RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FERRUCCIO COLOMBINI
DANIELE DEL SANTO

Some uniqueness results for degenerate elliptic operators in two variables

Rendiconti del Seminario Matematico della Università di Padova, tome 86 (1991), p. 111-129
<http://www.numdam.org/item?id=RSMUP_1991__86__111_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1991, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

REND. SEM. MAT. UNIV. PADOVA, Vol. 86 (1991)

Some Uniqueness Results for Degenerate Elliptic Operators in Two Variables.

FERRUCCIO COLOMBINI - DANIELE DEL SANTO (*)

SUMMARY - We prove the uniqueness in the Cauchy problem for degenerate elliptic operators in two variables of the type:

$$P(x, t, D_x, D_t) = D_t^2 + a(t) D_x^2 + b(t) D_x + c(x, t) D_t + d(x, t),$$

with respect to the surface $\{t = 0\}$ in the origin, under some assumptions involving the coefficients $a(t)$ and $b(t)$. These results can be applied to the operators:

$$P_1 = D_t^2 + e^{-1/t^2} D_x^2 + e^{-1/t} [1 + it^2] D_x + d_1(x, t),$$

$$P_2 = D_t^2 + e^{-1/t^2} D_x^2 + e^{-1/t} \left[t^2 \sin \frac{1}{t} + i \right] D_x + d_2(x, t).$$

1. Introduction.

In this paper we give some sufficient conditions for the uniqueness in the Cauchy problem for linear partial differential operators in two variables of the type:

(1) $P(x, t, D_x, D_t) = D_t^2 + a(x, t) D_x^2 + b(x, t) D_x + c(x, t) D_t + d(x, t),$

(D_y means as usual $-i(\partial/\partial y)$) relatively to the surface $\{(x, t) \in \boldsymbol{R}^2 \,|\, t = 0\}$

(*) Indirizzo degli AA.: F. COLOMBINI: Dipartimento di Matematica, Università di Pisa, Via Buonarroti 2, I-56127 Pisa; D. DEL SANTO: Dipartimento di Scienze Matematiche, Università di Trieste, Piazzale Europa 1, I-34127 Trieste.

in the origin. More precisely we will consider operators like (1) which are degenerate elliptic with real principal part, i.e.:

$$a(x,t) \geqslant 0\,.$$

Let us briefly recall what we mean by uniqueness in the Cauchy problem. Let Q be a linear partial differential operator:

$$Q(y, D_y) = \sum_{|\alpha| \leqslant m} a_\alpha(y) D_y^\alpha\,,$$

where the a_α are complex valued functions defined in Ω, open subset of $\boldsymbol{R}^n$. Let φ be a real valued $\mathcal{C}^1$ function defined in Ω and let y_0 be a point of Ω such that $\varphi(y_0) = 0$ and $\nabla\varphi(y_0) \neq 0$.

DEFINITION 1. *We will say that the operator Q has the uniqueness in the Cauchy problem with respect to the surface $\{y \in \Omega \mid \varphi(y) = 0\}$ in y_0 if given $u \in \mathcal{C}^\infty(\Omega')$, $\Omega'(\subseteq \Omega)$ open neighborhood of y_0, such that* $\operatorname{supp}(u) \subseteq \{y \in \Omega' \mid \varphi(y) \geqslant 0\}$ *and $Qu \equiv 0$ in Ω', it follows that $u \equiv 0$ in a neighborhood of y_0.*

Dealing with operator (1), the classical Calderón's uniqueness theorem ([4]) can be applied only in the simple case of $a(0,0) \neq 0$, i.e. when the operator is elliptic or strictly hyperbolic. The degenerate elliptic case has been considered by several authors ([1], [2], [6], [8], [10]).

In [6] a uniqueness result for the operator (1) with respect to the surface $\{(x,t) \in \Omega \mid t = 0\}$ in 0, is proved under the following conditions:

there exist two strictly positive constants C, ε such that:

$$a(x,t) \geqslant 0\,, \tag{2}$$

$$(2-\varepsilon)\, a(x,t) + t \frac{\partial}{\partial t} a(x,t) \geqslant C t^2 |b(x,t)|^2\,, \tag{3}$$

for all $(x,t) \in \{(x,t) \in \Omega \mid t \geqslant 0\}$.

Condition (3) can be interpreted as a combination of a «quasi-monotonicity» property in the variable t for the coefficient $a(x,t)$ and a «Levi type» condition on the term of order one.

In [10] K. Watanabe shows that if (2) holds and the function $t \mapsto a(0,t)$ has a finite order zero in 0, then operator (1) has the uniqueness property without any condition on the term of order one.

S. Alinhac and C. Zuily consider in [2] the case of $a(x,t) \equiv 0$, in which the recalled result of [6] is not meaningful. They give some sufficient conditions for the uniqueness, in particular they prove that uniqueness holds if $b(x,t) = b(t) = b_1(t) + ib_2(t)$ with $b_1(t)$ and $b_2(t)$ real valued $\mathcal{C}^\infty$ functions and $b_2(t)$ has a finite order zero in 0.

