

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 87 (1992), p. 139-149

http://www.numdam.org/item?id=RSMUP_1992__87__139_0

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On the Structure of the Normal Subgroups of a Group: Supersolubility.

JAMES C. BEIDLEMAN - DEREK J. S. ROBINSON (*)

1. Introduction.

A group G will be said to have *property* σ if, whenever N is a non-supersoluble normal subgroup of G , there is a normal subgroup L of G such that $L \leq N$ and N/L is finite and non-supersoluble. That is, every non-supersoluble normal subgroup must have a finite non-supersoluble G -quotient.

In a previous paper [3], of which the present work is a continuation, a weaker property called ν was studied: a group G has *property* ν if every non-nilpotent normal subgroup of G has a finite non-nilpotent G -quotient. In particular, groups with property ν were characterized in terms of their Fitting subgroups and φ_f -subgroups: here $\varphi_f(G)$ is the intersection of the maximal subgroups of finite index in a group G , or G itself, should there be no such subgroups.

Groups with property σ are of interest from several points of view. In the first place there is an algorithm to decide if normal subgroups are supersoluble (see Corollary 1). In addition they exhibit extremely good φ_f -behaviour, by which we mean that many of the standard properties of the Frattini subgroup of a finite group that relate to supersolubility or nilpotency also hold for groups with σ . Finally there are several interesting characterizations of groups with property σ , as well as a sometimes elusive connection with property ν .

Obvious examples of groups with property σ are finite groups and supersoluble groups. Less obviously all polycyclic-by-finite groups have σ ; this is a consequence of the theorem of Baer ([1] or [10], p. 162) that a polycyclic group whose finite quotients are supersoluble is itself

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supersoluble. Baer's theorem will play a fundamental role in our investigation.

Other examples of groups with property σ are free groups (see Theorem 5) and \mathbb{Z} -linear groups (see Corollary 3).

We shall mention the main characterizations of property σ obtained here. As in the case of property ν the crucial invariant of group G are the Fitting and φ_f -subgroups, $\text{Fit}(G)$ and $\varphi_f(G)$ respectively.

THEOREM 1. A group G has property σ if and only if the following conditions hold:

- (a) $\varphi_f(G) \leq \text{Fit}(G)$ and $\text{Fit}(G)$ is finitely generated;
- (b) $\text{Fit}(G/\varphi_f(G)) = \text{Fit}(G)/\varphi_f(G)$.

This follows from the main characterization of property ν obtained in [3], together with

THEOREM 2. A group G has property σ if and only if it has property ν and $\text{Fit}(G)$ is finitely generated.

An interesting feature of these characterizations is the absence of the word «supersoluble» from the statements. This paradox is explained by a further result, which may be regarded as a reformulation of the definition of property σ .

THEOREM 3. A group G has property σ if and only if each normal subgroup which is not finitely generated nilpotent has a finite non-nilpotent G -quotient.

Property σ is much more restrictive than property ν . One indication of this is the fact that the only soluble groups with σ are the polycyclic groups. On the other hand, it was shown in [3] that there are many soluble groups with ν which are not polycyclic.

Another significant difference between σ and ν occurs in the behaviour of linear groups.

THEOREM 7. Let R be a finitely generated domain, and let G be an R -linear group. Then G has property σ if and only if the unipotent radical of G is finitely generated.

Thus, only if the additive group of R is finitely generated, can one expect all R -linear groups to have property σ . On the other hand, every R -linear group has property ν ([10], p. 60).

Each group with σ has a unique maximum soluble normal subgroup, and this is polycyclic. In fact the study of groups with σ reduces to that of groups in which this soluble radical is trivial.

THEOREM 12. A group G has property σ if and only if it has a polycyclic normal subgroup S such that $\text{Fit}(G/S) = 1 = \varphi_f(G/S)$.

An even stronger reduction can be made for locally finite groups.

THEOREM 9. A locally finite group G has property σ if and only if there is a finite normal subgroup F such that G/F is residually finite and $HP(G/F)$, the Hirsch-Plotkin radical, is trivial.

Thus residually finite, locally finite groups without non-trivial locally nilpotent normal subgroups are the objects that need to be studied.

Notation.

$\varphi_f(G)$: the intersection of the maximal subgroups of finite index in G .

$\text{Fit}(G)$: the Fitting subgroup.

$HP(G)$: the Hirsch-Plotkin radical.

$Z(G)$: the centre.

