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Chain conditions and continuous mappings on $C_p(X)$

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Chain Conditions and Continuous Mappings on $C_p(X)$.

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ABSTRACT - Let $X$, $Y$ Tychonoff spaces and $\mathcal{B}: C_p(X) \rightarrow C_p(Y)$ a one-to-one, continuous linear mapping. We prove that if $Y$ satisfies a certain kind of chain conditions (caliber, c.c.c. e.t.c.) so does $X$. As a consequence of this, we prove $\{0, 1\}^\tau$ ($\tau$ regular) cannot be embedded into $C_p(X)$, if $X$ has $\tau$ caliber. More generally, we prove that if $X$ has $\tau$ caliber then $C_p(X)$ does not contain compact subspaces of weight $\tau$. It follows, subject to GCH, that if $B$ is a Banach space and $(B, w)$ has $\omega_1$ and $\omega_2$ calibers then $B$ is separable. Finally we prove that $C_p(X)$ with $X$ dyadic of weight $\tau$ (of uncountable cofinality) does not admit a strictly positive measure.

All topological spaces are assumed to be infinite Tychonoff spaces. In the notations and terminology left unexplained below, we follow [4]. The symbols $X$, $Y$, $Z$ always denote spaces and the symbols $\tau$, $\lambda$ denote infinite cardinals. The cofinality of a cardinal $\tau$, denoted by $\text{cf} \tau$, is the least ordinal $\beta$, such that $\tau$ is the cardinal sum of $\beta$ many cardinals each smaller than $\tau$. A cardinal $\tau$ is regular if $\tau = \text{cf} \tau$. The symbol $\mathbb{N}$ stands for the set of all positive integers and the symbols $k$, $l$, $m$, $n$ are used only to denote members of $\mathbb{N}$. Further $d$ is the density, $w$ is the weight and $|\cdot|$ is the cardinality. A space $X$ satisfies $\tau$.c.c. if there is no family $\gamma \subset \mathcal{F}^*(X)$ (the set of all non-empty, open subsets of $X$) of pairwise disjoint elements with $|\gamma| = \tau$. We set c.c.c. for $\omega_1$.c.c.. A space $X$ has $(\tau, \lambda)$ caliber (pre-caliber) if for every family $\gamma \subset \mathcal{F}^*(X)$ with $|\gamma| = \tau$ there is a subfamily $\gamma_1 \subset \gamma$ with $|\gamma_1| = \lambda$ and $\cap \gamma_1 \neq \emptyset$ ($\gamma_1$ is centered). We set $\tau$ caliber for $(\tau, \tau)$ caliber. A space $X$ satisfies property $K$, if for every family $\gamma \subset \mathcal{F}^*(X)$ with $|\gamma| = \tau$, there is a subfamily $\gamma_1 \subset \gamma$ with $|\gamma_1| = \tau$ with the 2-intersection property. It is well known that if $X$ has

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\( \tau \) caliber then \( X \) has also \( cf \tau \) caliber, and so \( cf \tau > \omega \). If \( F = \{ U_1, \ldots, U_n \} \) is a non-empty, finite subfamily of \( \mathcal{F}^*(X) \), then \( \text{cal}(F) \) is the largest \( \kappa \) such that, there is \( S \subset F \) with \( |S| = \kappa \) and \( \cap S \neq \emptyset \). If \( J \subset \mathcal{F}^*(X) \) then \( \kappa(J) = \inf \{ \text{cal}(F)/|F| : F \subset J, \text{finite} \} \). A space \( X \) satisfies property (**) if \( \mathcal{F}^*(X) \) can be written in the form, \( \mathcal{F}^*(X) = \bigcup_{n \in \mathbb{N}} \mathcal{J}_n \), with \( \kappa(\mathcal{J}_n) > 0 \), for all \( n = 1, 2, \ldots \). A space \( X \) admits a strictly positive measure, if there is a non-negative Borel probability measure \( \mu \) on \( X \) with \( \mu(U) > 0 \) for all non-empty open \( U \). For a compact space, Kelley (see [4]), proved that property (**) is equivalent with the existence of a strictly positive measure on the space. It is well known that if \( X \) admits a strictly positive measure then it has \((\tau, \omega)\) caliber for every cardinal \( \tau \) with \( cf \tau > \omega \), [4].

