(i \geqslant 2)$ then $(e_i + pI_G)(1 - x + pI_G) \in$

$\in M$. But by ($*$) $(e_i + pI_G)(1 - x + pI_G) = |H_i|(1 - x) + pI_G$; on the other hand $(p, |H_i|) = 1$; so we may conclude again $1 - x + pI_G \in M$. Therefore M contains all the elements $1 - x + pI_G$, $x \in H_i$, $1 \leq i \leq s$. Since $G = \langle H_1, \ldots, H_s \rangle$, this implies immediately $M = I_G / pI_G$. ∎

Proof of Theorem 2.

We use again that I_G is a Swan module, so we have only to prove that $d_G(I_G/pI_G) \leq d + 1$ for every prime divisor p of $|G|$. We are assuming that G has a family of d-generator subgroups whose indices have no common divisor, so we may assume that G contains a d-generator subgroup, say H, whose index is not divisible by p.

The conclusion now follows from the following result:

PROPOSITION 2. *If a finite group G contains a d-generator subgroup H whose index is not divisible by p, then $d_G(I_G/pI_G) \leq d+1$.*

PROOF. By hypothesis I_H is generated as H-module by d elements $f_1, \ldots, f_d$.

Let $n = |G:H|$ and let $t_1 = 1, t_2, \ldots, t_n$ be a right transversal for H in G.

Define $f = \sum_{1 \leq i \leq n} 1 - t_i$.

Let M be the G-submodule generated by the $d + 1$ elements $f + pI_G$, $f_1 + pI_G, \ldots, f_d + pI_G$. Since $f_1, \ldots, f_d$ is a set of generators for I_H, M contains $1 - h + pI_G$ for any $h \in H$.

For each j such that $1 \leq j \leq n$, the module M also contains the elements $(f + pI_G)(1 - t_j + pI_G)$. Moreover, there exist elements $x_{i,j}$ in H and permutations $\tau(j)$ on $\{1, \ldots, n\}$ such that $t_i t_j = x_{i,j} t_{i\tau(j)}$ for all relevant i, j, and then

$$(f + pI_G)(1 - t_j + pI_G) = \sum_{1 \leq i \leq n} (1 - t_i)(1 - t_j) + pI_G =$$

$$= \sum_{1 \leq i \leq n} (1 - t_i - t_j + t_i t_j) + pI_G = \sum_{1 \leq i \leq n} (1 - t_{i\tau(j)} - t_j + x_{i,j} t_{i\tau(j)}) + pI_G =$$

$$= n(1 - t_j) - \sum_{1 \leq i \leq n} (1 - x_{i,j}) t_{i\tau(j)} + pI_G.$$

Since $1 - x + pI_G \in M$ for any x in H, each summand $(1 - x_{i,j}) t_{i\tau(j)} + pI_G$ lies in M, and so $n(1 - t_j) + pI_G \in M$; as $(n, p) = 1$ we may now conclude that $1 - t_j + pI_G \in M$.

Therefore M contains all the elements $1 - h + pI_G$, $h \in H$ and $1 - t_j + pI_G$, $1 \leq j \leq n$. On the other hand $G = \langle H, t_1, \ldots, t_n \rangle$ so we conclude $M = I_G / pI_G$. ∎

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