$Z_i(G)$: the i -th term of the upper central series.

H_G : the normal core of H in G .

2. An algorithm to detect supersolubility.

We begin the study of property σ with a reformulation of the definition which is advantageous for the development of an algorithm to detect supersolubility of normal subgroups.

LEMMA 1. A group G has property σ if and only if whenever N is a non-supersoluble normal subgroup of G , there is a normal subgroup F with finite index in G such that NF/F is not supersoluble.

PROOF. The condition certainly implies that G has σ . Conversely, let G have σ and suppose that $N \triangleleft G$ is not supersoluble. Then there exists $L \triangleleft G$ with $L \leq N$, and N/L finite and non-supersoluble. Put $F = C_G(N/L)$; then $F \triangleleft G$ and G/F is finite. If NF/F were supersoluble, N/L would be centre-by-supersoluble, and hence supersoluble.

COROLLARY 1. Let G be a finitely presented group with property σ . Then there is an algorithm which, when a finite subset $\{x_1, \dots, x_k\}$ of G is given, together with the information that $N = \langle x_1, \dots, x_k \rangle$ is normal in G , decides whether N is supersoluble.

PROOF. We shall describe two recursive procedures designed to detect non-supersolubility and supersolubility of N . The first procedure simply enumerates all finite quotients G/F and tests NF/F for supersolubility. If this ever fails to be true, the procedure stops; N is not supersoluble.

The second procedure attempts to construct a cyclic normal series in N . It enumerates all finite sets of words $\{w_1, \dots, w_m\}$ in x_1, \dots, x_k . At the same time it enumerates all words of the form $w_i^{-l} w_i^{w_j} u$ and $w_i^{-l'} w_i^{w_j^{-1}} u'$ for $i < j = 1, 2, \dots, m, l, l' \in \mathbb{Z}, u, u' \in \langle w_1, \dots, w_{i-1} \rangle$, and also all consequences of the defining relators. If it is found that for all i, j some $w_i^{-l} w_i^{w_j} u$ and $w_i^{-l'} w_i^{w_j^{-1}} u'$ are relators then $W = \langle w_1, \dots, w_m \rangle$ is supersoluble. The next step is to test if $W = N$. To do this enumerate all $x_i w^{-1}$ with $i = 1, \dots, k$ and w in W , and also all consequences of the defining relators. If for each i some $x_i w^{-1}$ is a relator, then $W = N$ and the procedure stops; N is supersoluble.

By Lemma 1 one of these procedure will terminate.

3. Characterizations of property σ .

The first two theorems of this section provide the principal characterizations of groups with property σ .

THEROEM 1. A group G has property σ if and only if the following conditions hold:

- (a) $\varphi_f(G) \leq \text{Fit}(G)$, and $\text{Fit}(G)$ is finitely generated;
- (b) $\text{Fit}(G/\varphi_f(G)) = \text{Fit}(G)/\varphi_f(G)$.

It was shown in Theorem 1 of [3] that if the requirement that $\text{Fit}(G)$ be finitely generated is omitted from (a), then what remains are necessary and sufficient conditions for G to have property ν . Therefore Theorem 1 will be a consequence of the following result.

THEOREM 2. A group G has property σ if and only if it has property ν and $\text{Fit}(G)$ is finitely generated.

PROOF. Assume that G has property σ , and let N be a non-nilpotent normal subgroup of G . If N is not supersoluble, then it has a finite G -

quotient which is not supersoluble, and hence not nilpotent. On the other hand, if N is supersoluble, a well-known theorem of Hirsch ([4], Theorem 3.26) implies that there is a finite non-nilpotent quotient N/L , of order r say. Then N/N^r is a finite non-nilpotent G -quotient. Hence G has property ν . Since all finite quotients of $\text{Fit}(G)$ are nilpotent, and hence supersoluble, it follows that $\text{Fit}(G)$ is supersoluble, and therefore finitely generated.

Conversely, assume that G has property ν , and that $\text{Fit}(G)$ is finitely generated. Thus $\text{Fit}(G)$ is nilpotent. Let N be a non-supersoluble normal subgroup of G all of whose finite G -quotients are supersoluble. We first show that N' is nilpotent. To this end, consider a finite G -quotient of N' , say N'/L , and put $C = C_N(N'/L)$. Then $C \triangleleft G$ and N/C is finite, and hence supersoluble. Consequently $N'/N' \cap C$ is nilpotent. But $[N' \cap C, C] \leq L$, so that N'/L is nilpotent. Since G has property ν , it follows that N' is nilpotent, as claimed. Therefore $N' \leq \text{Fit}(G)$, and N' is finitely generated.