Nevertheless it is not so hard a work to find operators like (1) to which no one of the previous uniqueness results can be applied, as it can be seen in the two following examples:

$$P_1 = D_t^2 + e^{-1/t^2} D_x^2 + e^{-1/t}[1 + it^2] D_x + d_1(x,t), \tag{4}$$

$$P_2 = D_t^2 + e^{-1/t^2} D_x^2 + e^{-1/t}\left[t^2 \sin\frac{1}{t} + i\right] D_x + d_2(x,t). \tag{5}$$

The aim of this work is to prove some uniqueness theorems such that they can be applied to operators like (4) and (5) i.e. to degenerate elliptic operators of type (1) where the coefficients $a(x,t)$ and $b(x,t)$ depend only on t, the first order part doesn't satisfy a «Levi type» condition and $a(t)$ is not identically zero. We will obtain two results depending on the fact that a «quasi-monotonicity» property is satisfied by the real (Theorem 1) or the imaginary part (Theorem 2) of the coefficient of order one. The proofs are based on Carleman's estimates in which we use an auxiliary function modeled on the «leading» part of the first order coefficient of the operator, together with a usual weight function.

2. Statements of the results and remarks.

Let Ω be an open neighborhood of the origin in $\mathbf{R}^2$. Let P be the following operator:

$$P(x,t,D_x,D_t) = \tag{6}$$

$$= D_t^2 + a(t) D_x^2 + [b_1(t) + ib_2(t)] D_x + c(x,t) D_t + d(x,t),$$

where a, b_1, b_2 are real valued $\mathcal{C}^1$ functions, while c and d are complex valued L^∞ functions, defined in Ω. We require that for each $t \geq 0$:

$$a(t) \geq 0. \tag{7}$$

We state now the main results (in the following «$'$» will mean $\partial/\partial t$).

THEOREM 1. *Let P be like* (6) *and let* (7) *hold. Suppose that there exist $\varepsilon > 0$, $T > 0$ and $\gamma \in L^\infty(]0,T])$ such that for each $t \in]0,T]$*:

$$(2-\varepsilon)\, b_1(t) + t b_1'(t) \geq 0; \tag{8}$$

$$\left(\frac{b_2}{b_1}\right)'(t) = \frac{\gamma(t)}{t}\,\frac{b_2(t)}{b_1(t)}; \tag{9}$$

$$2\gamma(t) + 2 + t\frac{b_1'(t)}{b_1(t)} \geqslant \varepsilon; \tag{10}$$

$$a(t) \equiv 0 \quad \text{or} \quad t\left[\frac{a'(t)}{a(t)} - \frac{b_1'(t)}{b_1(t)}\right] \geqslant \varepsilon. \tag{11}$$

Then P has the uniqueness in the Cauchy problem with respect to $\{(x,t) \in \Omega \mid t = 0\}$ in 0.

THEOREM 2. *As in Theorem* 1, *let P be like* (6) *and let* (7) *hold. Suppose that there exist $\varepsilon > 0$, $T > 0$ and $\sigma \in L^\infty(]0,T])$ such that for each $t \in]0,T]$*:

$$(2-\varepsilon)\,b_2(t) + tb_2'(t) \geqslant 0; \tag{12}$$

$$\left(\frac{b_1}{b_2}\right)'(t) = \frac{\sigma(t)}{t}; \tag{13}$$

$$\left[(2-\varepsilon) + t\frac{b_2'(t)}{b_2(t)}\right] \geqslant |\sigma(t)| + \varepsilon; \tag{14}$$

$$a(t) \equiv 0 \quad \text{or} \quad t\left[\frac{a'(t)}{a(t)} - \frac{b_2'(t)}{b_2(t)}\right] \geqslant \varepsilon. \tag{15}$$

Then the same conclusion of Theorem 1 *holds.*

Condition (8) implies that $b_1(t)$ is identically zero or strictly positive for $t > 0$. If in Theorem 1 we replace condition (8) with an analogous condition with the opposite sign the thesis is the same. In fact it is sufficient to change x with $-x$ to obtain again the hypotheses of Theorem 1. The same remark is valid for condition (12) in Theorem 2.

Let us observe that from conditions (8) and (11) it follows that also $a(t)$ satisfies a «quasi-monotonicity» condition like (8). This fact is not surprising if we look at (3). Nevertheless we know something about the necessity of such a condition on $a(t)$ only if the space dimension is greater or equal than 2 ([6], Theorem 3). In fact in the non-uniqueness examples known for operators like (1) ([5], [7], [9]), the non-uniqueness phenomenon seems to be caused only by the rapidly oscillatory behavior of the coefficient of the first order part, without any relation with

$a(t)$. We don't know any example of non-uniqueness for operators like (1) with $a(x, t) \geqslant 0$ and $b(x, t) \equiv 0$.

If $b_1(t)$ verifies (8) then condition (9) implies that $b_2(t)$ verifies condition (12). Nevertheless there exist operators for which the hypotheses of Theorem 1 hold while this is not true for the ones of Theorem 2: the operator P_1 given by (4) is an example. Moreover condition (9) implies that $b_2(t) \equiv 0$ or there exist $M > 0$, $C_1 > 0$ and $C_2 > 0$ such that for each $t \in]0, T]$:

$$C_1 b_1(t) t^M \leqslant |b_2(t)| \leqslant C_2 b_1(t) t^{-M}, \tag{16}$$

but obviously the opposite implication doesn't hold.

As far as the Theorem 2 is concerned, we observe that if condition (12) holds and there exists a $\mathcal{C}^1([0, T])$ function f with $f' \in L^\infty$, such that:

$$b_1(t) = f(t) b_2(t),$$

then the conditions (13) and (14) are verified.

3. Proofs.

a) Preliminary results.

First of all we state a lemma that will be useful in the following.