If \( X \) is a space, \( C_p(X) \) is the space of all real-valued continuous functions on \( X \) with the topology of pointwise convergence. For different points \( x_1, \ldots, x_\kappa \) in \( X \) and \( E_1, \ldots, E_\kappa \) non-empty, open intervals of \( \mathbb{R} \), let

\[
V(x_1, \ldots, x_\kappa : E_1, \ldots, E_\kappa) = \{ f \in C_p(X) : f(x_i) \in E_i, \text{ for } i = 1, \ldots, \kappa \}.
\]

It is clear that \( V(x_1, \ldots, x_\kappa : E_1, \ldots, E_\kappa) \) form a base of \( C_p(X) \). It is well known that \( C_p(X) \) is a dense subspace of \( \mathbb{R}^{[X]} \) the set of all real-valued functions on \( X \) with the topology of pointwise convergence. It follows from well known properties of \( \mathbb{R}^{[X]} \), that \( C_p(X) \) has pre-caliber \( \tau \), for every cardinal \( \tau \) with \( cf \tau > \omega \) and also satisfies property (**) In the case of a compact space \( X \) with \( w(X) = \tau \) and \( cf \tau > \omega \), Arhangel’skii and Tkacuk in [3], proved that \( C_p(X) \) does not have \( cf \tau \) caliber and also by a result of Tulcea [9], it follows that \( C_\omega(X) \) (the space \( C(X) \) with the weak topology) does not admit a strictly positive measure, although it satisfies property (**).

For \( A \subset X \) and \( f \in C_p(X) \) we set \( f|_A \) for the restriction of \( f \) on \( A \), and \( \text{supp} f = \{ x \in X : f(x) \neq 0 \} \) for the support of \( f \).

**THEOREM 1.** Let \( \vartheta : C_p(X) \to C_p(Y) \) be a 1-1, continuous mapping. Then we have the following:

(a) \( d(X) \leq d(Y) \),

(b) Let \( \tau, \lambda \) be cardinals with \( \tau \) regular, \( \tau \geq \lambda \) and \( cf \lambda > \omega \).

We suppose that \( Y \) has \((\tau, \lambda)\) caliber. Then \( X \) has \((\tau, \lambda)\) caliber.

**PROOF.** (a) We can suppose that \( \vartheta(0) = 0 \). Let \( D \) be a dense subset of \( Y \). For every \( y \in D \) and \( n \in \mathbb{N} \), it follows from the continuity of \( \vartheta \) at \( 0 \in C_p(X) \) that there exist \( x_1^{y, n}, \ldots, x_{2^n}^{y, n} \), pairwise different elements of
We claim that the set $A = \bigcup_{y \in D} \bigcup_{n \in \mathbb{N}} \{x^{y, n}_{k, n}, \ldots, x^{y, n}_{k, n}\}$ is dense in $X$ and since $|A| \leq |D|$, $(a)$ follows. Indeed, let $x_0 \in X \setminus \overline{A}$, then there exists $f \in C_p(X)$ with $f(x_0) \neq 0$ and $f|_{\overline{A}} = 0$. But then $f|_{\{x^{y, n}_{k, n}, \ldots, x^{y, n}_{k, n}\}} = 0$ for every $y \in D$ and $n \in \mathbb{N}$ and so $\partial(f)|_D = 0$ so $\partial(f) = 0$. This is contradiction since $\partial$ is one-to-one.