Next write $K = C_N(N')$. Now N/K is a soluble group of automorphisms of a finitely generated nilpotent group, so it is polycyclic by a theorem of Mal'cev (see [8], 3.27). Clearly K is nilpotent, so it is finitely generated and therefore N is polycyclic. It now follows from the theorem of Baer ([1], or [10], p. 162) that N is supersoluble. Hence G has property σ .

The next characterization of σ sheds some light on the question of why Theorems 1 and 2 make no mention of supersolubility.

THEOREM 3. A group G has property σ if and only if whenever N is a normal subgroup of G which is not finitely generated nilpotent, there is a finite non-nilpotent G -quotient of N .

PROOF. Let us temporarily name the property stated in the theorem ν' . Assume that G has property σ , and let N be a normal subgroup of G which is not finitely generated nilpotent. If N is not supersoluble, then it has a finite non-supersoluble G -quotient, and this is non-nilpotent. On the other hand, if N is supersoluble, Hirsch's theorem can be applied to produce a finite non-nilpotent G -quotient of N . Hence G has property ν' .

Conversely, suppose that G has property ν' . Then, obviously, G has property ν , and therefore $\text{Fit}(G)$ is nilpotent, by Proposition 1 of [3]. Property ν' implies immediately that $\text{Fit}(G)$ is finitely generated. Finally, G has property σ by Theorem 2.

For our final characterization of property σ , which can be regarded

as a statement of optimum φ_f -behaviour, we need a result about finite groups due to Mukherjee and Bhattacharya [7] (Theorem 9).

LEMMA 2. Let H be a normal subgroup of a finite group G . If $H\varphi(G)/\varphi(G)$ is supersoluble, then H is supersoluble.

THEOREM. 4. A group G has property σ if and only if the following conditions hold:

- (a) $\text{Fit}(G/\varphi_f(G))$ is finitely generated;
- (b) if $H \triangleleft G$ and $H/\varphi_f(G)$ is supersoluble, then H is supersoluble.

PROOF. Assume that conditions (a) and (b) hold. Then $G/\varphi_f(G)$ has property σ by Theorem 1. Suppose that N is a normal subgroup of G all of whose finite G -quotients are supersoluble. Then $N\varphi_f(G)/\varphi_f(G)$ must be supersoluble. It follows from (b) that N is supersoluble, and so G has σ .

Conversely, let G have property σ . Then Theorem 1 shows that condition (a) holds. Let $H \triangleleft G$ be such that $H/\varphi_f(G)$ is supersoluble. Then H is polycyclic since $\varphi_f(G)$ is finitely generated and nilpotent. If H is not supersoluble, Baer's theorem shows that H has a finite non-supersoluble G -quotient. The argument of the proof of Lemma 1 allows us to conclude that there is a finite quotient G/L such that HL/L is not supersoluble. But $\varphi(G/L) \supseteq \varphi_f(G)L/L$, and $HL/\varphi_f(G)L$ is supersoluble. Therefore Lemma 2 may be applied to give the contradiction: HL/L is supersoluble.

REMARK. If G is any group, then $G/\varphi_f(G)$ always has property ν , as is shown by the argument used to prove Theorem 1 of [3]. However such a quotient may fail to have property σ . For example, let $G = \langle x, y \mid x^y = x^2 \rangle$, a finitely generated metabelian group of finite rank. Here $\varphi_f(G) = \varphi(G) = 1$, but of course G does not have σ since $\text{Fit}(G)$ is infinitely generated.

4. Applications of the main theorems.

We shall now apply the characterizations of the last section to some special types of group.

THEOREM 5. All free groups have property σ .

For a free group has property ν by [3], Corollary 6; also, if

the group is non-cyclic, its Fitting subgroup is trivial. Thus the result follows from Theorem 2.

PROPOSITION 1. In any group with property σ the union of the upper Hirsch-Plotkin series is polycyclic. Thus a radical group has σ if and only if it is polycyclic.

PROOF. Let G be a group with σ , and let S be the union of the upper Hirsch-Plotkin series of G . Then $HP(S) = HP(G)$ is supersoluble by the definition of σ , and therefore finitely generated.