LEMMA 1. *Let u be the solution of the following Cauchy problem for O.D.E.:*

$$\begin{cases} \dfrac{d^2}{dt^2} u + c(x, t) \dfrac{d}{dt} u + d(x, t) u = r(t) h(x, t), \\ u(x, 0) = \dfrac{d}{dt} u(x, 0) = 0, \end{cases}$$

where x is a parameter varying in $[-C, C]$, $c(x, t)$, $d(x, t)$, $h(x, t)$ are complex valued L^∞ functions defined in $[-C, C] \times [0, T]$, $r(t)$ is a continuous real valued increasing function with $r(0) = 0$ and $r(t) > 0$ for $t > 0$.

Then:

$$u(x,t) = r(t)\, z_1(x,t)\,,$$

$$\frac{d}{dt} u(x,t) = r(t)\, z_2(x,t)\,,$$

where $z_1(x,t)$ *and* $z_2(x,t)$ *are* $L^\infty([-C,C]\times]0,T])$ *functions.*

Proof. It is sufficient to write u like a linear combination of independent solutions of the correspondent homogeneous Cauchy problem. So it is easy to obtain the explicit expression for the coefficients of the linear combination. Then we use the monotonicity of $r(t)$.

Let us now consider the operator (6). Suppose that the hypotheses of Theorem 1 hold. Using (16) it is not difficult to obtain that:

$$b_2(t) = t^{-M_1} g_1(t)\, b_1(t)\,, \tag{17}$$

for some $g_1 \in \mathcal{C}^0([0,T]) \cap \mathcal{C}^1(]0,T])$ and for some $M_1 \geqslant 0$.

In a similar way from (11) we get:

$$a(t) = g_2(t)\, b_1(t)\,, \tag{18}$$

for some $g_2 \in \mathcal{C}^0([0,T]) \cap \mathcal{C}^1(]0,T])$.

Let $\Omega'(\subseteq \Omega)$ be an open neighborhood of the origin and let $u(x,t) \in \mathcal{C}^\infty(\Omega')$ be a *null solution* for the operator (6) (this means simply that $Pu \equiv 0$ in Ω' and $\operatorname{supp}(u)$ is contained in $\{(x,t) \in \Omega' \mid t \geqslant 0\}$). So that:

$$\begin{cases} D_t^2 u + cD_t u + du = -aD_x^2 u - [b_1 + ib_2] D_x u\,, \\ u(x,0) = D_t u(x,0) = 0\,. \end{cases}$$

By (17), (18) and the fact that $D_x u$ and $D_x^2 u$ are $\mathcal{C}^\infty$ with support in $\{(x,t) \in \Omega' \mid t \geqslant 0\}$ we obtain:

$$D_t^2 u + c(x,t) D_t u + d(x,t)\, u = t^k b_1(t)\, h_k(x,t)\,,$$

where $h_k(x,t)$ is in L^∞ and by (8) we have that if we define $r_k(t) = t^k b_1(t)$ then r_k is a continuous increasing function with $r_k(0) = 0$, $r_k(t) > 0$ for each $t > 0$, if k is an integer $\geqslant 2$. We use the Lemma 1 and we get:

$$
(19) \qquad \begin{aligned} u(x,t) &= t^k b_1(t)\, z_{1,k}(x,t)\,, \\ D_t u(x,t) &= t^k b_1(t)\, z_{2,k}(x,t)\,, \end{aligned}
$$

where $z_{1,k}$ and $z_{2,k}$ are L^∞ functions for each $k \geqslant 2$.

Similarly if we are in the hypotheses of Theorem 2 we obtain that:

$$
(20) \qquad \begin{aligned} u(x,t) &= t^k b_2(t)\, \zeta_{1,k}(x,t)\,, \\ D_t u(x,t) &= t^k b_2(t)\, \zeta_{2,k}(x,t)\,, \end{aligned}
$$

where $\zeta_{1,k}$ and $\zeta_{2,k}$ are L^∞ functions for each $k \geqslant 2$.

b) Proof of the Theorem 1.

Consider the following *singular change of variables* ([3]):

$$
\begin{cases} y = x\,, \\ s = \dfrac{t}{\delta - |x|^2}\,. \end{cases}
$$

We set:

$$
\begin{aligned} \alpha(y,s) &= (\delta - |y|^2)^2 a(s(\delta - |y|^2))\,, \\ \beta_j(y,s) &= (\delta - |y|^2)^2 b_j(s(\delta - |y|^2))\,, \qquad j = 1, 2\,. \end{aligned}
$$

We define:

$$
X = D_y + \frac{2ys}{\delta - |y|^2} D_s\,.
$$

We will deal with the following operator:

$$
(21) \qquad Q(y, s, D_y, D_s) = D_s^2 + X(\alpha X) + (\beta_1 + i\beta_2) X\,.
$$

The relation between P in (6) and Q in (21) is clear; if we call $\tilde{P}$ the operator P in the new variables, we obtain:

$$
(22) \qquad \tilde{Q} = (\delta - |y|^2)^2 \tilde{P} = Q - 4i\alpha \frac{y}{\delta - |y|^2} X + \eta D_s + \omega\,,
$$

where η and ω are complex valued L^∞ functions.

Uniqueness for P will be deduced by an estimate on $\tilde{Q}$. To get such an estimate we will use Q.