(b) Let $\{U_i : i < \tau\} \subset \partial^*(X)$. For every $i < \tau$ we choose $f_i \in C_p(X)$ with $f_i \neq 0$ and $\text{supp} f_i \subset U_i$ and set $V_i = \{y \in Y : \partial(f_i)(y) \neq 0\}$. From the regularity of $\tau$, it follows that either exists $\tau V_i$'s equal elements or $\tau$ pairwise different. In both cases, since $Y$ has $(\tau, \lambda)$ caliber, it follows that there exists $\Lambda \subset \tau$, $|\Lambda| = \lambda$ and $y_0 \in \cap \{V_i : i \in \Lambda\}$. Since $\partial f_i(y_0) \neq 0$ for every $i \in \Lambda$ it follows that either there exists $\Lambda_1 \subset \Lambda$, $|\Lambda_1| = \lambda$ and $r_1 > 0$ such that $\partial(f_i)(y_0) \geq r_1$, for every $i \in \Lambda_1$, or $\Lambda_2 \subset \Lambda$, $|\Lambda_2| = \lambda$ and $r_2 < 0$ such that $\partial(f_i)(y_0) \leq r_2$ for every $i \in \Lambda_2$. We can suppose that we have the first. From the continuity of $\partial$ at $0 \in C_p(X)$ there exist $x_1, \ldots, x_\kappa$ pairwise different elements of $X$, and $E_1, \ldots, E_\kappa$ open intervals of $\mathbb{R}$ containing $0 \in \mathbb{R}$, such that

$$
\partial(V(x_1, \ldots, x_\kappa : E_1, \ldots, E_\kappa)) \subset V(y_0 : (-r_1, r_1)).
$$

Then $f_i \notin V(x_1, \ldots, x_\kappa : E_1, \ldots, E_\kappa)$ for every $i \in \Lambda_1$ and so $\{x_1, \ldots, x_\kappa\} \cap U_i \neq 0$ for every $i \in \Lambda_1$. Now (b) follows immediately.

REMARK. We note that the $(a)$ of the above theorem follows also from well known results [7]. We also note that in the (b) of the above theorem the assumption that $Y$ has $(\tau, \lambda)$ caliber cannot be relaxed to have pre-caliber. Indeed, since $C_p(X)$ is contained homeomorphically into $C_p(C_p(C_p(X)))$ and $C_p(C_p(X))$ has $\tau$ pre-caliber if $\text{cf} \tau > \omega$, however $X$ in general does not satisfy c.c.c..
$C_p(C_p(X))$. The «if» part follows from the fact that $C_p(X)$ is contained homeomorphically into $C_p(C_p(C_p(X)))$ and Th. 1(b).

**COROLLARY 3.** Let $X$ be a space with $\tau$ caliber, $\tau$ regular. Let, also, $\delta_i \in \mathbb{R}^\tau$ with $\delta_i = (\delta_{ij})_{j < \tau}$, $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Then there is no, one-to-one continuous mapping from the compact subspace $\{\delta_i: i < \tau\} \cup \{0\}$ of $\mathbb{R}^\tau$ into $C_p(X)$ and so there is no, one-to-one, continuous mapping from $\{0, 1\}^\tau$ into $C_p(X)$.

**PROOF.** On the discrete space $\tau$, we consider the family $\{\{i\}: i < \tau\}$ and the set of continuous functions $\{\delta_i: i < \tau\}$ and repeat the argument of Th. 1(b).

**REMARK.** In the case that $X$ is compact the above corollary follows also from well known arguments. Indeed in the case that $X$ is compact and has $\tau$ caliber, if there exists $\delta: \{\delta_i: i < \tau\} \cup \{0\} \to C_p(X)$, a one-to-one continuous mapping then $\delta(\{\delta_i: i < \tau\} \cup \{0\})$ would be a compact subspace of $C_p(X)$, of weight $\tau$, contradiction (see [3]). Also, if $X$ is compact and $\{0, 1\}^\tau \subseteq C_p(X)$ homeomorphically then $\{0, 1\}^\tau$ would be Eberlein compact and since it satisfies c.c.c., would be metrizable [6].