It is well-known that a radical group whose Hirsch-Plotkin radical is finitely generated is polycyclic (see [8], Theorem 3.3.1). Hence S is polycyclic.

The previous theorem implies that every group with property σ has a soluble radical.

COROLLARY 2. Let G be a group with property σ . Then G has a unique maximum soluble normal subgroup S , and S is polycyclic. Furthermore the finite residual of G is contained in $Z(S)$.

PROOF. Let S be the union of the upper Hirsch-Plotkin series of G . Then S is polycyclic by Proposition 1, and $HP(G/S)$ is trivial; thus S is the maximum soluble normal subgroup of G .

Now let R be the finite residual of G . Then R is nilpotent, so $R \leq S$. But $G/C_G(S)$ is residually finite since the automorphism group of a polycyclic group is residually finite (see [2], [9] or [8], 9.12). Therefore $R \leq C_G(S) \cap S = Z(S)$.

This result reveals another interesting difference between properties ν and σ . For *the finite residual of a group with property ν may have arbitrary nilpotent class*. Indeed let G be the standard wreath product $H \text{ wr } T$ where H is finitely generated nilpotent and T is infinite cyclic. Then Theorem 9 of [3] shows that G has property ν . However an easy calculation reveals that the finite residual of G equals the normal closure of H' in G .

Turning to groups with finite rank, we find that few such groups have property σ . The next result rests ultimately on a deep theorem of Lubotzky and Mann [5] on residually finite groups with finite rank.

THEOREM 6. Let G be a group with finite rank. Then G has property σ if and only if it is polycyclic-by-finite.

PROOF. Assume that G has σ . Then G has ν , by Theorem 2, and hence G is soluble-by-finite, by [3], Theorem 3. Therefore G is polycyclic-by-finite by Corollary 2. The converse is known.

It has been seen that no non-polycyclic soluble group has property σ , which contrasts with property ν since many types of finitely generated soluble group have ν ([3]). However, as soon as we pass to locally soluble groups the situation changes; there are insoluble locally soluble groups with property σ .

EXAMPLE. Let G be McLain's example of an insoluble, locally soluble group with the maximal condition on normal subgroups (see [6] or [8], 5.3). McLain proved that the only non-trivial normal subgroups of G are the terms of the derived series. Moreover

$$G/G^{(j+1)} \simeq G_j = G_{j-1} \ltimes N_j,$$

where N_j is a faithful simple module for the finite group G_{j-1} . Thus $\varphi(G_j) = 1$, from which it follows that $\varphi_f(G) = 1$ since the intersection of the $G^{(j+1)}$ is trivial. Of course $\text{Fit}(G) = 1$, so Theorem 1 shows that G has property σ .

Next we examine the behaviour of linear groups.

THEOREM 7. Let R be a finitely generated domain and let G be an R -linear group. Then G has property σ if and only if the unipotent radical of G is finitely generated.

PROOF. As already mentioned G has property ν . By Theorem 2 the group G will have σ if and only if $F = \text{Fit}(G)$ is finitely generated. Denote by U the unipotent radical of G ; then $U \leq F$. If F is finitely generated, then, of course, so is U .

Conversely, assume that U is finitely generated. Now F is nilpotent. Hence Theorem 13.29 of [10] may be applied to show that F has a normal unipotent subgroup V such that F/V is finitely generated. Clearly $V \leq U$, so V is finitely generated, as must be F .

COROLLARY 3. Every \mathbb{Z} -linear group has property σ .

For by a well-known theorem of Mal'cev (see [8], 3.26) every soluble \mathbb{Z} -linear group is polycyclic. Thus Theorem 7 can be applied to give the result.

A simple example of a $\mathbb{Z}[1/2]$ -linear group that does not have σ is

the group G generated by the matrices

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Of course $G \simeq \langle x, y \mid x^y = x^2 \rangle$, which does not have σ .

Locally finite groups.

One can deduce directly from Theorem 2 and the characterization of locally finite groups with ν ([3], Theorem 5) the following result.

THEOREM 8. Let G be a locally finite group with finite residual R . Then G has property σ if and only if

- (a) $R \leq HP(G)$ and $HP(G)$ is finite;
- (b) $HP(G/R) = HP(G)/R$.

In one direction this can be strengthened.

THEOREM 9. A locally finite group G has property σ if and only if it has a finite normal subgroup F such that G/F is residually finite and $HP(G/F)$ is trivial.