Suppose to be in the hypotheses of Theorem 1. We set:

$$\psi(y, s) = \left(\frac{s(\delta - |y|^2)}{b_1(s(\delta - |y|^2))} \right)^{1/2},$$

so that $\psi \in \mathcal{C}^1(]-\sqrt{\delta}, \sqrt{\delta}[\times]0, S_0])$. Let $w(x, t)$ be a null solution for P in (6). Let us consider the function:

$$\tilde{w}(y, s) = w(y, s(\delta - |y|^2)).$$

So $\tilde{w}(y, s)$ is a null solution for $\tilde{P}$ and consequently for $\tilde{Q}$. We point out that $\tilde{w}$ can be considered having support contained in $\{(y, s) \in \Omega \mid |y|^2 \leq \delta, s \geq 0\}$.

By (19) we obtain:

$$\begin{aligned} &\tilde{w}(y, s) = s^k(\delta - |y|^2)^k b_1(s(\delta - |y|^2))\, \tilde{z}_{1,k}(y, s), \\ (23) \qquad &D_s \tilde{w}(y, s) = s^k(\delta - |y|^2)^{k+1} b_1(s(\delta - |y|^2))\, \tilde{z}_{2,k}(y, s), \\ &D_s^2 \tilde{w}(y, s) = s^k(\delta - |y|^2)^{k+2} b_1(s(\delta - |y|^2))\, \tilde{z}_{3,k}(y, s), \end{aligned}$$

where $\tilde{z}_{1,k}$, $\tilde{z}_{2,k}$, $\tilde{z}_{3,k}$ are L^∞ functions.

Let $\chi(s)$ be a real valued $\mathcal{C}^\infty$ function defined on $\boldsymbol{R}$ such that:

$$\chi(s) = 1 \quad \text{for } s \leq \frac{S_0}{3},$$
$$\chi(s) = 0 \quad \text{for } s \geq \frac{2}{3} S_0.$$

We define $u(y, s) = \chi(s)\, \tilde{w}(y, s)$. Obviously u is a $\mathcal{C}_0^\infty$ function with support in $[-\sqrt{\delta}, \sqrt{\delta}] \times [0, S_0]$.

We will compute:

$$\|\psi(y, s)\, s^{-\tau} Q u(y, s)\|^2_{L^2(\Omega)}.$$

Let us set:

$$v = s^{-\tau} u.$$

We obtain:

$$(24) \qquad \|\psi s^{-\tau} Q u\|^2_{L^2(\Omega)} = \|J_1 v + J_2 v + \tau \psi \mu(y, s)\, v\|^2_{L^2(\Omega)}$$

where $\mu \in L^{\infty}(\Omega)$ and:

$$(25)\quad \begin{aligned} J_1 v &= \psi D_s^2 v - \tau(\tau-1)\psi \frac{v}{s^2} + \psi X(\alpha X v) - 4\tau^2 \psi\alpha \frac{y^2}{(\delta - |y|^2)^2} v + \psi\beta_1 X v, \\ J_2 v &= -2i\tau\psi \frac{D_s v}{s} - 4i\tau\psi\alpha \frac{y}{\delta - |y|^2} X v + i\psi\beta_2 X v. \end{aligned}$$

First of all we must verify that $J_1 v$ and $J_2 v$ defined in (25) are in L^2. We show only that $\psi D_s^2 v$, $\psi X(\alpha X v)$ and $i\psi\beta_2 X v$ are in L^2, the other functions being similar to these:

$$\psi D_s^2 v = s^{-\tau-3/2+k}(\delta - |y|^2)^{k+1/2}(b_1(s(\delta - |y|^2)))^{1/2}\varphi_k(\tau, y, s),$$

where $\varphi_k(\tau, y, s)$ is bounded for each fixed (k, τ); so far each $\tau \geqslant 0$, $\psi D_s^2 v \in L^2(\Omega)$.

By the definition of α have that:

$$X(\alpha) = 4i \frac{y}{\delta - |y|^2}\alpha,$$

so that:

$$\psi X(\alpha X v) = \psi\alpha X^2 v + 4i\psi \frac{y}{\delta - |y|^2}\alpha X v =$$

$$= (s(\delta - |y|^2))^{1/2}\left(\frac{a(s(\delta - |y|^2))}{b_1(s(\delta - |y|^2))}\right)^{1/2}(a(s(\delta - |y|^2)))^{1/2}\tilde{\varphi}_k(\tau, y, s),$$

where $\tilde{\varphi}_k(\tau, y, s)$ is bounded for each fixed (k, τ). Using (18) it is easy to get that:

$$0 < \frac{a(t)}{b_1(t)} < K,$$

for each $t \in]0, T]$. So $\psi X(\alpha X v) \in L^2(\Omega)$.

Finally:

$$i\psi\beta_2 X v = i s^{1/2}(s(\delta - |y|^2))^{5/2}\left(\frac{b_2(s(\delta - |y|^2))}{b_1(s(\delta - |y|^2))}\right)^{1/2}(b_2(s(\delta - |y|^2)))^{1/2} X v.$$

By (17) we know that:

$$\frac{b_2(s(\delta-|y|^2))}{b_1(s(\delta-|y|^2))}=(s(\delta-|y|^2))^{-M_1}g_1(s(\delta-|y|^2)),$$

where g_1 is a continuous function. But:

$$(s(\delta-|y|^2))^{-M_1}Xv=X((s(\delta-|y|^2))^{-M_1}v),$$

and $(s(\delta-|y|^2))^{-M_1}v\in\mathcal{C}_0^\infty$. So $i\psi\beta_2Xv\in L^2(\Omega)$.