In connection with the above we prove the following stronger result.

**THEOREM 4.** Let $X$ be a space and we suppose that there exists some $F \subset C_p(X)$ compact with $\omega(F) = \tau$ and $\text{cf} \tau > \omega$. Then $X$ has not $\text{cf} \tau$ caliber.

**PROOF.** Let $\{\mu_j: j < \tau\}$ be a $||\cdot||$-dense subset of $C(F)$. We claim that for every $i < \tau$, there exist $f_i, g_i \in F$, $f_i \neq g_i$ and $\mu_j(f_i) = \mu_j(g_i)$ for all $j < i$. This follows easily from Stone-Weierstrass Theorem. Since $f_i \neq g_i$, there exist $r_i \in \mathbb{Q}$, $\delta_i > 0$ such that either

$$f_i^{-1}(-\infty, r_i) \cap g_i^{-1}(r_i + \delta_i, +\infty) \neq \emptyset,$$

or

$$g_i^{-1}(-\infty, r_i) \cap f_i^{-1}(r_i + \delta_i, +\infty) \neq \emptyset.$$

Since $\text{cf} \tau > \omega$, without loss of generality, we can suppose that there exist $A \subset \tau$, $|A| = \tau$ and $r \in \mathbb{Q}$, $\delta > 0$ such that

$$U_i = f_i^{-1}(-\infty, r) \cap g_i^{-1}(r + \delta, +\infty) \neq \emptyset,$$

for every $i \in A$. Let $\{i_\eta: \eta < \text{cf} \tau\} \subset A$ with $i_\eta < i_{\eta'}$, if $\eta < \eta' < \text{cf} \tau$ and $\sum_{\eta < \text{cf} \tau} i_\eta = \tau$. We
suppose, if possible, that $X$ has $\text{cf} \tau$ caliber. Then there is a cofinal $B \subset \{i_\gamma: \gamma < \text{cf} \tau\}$ with $|B| = \text{cf} \tau$ and $\cap \{U_i: i \in B\} \neq \emptyset$. In case that there are no $\text{cf} \tau$ pairwise different elements in $\{U_i: \gamma < \text{cf} \tau\}$, this follows immediately. Otherwise this follows from the assumption that $X$ has $\text{cf} \tau$ caliber. Let $x \in \cap \{U_i: i \in B\}$. Then $\delta_x \in C(F)$ and so there exists $i_0 < \tau$ such that

$$\|\mu_{i_0} - \delta_x\| < \frac{\delta}{4}.$$ 

If $i \in B$ with $i > i_0$ we have $\mu_{i_0}(f_i) = \mu_{i_0}(g_i)$ and so $|f_i(x) - g_i(x)| < \delta/2$ contradiction, since $f_i(x) \in (-\infty, r)$ and $g_i(x) \in (r + \delta, +\infty)$.

**Corollary 5.** (Arhangel’skii and Tkacuk, [3]). Let $X$ be a compact space with $w(X) = \tau$ and $\text{cf} \tau > \omega$. Then $C_p(X)$ does not have $\text{cf} \tau$ caliber.

**Proof.** It follows from the fact that $X$ is contained homeomorphically into $C_p(C_p(X))$ and Th. 4.

**Corollary 6.** Suppose that $2^{\omega_1} = \omega_2$. Then we have the following:

(a) If $X$ has $\omega_1$ and $\omega_2$ calibers, every compact $F \subset C_p(X)$ is metrizable.

(b) If $X$ is compact and $C_p(X)$ has $\omega_1$ and $\omega_2$ calibers then $X$ is metrizable.