PROOF. Suppose that the conditions hold, and let R be the finite residual of G . Then $R \leq F \cap C_G(F) = Z(F)$. Writing $HP(G/R) = H/R$, we see that $H \leq F$. Therefore H is nilpotent, and $H = HP(G)$. Finally $HP(G) \leq F$, so $HP(G)$ is finite. Therefore G has property σ by Theorem 8.

Conversely, assume that G has σ and let F be the union of the upper Hirsch-Plotkin series of G . By Proposition 1, F is finite and clearly $HP(G/F)$ is trivial. By Theorem 8 the finite residual lies in F and it is easy to see that G/F is residually finite.

5. Closure properties.

Property σ has more closure properties than ν does, although these are still quite limited. The main difference is that σ is preserved under finite extensions unlike ν (cf. [3], Section 5).

LEMMA 3. Property σ is preserved under the formation of ascendant subgroups and finite subdirect products.

PROOF. Let G be a group with σ , and suppose that A is an ascendant subgroup of G . Then G has ν , and by Lemma 2 of [3], so does A .

Now $\text{Fit}(A) \leq HP(G) = \text{Fit}(G)$, and $\text{Fit}(G)$ is finitely generated. Therefore $\text{Fit}(A)$ is finitely generated. Theorem 2 now implies that A has σ .

The second statement is proved in the same way as Lemma 2 of [3].

Next we show that property σ is preserved under finite extensions.

THEOREM 10. Let G be a group with a subgroup H of finite index. Then G has property σ if and only if H does.

PROOF. Suppose first that $H \triangleleft G$ and H has σ ; we argue that G has σ . To this end, let $N \triangleleft G$ be non-supersoluble with all its finite G -quotients supersoluble. Then $N/H \cap N$ is supersoluble. Now $H \cap N$ is not supersoluble, otherwise Baer's theorem would imply that N has a finite non-supersoluble quotient. Since H has σ , it follows that there is a finite non-supersoluble H -quotient $H \cap N/U$. But $H \leq N_G(U)$, and thus $H \cap N/U_G$ is finite (where U_G is the normal core of U in G). Therefore N/U_G is finite and hence supersoluble, a contradiction.

In the general case, we can apply Lemma 3, together with the result of the last paragraph, to H_G and deduce the theorem.

The following simple closure property may be established in exactly the same way as Lemma 3 of [3].

LEMMA 4. Let G be a group with a normal subgroup M satisfying $M \leq Z_i(G)$ for some i . Then G has property σ if and only if M is finitely generated and G/M has property σ .

Next we record a further extension property of σ not shared by ν .

THEOREM 11. Let G be a group with a normal polycyclic-by-finite subgroup A . Then G has property σ if and only if G/A does.

PROOF. Suppose first that G/A has σ . Let $N \triangleleft G$ be non-supersoluble with all its finite G -quotients supersoluble. Then NA/A is supersoluble and N is polycyclic-by-finite. Baer's theorem now gives a contradiction. Hence G has σ .

Conversely, let G have σ . Clearly one can assume that A is either finite or else finitely generated free abelian. Suppose first that A is finite, and put $C = C_G(A)$. Then G/C is finite and $C \cap A \leq Z(C)$. By Lemmas 3 and 4 the group CA/A has σ ; hence G/A has σ by Theorem 10.

Now assume that A is finitely generated free abelian, and again

write $C = C_G(A)$. Then G/C is \mathbb{Z} -linear, so by Corollary 3 it has σ . Suppose that $N \triangleleft G$, with N/A non-supersoluble and all its finite G -quotients supersoluble. Then NC/C is supersoluble since $A \leq C$. Therefore $N_1 = N \cap C$ is not supersoluble, and so it must have a finite non-supersoluble G -quotient N_1/L . Put $C_1 = C_N(N_1/L)$. Then $C_1 \triangleleft G$ and N/C_1 is finite. Since $A \leq C_1$, it follows that N/C_1 is supersoluble, whence so is N_1/L since it is centre-by-supersoluble. This is a contradiction.

In conclusion we apply Theorem 11 and Theorem 1 to produce a result which highlights the role of the soluble radical in a group with σ .

THEOREM 12. A group G has property σ if and only if there is a polycyclic normal subgroup S such that $\text{Fit}(G/S) = 1 = \varphi_f(G/S)$.

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Manoscritto pervenuto in redazione il 27 agosto 1990 e, in versione corretta, il 13 febbraio 1991.