We want now to compute $2\,\mathrm{Re}\,(J_1v,J_2v)_{L^2(\Omega),L^2(\Omega)}$. By (25) this term splits into fifteen integrals. Let us calculate them (from now on «$'$» will mean $\partial/\partial s$)

$$M_1=2\,\mathrm{Re}\left(\psi D_s^2v,\,-2i\tau\psi\frac{D_sv}{s}\right)_{L_2(\Omega),L_2(\Omega)}=$$

$$=\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}-2i\tau\frac{\psi^2}{s}D_s^2v\,D_s\bar v-2i\tau\frac{\psi^2}{s}D_sv\,D_s^2\bar v\,ds\,dy.$$

Let us consider:

$$D_s\left(\frac{\psi^2}{s}|D_sv|^2\right)=D_s\left(\frac{\psi^2}{s}\right)|D_sv|^2-\frac{\psi^2}{s}D_s^2v\,D_s\bar v-\frac{\psi^2}{s}D_sv\,D_s^2\bar v.\tag{26}$$

All of the three functions on the right-hand side in (26) are in $L^1(\Omega)$, then also the term of the left-hand side is in $L^1(\Omega)$. By (23) we easily obtain that:

$$D_sv=s^{-\tau-1+k}(\delta-|y|^2)^kb_1(s(\delta-|y|^2))\,\tilde{\tilde\varphi}_k(\tau,y,s),$$

where $\tilde{\tilde\varphi}_k$ is bounded for each fixed (k,τ).

This implies that:

$$\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}D_s\left(\frac{\psi^2}{s}|D_sv|^2\right)ds\,dy=0.$$

By (26) we finally get:

$$M_1 = 2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} s \frac{\beta_1'}{\beta_1} \left| \psi \frac{D_s v}{s} \right|^2 ds\, dy .$$

Using a similar technique we have:

$$M_2 = 2 \operatorname{Re}\left(- \tau(\tau-1)\psi \frac{v}{s^2}, -2i\tau\psi \frac{D_s v}{s} \right) =$$

$$= 2\tau^2(\tau-1) \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(2 + s\frac{\beta_1'}{\beta_1} \right) \left| \psi \frac{v}{s^2} \right|^2 ds\, dy .$$

$$M_3 = 2 \operatorname{Re}\left(\psi X(\alpha X v), -2i\tau\psi \frac{D_s v}{s} \right) =$$

$$= -2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(\frac{\psi^2 \alpha}{s} \right) Xv\, X\bar{v}\, ds\, dy - 8\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \frac{\psi^2 \alpha}{s} \frac{y}{\delta - |y|^2} Xv\, D_s \bar{v}\, ds\, dy ,$$

where we have used the fact, deduced by (11), that $\dfrac{d}{dt}\left(\dfrac{a}{b_1}\right) \in$ $\in L^1(]0,T])$. So:

$$M_3 \geqslant 2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \alpha s \left[\frac{\alpha'}{\alpha} - \frac{\beta_1'}{\beta_1} \right] \left| \psi \frac{Xv}{s} \right|^2 ds\, dy -$$

$$- \tau\rho_1 \left\| \psi \frac{D_s v}{s} \right\|^2 - \tau C_1 \left\| \psi\alpha \frac{y}{\delta - |y|^2} Xv \right\|^2 .$$

$$M_4 = 2 \operatorname{Re}\left(- 4\tau^2 \psi\alpha \frac{y^2}{(\delta - |y|^2)^2} v, - 2i\tau\psi \frac{D_s v}{s} \right) =$$

$$= - 8\tau^3 \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(\frac{\psi^2 \alpha}{s} \right)' \frac{y^2}{(\delta - |y|^2)^2} |v|^2 ds\, dy ,$$

and this implies that:

$$M_4 \geqslant -\tau^3 C_2 \left\| \psi \frac{v}{s} \right\|^2.$$

$$M_5 = 2 \operatorname{Re}\left(\psi\beta_1 Xv, -2i\tau\psi\frac{D_s v}{s}\right) = -2\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(\frac{\psi^2\beta_1}{s}\right)' vX\bar{v}\, ds\, dy -$$

$$-8\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \frac{\psi^2}{s}\beta_1 \frac{y}{\delta - |y|^2} v D_s \bar{v}\, ds\, dy,$$

consequently, by the choice of ψ we have:

$$M_5 \geqslant -\tau\rho_3 \left\| \psi \frac{D_s v}{s} \right\|^2 - \tau C_3 \|\psi v\|^2.$$

In the following $\nu_1, \dots, \nu_6$ are functions in $L^\infty(\Omega)$. Then:

$$M_6 = 2 \operatorname{Re}\left(\psi D_s^2 v, -4i\tau\psi\alpha \frac{y}{\delta - |y|^2} Xv\right) =$$

$$= 8\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} (\psi^2\alpha)' \frac{y}{\delta - |y|^2} Xv D_s \bar{v}\, ds\, dy - \tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \nu_1 |\psi D_s v|^2\, ds\, dy.$$

So:

$$M_6 \geqslant -\tau\rho_4 \left\| \psi \frac{D_s v}{s} \right\|^2 - \tau C_4 \left\| \psi\alpha \frac{1}{\delta - |y|^2} Xv \right\|^2 +$$

$$-\tau C_4 \left\| \alpha \frac{s}{\delta - |y|^2} \left[\frac{\alpha'}{\alpha} - \frac{\beta_1'}{\beta_1} \right] \psi Xv \right\|^2 - \tau C_4 \|\psi D_s v\|^2,$$

$$M_7 = 2 \operatorname{Re}\left(-\tau(\tau-1)\psi\frac{v}{s^2}, -4i\tau\psi\alpha\frac{y}{\delta - |y|^2} Xv\right) =$$

$$-\tau^2(\tau-1) \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \nu_2 \left| \psi \frac{v}{s} \right|^2 ds\, dy.$$

