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A property equivalent to commutativity for infinite groups

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A Property Equivalent to Commutativity for Infinite Groups.

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According to a well-known result of B. H. Neumann, the class of groups with centre of finite index can be characterized as the class of groups all whose infinite subsets contain a pair of permutable elements [2].

Similar, possibly more general classes have been introduced by J. C. Lennox, A. M. Hassanabadi and J. Wiegold in [1]. If $G$ is a group and $n$ is a positive integer, they define an $n$-set in $G$ to be a subset of $G$ of cardinality $n$. The class $P_n^*$ is then defined by

$$G \in P_n^* \text{ if and only if every infinite set of } n\text{-sets in } G \text{ contains a pair } X, Y \text{ of different members such that } XY = YX.$$ 

In this terminology, $P_n^*$ is the class of centre-by-finite groups. In [1] it is shown that infinite groups in $P_n^*$ are abelian if $n = 2$ or $n = 3$. However, this result holds in general; in fact, we will prove the following

**Theorem.** Suppose $G$ is an infinite non-abelian group. For every integer $n > 1$ there is an infinite set of pairwise non-permutable $n$-sets in $G$.

1. – We will look first at a special case.

**Proposition.** If $n > 1$ and $G \in P_n^*$ has an element with finite centralizer, then $G$ is finite.

**Proof.** Assume, by contradiction, that $G \in P_n^*$ is an infinite group, $a \in G$, and $C_G(a)$ is finite. For $g \in G$, we denote $S(a, g)$ the set $\{x \in$ (*) Indirizzo dell'A.: Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, I-35131 Padova.
$E \in G | a^2 = g$: $S(a, g)$ is either empty or a right coset of $C_G(a)$, and in any case it is finite. Let $F$ be a fixed subset of $G$ of cardinality $n - 1$ and such that $a \in F$; the set $B = a^{-1}FF \cup \left( \bigcup_{b \in F} S(a, b) \right)$ is finite. We now define by induction a sequence of elements of $G$, beginning with an element $y_1 \notin F$: if $y_1, \ldots, y_{m-1}$ have already been defined, we choose $y_m$ in the complement of the finite set $B \cup \left( \bigcup_{1 \leq k < m} a^{-1}Fy_k \right) \cup \left( \bigcup_{1 \leq k < m} S(a, y_k) \right)$. Finally, for every integer $m$ we set $X_m = F \cup \{y_m\}$. All these $X_m$'s are $n$-sets in $G$, and $X_s \neq X_t$ if $s \neq t$. Fix now $1 \leq s < t$; certainly $ay_t \in X_sX_t$. On the other hand, $ay_t \notin X_sX_t = FF \cup Fy_s \cup \cup y_tF \cup \{y_ty_s\}$, since

$ay_t \in FF$ implies $y_t \in a^{-1}FF$,
$ay_t \in Fy_s$ implies $y_t \in a^{-1}Fy_s$ with $s < t$,
$ay_t \in y_tF$ implies $y_t \in S(a, b)$ for some $b \in F$,
$ay_t = y_ty_s$ implies $y_t \in S(a, y_s)$ with $s < t$,

and all of the above possibilities contradict our choice of the $y_m$'s.

2. - We come now to the proof of our theorem. Let $G$ be any infinite non-abelian group, and $n$ an integer, $n > 1$. By the Proposition, we may assume that $G$ satisfies the following condition: for every infinite subgroup $H$ of $G$ and every $h \in H$, $C_H(h)$ is infinite. It is immediate to check that this condition implies that every element of $G$ belongs to an infinite abelian subgroup of $G$. At this point, one can apply Theorem C of [1] and conclude. To get a self-contained proof, one can proceed as follows. Suppose $x, y \in G$ and $xy \neq yx$, and let $A$ be an infinite abelian subgroup of $G$ containing $y$; of course, $x \notin A$. Choose $a_1, \ldots, a_{n-1} \in A$ such that $a_1 = y$ and $a_1, a_2, \ldots, a_{n-1}$ are different elements of $G$. Now define a sequence of elements of $G$ by induction: take $c_1 = 1$ and, if $c_1, \ldots, c_{m-1}$ have already been defined, choose $c_m$ in the complement in $A$ of the finite set $\bigcup_{1 \leq k < m} a_{i-1}^{-1}\{a_1, \ldots, a_{n-1}\}c_k$.

For every $m$, set $Y_m = \{a_1, \ldots, a_{n-1}, c_mx\}$; then $|Y_m| = n$ for every $m$, and $Y_i \neq Y_j$ if $i \neq j$. Now for $i < j$ we have $a_1c_jx \in Y_iY_j$ but

$a_1c_jx \notin Y_iY_i = \{a_1, \ldots, a_{n-1}\}^2 \cup c_jx\{a_1, \ldots, a_{n-1}\} \cup \cup \{a_1, \ldots, a_{n-1}\}c_ix \cup \{c_jx, c_ic_jx\}$;

in fact: $a_1c_jx \in \{a_1, \ldots, a_{n-1}\}^2$ implies $x \in A$; $a_1c_jx \in c_jx\{a_1, \ldots, a_{n-1}\}$ implies $a_i^r = a_r$ for some $r$, $1 \leq r \leq n - 1$; $a_1c_jx \in \{a_1, \ldots, a_{n-1}\}c_ix$ implies $c_i \in a_{i-1}^{-1}\{a_1, \ldots, a_{n-1}\}c_i$ with $i < j$; and finally $a_1c_jx = c_jx, c_ic_jx$ implies $x \in A$; and all these contradict some earlier assumption.
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