**Proof.** (a) We claim that $F$ is separable. If not, there exists $\{f_i: i < \omega_1\} \subset F$ such that $f_j \notin \{f_i: i < j\}$. Then $w(\{f_i: i < \omega_1\}) = 2^{\omega_1} = \omega_2$ and so $w(\{f_i: i < \omega_1\}) = \omega_1$ or $\omega_2$ contradiction by Corol. 5. Therefore $F$ is separable and so $w(F) \leq 2^{\omega_1} \leq 2^{\omega_1} = \omega_2$. It follows as before from Corol. 5 that $F$ is metrizable.

(b) It follows from the fact $X \subset C_p(C_p(X))$ homeomorphically and (a).

**Note 1.** I have recent information that Th. 4 follows also Corol. 5 and results in [8].

**Note 2.** The (b) of the above corollary has already been proved in [3].

**Note 3.** The above theorem is not valid if the assumption $\tau$ caliber is relaxed to pre-caliber. Indeed, in general $X \subset C_p(C_p(X))$ and $C_p(X)$ has $\tau$ pre-caliber if $\text{cf} \tau > \omega$, although $X$ may have weight $\tau$. 
Theorem 7 (GCH). If $B$ is a Banach space, such that the space $(B, w)$ has $\omega_1$ and $\omega_2$ calibers, then $B$ is separable.

Proof. It is well known that $(S_{B^*}, w^*)$ the unit ball of $B^*$ with the $w^*$-topology is contained homeomorphically into $C_p(B, w)$, and also that $B$ is contained isometrically into $C(S_{B^*}, w^*)$. The result follows from Corol. 6.

For a set $\Gamma$ we set $\Sigma(\mathbb{R}^\Gamma) = \{t \in \mathbb{R}^\Gamma : \{\gamma : t(\gamma) \neq 0\} \text{ is countable}\}$ with the relative topology in $\mathbb{R}^\Gamma$.

Proposition 7. We assume that $C_p(X)$ has $\omega_1$ caliber and there exists $\partial : C_p(X) \to \Sigma(\mathbb{R}^\Gamma)$, a 1-1, continuous mapping. Then $X$ is separable.

Proof. We claim that there exists a countable $A \subset \Gamma$ such that $\partial(f)(\gamma) = 0$ for every $\gamma \in \Gamma \setminus A$. Indeed, if not, there exists $\gamma_\xi, \xi < \omega_1$ in $\Gamma$ and $f_\xi, \xi < \omega_1$ in $C_p(X)$ with $\partial(f_\xi)(\gamma_\xi) > 0$ for every $\xi < \omega_1$. We may suppose that $\partial(f_\xi) \supset r$ for some $r \in \mathbb{R}$. Since $C_p(X)$ has $\omega_1$ caliber there exist $B \subseteq \omega_1, |B| = \omega_1$ and $f \in C_p(X)$ such that $\partial(f) \in \bigcap_{\xi \in B} V(\gamma_\xi : (r, + \infty))$ contradiction, because $\partial(f) \in \Sigma(\mathbb{R}^\Gamma)$.

It follows that the continuous mapping $f \to \partial(f)|_A$ remains 1-1, and the result follows from Th. 1(a).

In Corol. 2 we have that $C_p(\mathbb{R}^\tau)$ has not $\tau$ caliber, if $\tau$ is an uncountable regular cardinal. In connection with this we have the following stronger result.

Proposition 8. Let $\tau$ be a cardinal with $\text{cf}\; \tau > \omega_1$ then the space $C_p(\mathbb{R}^\tau)$, does not have $(\tau, \omega)$ caliber (and so it does not admit a strictly positive measure).

Proof. For every $i < \tau$, let $\delta_i \in \mathbb{R}^\tau$ with $\delta_i = (\delta_{ij})_{j < \tau}$, and $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Then the family

$$V(\delta_i, \delta_{i+1} : (0, 1), (2, 3))$$

does not contain an infinite subfamily with non-empty intersection. Indeed, let $A \subset \tau$ and $f \in \bigcap_{i \in A} V(\delta_i, \delta_{i+1} : (0, 1), (2, 3))$. We set $0 = (0)_i < \tau$.