We obtain:

$$M_7 \geqslant -\tau^3 C_5 \left\| \psi \frac{v}{s} \right\|^2,$$

$$M_8 = 2 \operatorname{Re}\left(\psi X(\alpha X v),\ -4 i \tau \psi \alpha \frac{y}{\delta - |y|^2} X v\right) =$$

$$= -4\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \psi^2 \alpha^2 \frac{\delta + 3y^2}{(\delta - |y|^2)^2} |Xv|^2 \, ds \, dy,$$

therefore:

$$M_8 \geqslant -\tau C_6 \left\| \psi \alpha \frac{1}{\delta - |y|^2} X v \right\|^2,$$

$$M_9 = 2 \operatorname{Re}\left(-4\tau^2 \psi \alpha \frac{y^2}{(\delta - |y|^2)^2} v,\ -4 i \tau \psi \alpha \frac{y}{\delta - |y|^2} X v\right) =$$

$$= \tau^3 \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \nu_3 |\psi v|^2 \, ds \, dy.$$

We have:

$$M_9 \geqslant -\tau^3 C_7 \|\psi v\|^2,$$

$$M_{10} = 2 \operatorname{Re}\left(\psi \beta_1 X v,\ -4 i \tau \psi \alpha \frac{y}{\delta - |y|^2} X v\right) = 0,$$

$$M_{11} = 2 \operatorname{Re}\,(\psi D_s^2 v,\ i \psi \beta_2 X v) =$$

$$= -2 \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} (\psi^2 \beta_2)' D_s v \overline{Xv} \, ds \, dy + \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \nu_4 |\psi D_s v|^2 \, ds \, dy.$$

We use (9) and we easily get:

$$\frac{\partial}{\partial s}(\psi^2 \beta_2) = (\tilde{\gamma} + 1) \psi^2 \frac{\beta^2}{s},$$

where $\tilde{\gamma}(y, s) = \gamma(s(\delta - |y|^2))$. Therefore:

$$-2 \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} (\psi^2 \beta_2)' D_s v \overline{Xv} \, ds \, dy = \operatorname{Re}\left(2 i \psi \frac{D_s v}{s},\ i(\tilde{\gamma} + 1) \psi \beta_2 X v\right).$$

By (25) we have:

$$i\psi\beta_2 Xv = J_2 v + 2i\psi\frac{D_s v}{s} + 4i\tau\psi\alpha\frac{y}{\delta - |y|^2}Xv\,.$$

Finally we reach:

$$M_{11} \geqslant 4\tau\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}(\tilde{\gamma}+1)\left|\psi\frac{D_s v}{s}\right|^2 ds\,dy - \frac{1}{2}\|J_2 v\|^2 - C_8\left\|\psi\frac{D_s v}{s}\right\|^2 +$$

$$- \tau\rho_9\left\|\psi\frac{D_s v}{s}\right\|^2 - \tau C_9\left\|\psi\alpha\frac{y}{\delta - |y|^2}Xv\right\|^2 - C_{10}\|\psi D_s v\|^2.$$

$$M_{12} = 2\ \mathrm{Re}\left(-\tau(\tau-1)\psi\frac{v}{s^2}, i\psi\beta_2 Xv\right) = \tau(\tau-1)\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}\nu_5\left|\psi\frac{v}{s}\right|^2 ds\,dy\,.$$

We obtain:

$$M_{12} \geqslant -\tau^2 C_{11}\left\|\psi\frac{v}{s}\right\|^2,$$

$$M_{13} = 2\ \mathrm{Re}\,(\psi X(\alpha Xv), i\psi\beta_2 Xv) = 2\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}\psi^2\alpha\beta_2\frac{y}{\delta - |y|^2}|Xv|^2 ds\,dy\,,$$

consequently:

$$M_{13} \geqslant -C_{12}\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}\alpha|\psi Xv|^2 ds\,dy\,.$$

$$M_{14} = 2\ \mathrm{Re}\left(-4\tau^2\psi\alpha\frac{y^2}{(\delta - |y|^2)^2}v, i\psi\beta_2 Xv\right) = \tau^2\int_{-\sqrt{\delta}}^{\sqrt{\delta}}\int_0^{S_0}\nu_6|\psi v|^2 ds\,dy\,.$$

We have:

$$M_{14} \geqslant -\tau^2 C_{13}\|\psi v\|^2.$$

Finally:

$$M_{15} = 2\ \mathrm{Re}\,(\psi\beta_1 Xv, i\psi\beta_2 Xv) = 0\,.$$

By (8) we get:

$$M_2 \geqslant \varepsilon\tau^3 \left\| \psi \frac{v}{s^2} \right\|^2.$$

By (10) we deduce:

$$M_1 + M_{11} \geqslant 2\varepsilon\tau \left\| \psi \frac{D_s v}{s} \right\|^2 - C_8 \left\| \psi \frac{D_s v}{s} \right\|^2 - \frac{1}{2}\|J_2 v\|^2 +$$
$$- \tau\rho_9 \left\| \psi \frac{D_s v}{s} \right\|^2 - \tau C_9 \left\| \psi\alpha \frac{y}{\delta - |y|^2} Xv \right\|^2 - C_{10}\|\psi D_s v\|^2.$$

