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PATRIZIA DONATO

GIOCONDA MOSCARIELLO

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On the Homogenization of Some Nonlinear Problems in Perforated Domains.

PATRIZIA DONATO - GIOCONDA MOSCARIELLO (*)

0. – Introduction.

In this paper we study the homogenization of a class of nonlinear elliptic Neumann problems in perforated domains of $\mathbb{R}^n$.

Let Ω_ε be a fixed bounded domain Ω from which a set T_ε of holes has been removed. The set T_ε is obtained in the following way: let T a fixed set properly contained in the basic cell Y , let D_ε be the hole homotetic by ratio ε to T . Let us suppose to have a periodic distribution of period εY of D_ε . Then T_ε is the set of the holes of this periodic distribution contained in Ω_ε .

Roughly speaking, let us consider the problem:

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} -\operatorname{div} a(x/\varepsilon, Du) = f & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega, \\ a(x/\varepsilon, Du) \cdot \nu = \varepsilon g(x, x/\varepsilon) & \text{on } \partial T_\varepsilon, \\ u \in H^{1,p}(\Omega_\varepsilon) & p > 1, \end{cases}$$

where ν denotes the exterior normal with respect to Ω_ε , $f \in L^{p'}(\Omega)$, g is Y -periodic in the second variable and $a(x, \xi)$ is a matrix periodic in x and satisfying suitable coerciveness and growth conditions in ξ .

(*) Indirizzo degli AA.: Dipartimento di Matematica e Applicazioni, Università di Napoli, via Mezzocannone 8, 80134 Napoli.

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Our aim is to study the asymptotic behaviour of the solutions u_ε of (P_ε) as $\varepsilon \rightarrow 0$. Indeed, we prove that the « limit » problem of $(\mathcal{F}_\varepsilon)$ is:

$$(\mathcal{F}_0) \quad \begin{cases} -\operatorname{div} b(Du) = f\theta + \mu_\sigma(x) & \text{in } \Omega, \\ u \in H_0^{1,p}(\Omega) & p > 1, \end{cases}$$

with $\theta = |Y \setminus T|/|Y|$ and

$$\mu_\sigma(x) = (1/|Y|) \int_{\partial T} g(x, y) d\sigma_y.$$

Moreover the matrix $b(\xi)$ is given by an explicit formula.

As a consequence, we are able in particular to describe the « homogenized » operator of the problem:

$$\begin{aligned} -\operatorname{div} (|Du|^{p-2} Du) &= f && \text{in } \Omega_\varepsilon, \\ u &= 0 && \text{on } \partial\Omega, \\ (|Du|^{p-2} Du) \cdot \nu &= g(x, x/\varepsilon) && \text{on } \partial T_\varepsilon, \\ u &\in H^{1,p}(\Omega_\varepsilon) && p > 1. \end{aligned}$$

The homogenization theory for linear elliptic operators goes back to De Giorgi-Spagnolo [9], Bensoussan-Lions-Papanicolau [2], Sanchez-Palencia [15].

For Dirichlet nonlinear problems of the type

$$-\operatorname{div} a(x, u, Du) = f$$

some homogenization results for $p = 2$ were first given by Tartar [19] (see also Suquet [16]). By different techniques, for $p > 1$, homogenization results have been recently given in [11].

On the other hand the homogenization of some linear problems in perforated domains has been studied in [6] and in Cioranescu-Saint Jean Paulin [7] by using energy method.

1. - Statement of the problem.

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$. Introduce the representative cell $Y = [0, l_1) \times \dots \times [0, l_n)$ and de-

note by T an open subset of Y , with smooth boundary ∂T , such that $\overline{T} \subset Y$. Denote by $\tau(\varepsilon \overline{T})$ the set of all traslated images $\varepsilon(k_1 + \overline{T})$, $k \in \mathbb{Z}^n$, $k_1 = (k_1 l_1, \dots, k_n l_n)$, $\varepsilon > 0$, of $\varepsilon \overline{T}$.