We may assume that $f(0) \notin (0, 1)$. We consider $r > 0$ with $(f(0) - r, f(0) + r) \cap (0, 1) = 0$. From the continuity of $f$ at $0$ there exist $i_1, \ldots, i_\kappa$ in $\tau$ and $I_1, \ldots, I_\kappa$ open intervals of $\mathbb{R}$, containing $0$ such that $f(\pi^{-1}_i(I_1) \cap \cdots \cap \pi^{-1}_\kappa(I_\kappa)) \subset (f(0) - r, f(0) + r)$. If $i \in A \setminus \{i_1, \ldots, i_\kappa\}$ then $\delta_i \in \pi^{-1}_i(I_1) \cap \cdots \cap \pi^{-1}_\kappa(I_\kappa)$ and so $f(\delta_i) \notin (0, 1)$, contradiction.
COROLLARY 9. Let $X$ be a dyadic space with $w(X) = \tau$ and $\text{cf} \tau > \omega$. Then $\mathbb{C}_p(X)$ does not have $(\tau, \omega)$ caliber, so it does not admit a strictly positive measure.

PROOF. By a result of Efimov, [5], it follows that $\{0, 1\}^\tau \subseteq X$. If we repeat the proof of Prop. 8 we can prove that $\mathbb{C}_p(\{0, 1\}^\tau)$ does not have $(\tau, \omega)$ caliber. Now the mapping $\varnothing: \mathbb{C}_p(X) \to \mathbb{C}_p(\{0, 1\}^\tau)$ with $\varnothing(f) = f|_{\{0, 1\}^\tau}$, is continuous, linear and onto. So if $\mathbb{C}_p(X)$ had $(\tau, \omega)$ caliber, then $\mathbb{C}_p(\{0, 1\}^\tau)$ would have $(\tau, \omega)$ caliber, contradiction.

The above theorem gives a partial answer to the following.

PROBLEM. Is there a non-metrizable compact Hausdorff space such that $\mathbb{C}_p(X)$ has a strictly positive measure?

D. Fremlin in note of 10 Oct. 1989 proved that this problem is connected with the following

PROBLEM. (A. Bellow). Is there a probability space $(Z, \Sigma, \nu)$ with a $Y \subseteq \ell^0(\Sigma)$ (= the space of real-valued measurable functions on $Z$) such that $Y$ is compact and non-metrizable in the topology of pointwise convergence and any pair of distinct members of $Y$ differ on a non-negligible set?

THEOREM 10. Let $\varnothing: \mathbb{C}_p(X) \to \mathbb{C}_p(Y)$ be a 1-1, continuous, linear mapping and $\tau$ be an uncountable regular cardinal. Then we have the following implications.

(a) If $Y$ has $(\tau, \omega)$ caliber, so does $X$.

(b) If $Y$ admits a strictly positive measure then $X$ has $(\tau, \omega)$ caliber and satisfies property $K_\tau$.

PROOF. (a) Let $\{U_i: i < \tau\}$ be a family of non-empty open sets in $X$. For every $i < \tau$ we find $f_i \in C_p(X)$ with $\text{supp} f_i \subseteq U_i, f_i \neq 0$ and set $V_i = \{y \in Y: \varnothing(f_i)(y) \neq 0\}$. From the regularity of $\tau$, it follows that either there exist $\tau$ $V_i$’s equal elements, or $\tau$ pairwise different. In both cases there exists $\Lambda \subseteq \tau$, infinite and $y_0 \in \cap \{V_i: i \in \Lambda\}$. Then there exist $x_1, \ldots, x_\kappa$ pairwise different elements in $X$, and $E_1, \ldots, E_\kappa$ open intervals of $\mathbb{R}$ each containing 0 such that

$$\varnothing(V(x_1, \ldots, x_\kappa: E_1, \ldots, E_\kappa)) \subseteq V(y_0: (-1, 1)).$$