Finally, by (11) we have:

$$M_3 \geqslant \varepsilon\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \alpha \left| \psi \frac{Xv}{s} \right|^2 ds\, dy + \tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \alpha s \left[\frac{\alpha'}{\alpha} - \frac{\beta_1'}{\beta_1} \right] \left| \psi \frac{Xv}{s} \right|^2 ds\, dy +$$
$$- \tau\rho_1 \left\| \psi \frac{D_s v}{s} \right\|^2 - \tau C_1 \left\| \psi\alpha \frac{y}{\delta - |y|^2} Xv \right\|^2.$$

Choosing $\sum \rho_j \leqslant \varepsilon/2$ and collecting all this information, we easily obtain that there exist $\tau_0 > 0$ and $S_0 > 0$ depending only on the coefficients of $\tilde{Q}$, such that if $\tau \geqslant \tau_0$ then:

$$\text{(27)} \quad \|J_1 v + J_2 v\|^2 = \|J_1 v\|^2 + \|J_2 v\|^2 + \sum_{j=1}^{15} M_j \geqslant \frac{\varepsilon}{2}\tau^3 \left\| \psi \frac{v}{s^2} \right\|^2 +$$
$$+ \frac{\varepsilon}{2}\tau \left\| \psi \frac{D_s v}{s} \right\|^2 + \frac{\varepsilon}{2}\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \alpha \left| \psi \frac{Xv}{s} \right|^2 ds\, dy\,.$$

Using (22), (24) and (27) it is trivial to obtain now the following Carleman's estimate:

$$\|\psi s^{-\tau} \tilde{Q} u\|^2 \geqslant C\tau^3 \|\psi s^{-\tau-2} u\|^2.$$

Remembering that $u(y, s) = \chi(s)\,\tilde{w}(y, s)$, with $\tilde{w}$ null solution for $\tilde{Q}$, we reach the thesis of Theorem 1 in the standard way (see [11], Ch. 1).

c) Proof of Theorem 2.

The proof of Theorem 2 is similar to that of Theorem 1. We set:

$$\psi(y, s) = \left(\frac{1}{b_2(s(\delta - |y|^2))}\right)^{1/2}.$$

and we use the same test function $u(y, s)$ defined at the point *b*). We obtain analogous estimates for the M_j. Let us point out the main differences:

$$M_1 = 2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(1 + s\frac{\beta_2'}{\beta_2}\right) \left|\psi \frac{D_s v}{s}\right|^2 ds\, dy,$$

$$M_2 = 2\tau^2(\tau - 1) \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(3 + s\frac{\beta_2'}{\beta_2}\right) \left|\psi \frac{v}{s^2}\right|^2 ds\, dy,$$

$$M_3 \geq 2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(\alpha s\left[\frac{\alpha'}{\alpha} - \frac{\beta_2'}{\beta_2}\right] - \alpha\right) \left|\psi \frac{Xv}{s}\right|^2 ds\, dy -$$

$$- \tau\rho_1 \left\|\psi \frac{D_s v}{s}\right\|^2 - \tau C_1 \left\|\psi\alpha \frac{y}{\delta - |y|^2} Xv\right\|^2,$$

$$M_5 = -2\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(\frac{\psi^2\beta_1}{s}\right)' vX\bar{v}\, ds\, dy -$$

$$- 12\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \frac{\psi^2}{s}\beta_1 \frac{y}{\delta - |y|^2} vD_s\bar{v}\, ds\, dy.$$

We use the condition (13) and we deduce:

$$\frac{\partial}{\partial s}\left(\frac{\psi^2\beta_1}{s}\right) = \psi^2[\tilde{\sigma}\beta_2 - \beta_1]\frac{1}{s^2},$$

where $\tilde{\sigma}(y, s) = \sigma(s(\delta - |y|^2))$. Therefore:

$$-2\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(\frac{\psi^2 \beta_1}{s} \right) vX\bar{v}\, ds\, dy =$$

$$= -2\tau \operatorname{Re} \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} [\tilde{\sigma}\beta_2 - \beta_1]\, \psi^2 \frac{v}{s^2} X\bar{v}\, ds\, dy =$$

$$= 2\tau \operatorname{Re}\left(i\tilde{\sigma}\psi \frac{v}{s^2}, i\psi\beta_2 Xv \right) + 2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, -\psi\beta_1 Xv \right).$$

We use now (25), and we get:

$$2\tau \operatorname{Re}\left(i\tilde{\sigma}\psi \frac{v}{s^2}, i\psi\beta_2 Xv \right) = 2\tau \operatorname{Re}\left(i\tilde{\sigma}\psi \frac{v}{s^2}, J_2 v \right) +$$

$$+ 2\tau \operatorname{Re}\left(i\tilde{\sigma}\psi \frac{v}{s^2}, 2i\tau\psi \frac{D_s v}{s} \right) + 2\tau \operatorname{Re}\left(i\tilde{\sigma}\psi \frac{v}{s^2}, 4i\tau\psi\alpha \frac{y}{\delta - |y|^2} Xv \right) \geqslant$$

$$\geqslant -\tau^2 C_1 \left\| \psi \frac{v}{s^2} \right\|^2 - \frac{1}{2} \|J_2\|^2 - 2\tau^3 \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} |\tilde{\sigma}| \left| \psi \frac{v}{s^2} \right|^2 ds\, dy +$$

$$-2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} |\tilde{\sigma}| \left| \psi \frac{D_s v}{s} \right|^2 ds\, dy - \tau^3 \rho_2 \left\| \psi \frac{v}{s^2} \right\|^2 - \tau C_2 \left\| \psi\alpha \frac{y}{\delta - |y|^2} Xv \right\|^2,$$