We claim that if $f|_{\{x_1, \ldots, x_\kappa\}} = 0$ then $\varnothing(f(y_0)) = 0$. Indeed, if $\varnothing(f(y_0)) \neq 0$, there is a $\lambda \in \mathbb{R}$ such that $\varnothing(\lambda f)(y_0) = \lambda \varnothing(f)(y_0) \notin (-1, 1)$, by the linearity of $\varnothing$, but $\lambda f \in V(x_1, \ldots, x_\kappa: O_1, \ldots, O_\kappa)$, contradiction.
(b) If $Y$ admits a strictly positive measure $\mu$, then $Y$ has $(\tau, \omega)$ caliber and so $X$ has $(\tau, \omega)$ caliber by (a).

In the following we shall prove that $X$ satisfies property $K_\tau$. We suppose, if possible, that there exists a family $\{U_i: i < \tau\}$ of non-empty, open subsets of $X$ which does not contain subfamily of the same cardinality with the 2-intersection property. Let $f_i$, $V_i$, $i < \tau$ as in (a). We can suppose that $\mu(V_i) \geq \delta$ for all $i < \tau$, for some $\delta > 0$.

For every $A \subset \tau$, $|A| = \tau$ we set

$$\mathcal{A}_A = \{B \subset A: \text{the family } \{U_i: i \in B\} \text{ has the 2-int-property}\}.$$ 

The set $\mathcal{A}_A$ is non-empty by (a), partially ordered by inclusion and satisfies the assumptions of Zorn’s Lemma. Let $B_A$ be maximal. Then $|B_A| < \tau$. For every $i \in A \setminus B_A$ there exists $j_i \in B_A$ with $U_i \cap U_{j_i} = \emptyset$. Then

$$A = B_A \cup \bigcup_{j \in B_A} \{i \in A: j_i = j\}.$$ 

From the regularity of $\tau$, there exists $j_1 \in B_A$, such that the set $A_1 = \{i \in A: j_i = j_1\}$ has cardinality $\tau$. We repeat the same argument with $A_1$ in place of $A$.

Inductively we find $j_1, j_2, \ldots, j_m, \ldots$ pairwise different elements of $\tau$, such that $U_{j_l} \cap U_{j_m} = \emptyset$, for every $l \neq m$, $l, m = 1, 2, \ldots$. Now since $\mu(V_{j_l}) \geq \delta$, $l = 1, 2, \ldots$ it follows that there exists a $B \subset \mathbb{N}$, infinite with $\bigcap_{l \in B} V_{j_l} \neq \emptyset$. Now similar arguments as in (a) lead to contradiction.

**Corollary 11.** Let $\tau$ be an uncountable regular cardinal. If $X$ admits a strictly positive measure, there is no, 1-1, linear continuous mapping from $\mathbb{R}^\tau$ into $C_p(X)$.

**Remark.** Let $X, Y$ be compact spaces and $\varphi: C_p(X) \to C_p(Y)$ be a one-to-one, continuous linear mapping, then the mapping $\varphi: (C(X), \|\|) \to (C(Y), \|\|)$ is also continuous (see Arhangel’skii [2]). However the existence of a 1-1, continuous linear mapping $\varphi: (C(X), \|\|) \to (C(Y), \|\|)$ does not imply the existence of a 1-1, continuous $\varphi: C_p(X) \to C_p(Y)$. By Dixmier’s Theorem $L^\infty[0, 1] = C(\Omega)$ where $\Omega$ is a compact extremelly disconnected space. On the other hand $L^\infty[0, 1]$ is isomorphic to $C(\beta\mathbb{N})$. The space $\Omega$ is not separable, so from Th. 1(a) there exists no a one-to-one, continuous $\varphi: C_p(\Omega) \to C_p(\beta\mathbb{N})$.
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