$$2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, -\psi\beta_1 Xv \right) = 2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, -J_1 v \right) +$$

$$+ 2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, \psi D_s^2 v \right) + 2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, -\tau(\tau - 1)\psi \frac{v}{s^2} \right) +$$

$$+ 2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, \psi X(\alpha Xv) \right) + 2\tau \operatorname{Re}\left(\psi \frac{v}{s^2}, -4\tau^2 \psi\alpha \frac{y^2}{(\delta - |y|^2)^2} v \right) \geqslant$$

$$\geqslant -\tau^2 C_3 \left\|\psi \frac{v}{s^2}\right\|^2 - \frac{1}{2}\|J_1 v\|^2 - \tau^2 \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(2 + s\frac{\beta_2'}{\beta_2}\right) \left|\psi \frac{v}{s^2}\right|^2 ds\, dy +$$

$$- \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left(2 + s\frac{\beta_2'}{\beta_2}\right) \left|\psi \frac{D_s v}{s}\right|^2 ds\, dy + 2\tau \left\|\psi \frac{D_s v}{s}\right\|^2 +$$

$$- 2\tau^2(\tau - 1)\left\|\psi \frac{v}{s^2}\right\|^2 + 2\tau \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \alpha \left|\psi \frac{Xv}{s}\right|^2 ds\, dy +$$

$$- C_4 \left\|\psi\alpha \frac{y}{\delta - |y|^2} Xv\right\|^2 - \rho_4 \tau^2 \left\|\psi \frac{v}{s^2}\right\|^2 - \tau^3 C_5 \left\|\psi \frac{v}{s}\right\|^2.$$

We use both of these estimates to calculate M_5. We obtain:

$$M_5 \geqslant -\frac{1}{2}\|J_1 v\|^2 - \frac{1}{2}\|J_2 v\|^2 - \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left[2\tau^3|\tilde{\sigma}| + \tau^2\left(2 + s\frac{\beta_2'}{\beta_2}\right)\right] \left|\psi \frac{v}{s^2}\right|^2 ds\, dy +$$

$$-(2\tau^2(\tau - 1) + \tau^3\rho_2 + \tau^2 C_1 + \rho_4\tau^2) \left\|\psi \frac{v}{s^2}\right\|^2 - \tau^3 C_5 \left\|\psi \frac{v}{s}\right\|^2 +$$

$$- \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \int_0^{S_0} \left[2\tau(|\tilde{\sigma}| - 1) + \left(2 + s\frac{\beta_2'}{\beta_2}\right)\right] \left|\psi \frac{D_s v}{s}\right|^2 ds\, dy +$$

$$- (\tau C_2 + C_4) \left\|\psi\alpha \frac{y}{\delta - |y|^2} Xv\right\|^2.$$

Finally:

$$M_{11} \geqslant - C_5 \|\psi D_s v\|^2.$$

The other M_j can be easily calculate and the result is very similar to that of point *b*). It is easy now to obtain (27) and then the thesis. This completes the proof.

REFERENCES

[1] S. ALINHAC, *Unicité du problème de Cauchy pour des opérateurs du second ordre à symboles réels*, Ann. Inst. Fourier, Grenoble, **34** (1984), pp. 89-109.

[2] S. ALINHAC - C. ZUILY, *Unicité et non unicité du problème de Cauchy pour des opérateurs hyperboliques à caractéristiques doubles*, Comm. in P.D.E., 7 (1981), pp. 799-828.

[3] M. S. BAOUENDI - E. C. ZACHMANOGLOU, *Unique continuation of solutions of partial differential equations and inequalities from manifolds of any dimension*, Duke Math. J., **45** (1978), pp. 1-13.

[4] A. P. CALDERÓN, *Uniqueness in the Cauchy problem for partial differential equations*, Amer. J. Math., **80** (1958), pp. 16-36.

[5] P. COHEN, *The non-uniqueness of the Cauchy problem*, O.N.R. Techn. Report, **93**, Stanford (1960).

[6] F. COLOMBINI - D. DEL SANTO - C. ZUILY, *Uniqueness and non-uniqueness in the Cauchy problem of degenerate elliptic operators*, to appear.

[7] F. COLOMBINI - S. SPAGNOLO, *A non-uniqueness result for the operators with principal part $\partial_t^2 + a(t)\,\partial_x^2$,*, in *Partial Differential Equations and the Calculus of Variations Essays in honor de Ennio De Giorgi*, Progress in Non-linear P.D.E. and their appl., vol. 1, Birkhäuser, Boston (1989), pp. 331-353.

[8] L. NIRENBERG, *Uniqueness in the Cauchy problem for a degenerate elliptic second order equation*, in *Differential Geometry and Complex Analysis*, Springer-Verlag, Berlin (1985), pp. 213-218.

[9] X. SAINT RAYMOND, *L'unicité pour les problèmes de Cauchy linéaires du premier ordre*, L'Ens. Math., **32** (1986), pp. 1-55.

[10] K. WATANABE, *L'unicité du prolongement des solutions des équations elliptiques dégénérées*, Tohoku Math. J., **34** (1982), pp. 239-249.

[11] C. ZUILY, *Uniqueness and non-uniqueness in the Cauchy problem*, Progress in Math., Vol. **33**, Birkhäuser, Boston (1983).

Manoscritto pervenuto in redazione il 7 settembre 1990.