By this way $\mathbb{R}^n$ is periodically perforated by holes of ε -size.

We make the following assumption:

The holes $\tau(\varepsilon \overline{T})$ do not intersect the boundary $\partial \Omega$.

This assumption restricts the geometry of the open set Ω (example: Ω is a finite union of rectangles homothetic to the representative cell).

Define now the perforated domain:

$$\Omega_\varepsilon = \Omega \cap \{\mathbb{R}^n \setminus \tau(\varepsilon \overline{T})\}.$$

Hence, by the previous assumption it follows that:

$$\partial \Omega_\varepsilon = \partial \Omega \cap \partial T_\varepsilon$$

where T_ε is the subset of $\tau(\varepsilon \overline{T})$ contained in Ω .

The following notations are used in the following:

- 1) $Y^* = Y \setminus \overline{T}$;
- 2) $\theta = |Y^*|/|Y|$;
- 3) $|\omega|$ = the Lebesgue measure of ω (for any measurable set of $\mathbb{R}^n$);
- 4) $\chi(\omega)$ = the characteristic function of the set ω ;
- 5) $\tilde{v}$ = the zero extension to the whole Ω , for any function v defined on Ω_ε ;
- 6) $\langle f \rangle_E = (1/|E|) \int_E f(x) dx$, for $f \in L^1_{n, \text{loc}}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ bounded open set of positive measure.

If $1 < p < +\infty$ and $p' = p/(p-1)$, we shall consider the following spaces:

$$H^{1,p}_{\text{per}}(Y) =$$

$$= \{ u(y) \in H^{1,p}(Y) : u \text{ has the same trace on the opposite faces of } Y \},$$

$$L^{p'}_{n, \text{r}}(Y) = \{ q(y) \in L^p_n(Y) : \int_Y q \cdot Du dy = 0 \text{ for any } u \in H^{1,p}_{\text{per}}(Y) \}.$$

Now, let $f \in L^{p'}(\Omega)$ and $g: \Omega \times Y \rightarrow \mathbf{R}$ verifying the following assumptions:

$$(1.1)_1 \quad g(x, \cdot) \text{ is } Y\text{-periodic and measurable for any } x \in \Omega,$$

$$(1.1)_2 \quad g(x, \cdot) \in H^{1-1/p', p'}(\partial T),$$

$$(1.1)_3 \quad |g(x_1, y) - g(x_2, y)| \leq \\ \leq c(1 + |\varphi(y)|) \omega(|x_1 - x_2|) \text{ for any } y \in Y \text{ and } x_1, x_2 \in \Omega,$$

where $\omega(t): [0, +\infty) \rightarrow [0, +\infty)$ is a bounded, concave and continuous function such that $\omega(0) = 0$ and $\varphi(y) \in L^{p'}(\partial T)$.

We shall consider the problem:

$$(I_\varepsilon) \quad \begin{cases} -\operatorname{div} a(x/\varepsilon, Du) = f & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega, \\ a(x/\varepsilon, Du) \cdot \nu = \varepsilon g_\varepsilon(x) & \text{on } \partial T_\varepsilon, \end{cases}$$

where ν denotes the exterior normal with respect to Ω_ε , $g_\varepsilon(x) = g(x, x/\varepsilon)$ and $a(y, \xi)$ verifies the following structure conditions:

$H_1)$ a is Y -periodic and measurable with respect to y ,

$H_2)$ for any y a.e. in $\mathbf{R}^n$ and any $\xi_1, \xi_2 \in \mathbf{R}^n$ then

if $p \geq 2$:

$$\text{i) } |a(y, \xi_1) - a(y, \xi_2)| \leq \beta(1 + |\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|$$

$$\text{ii) } (a(y, \xi_1) - a(y, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p, \quad \alpha > 0$$

or, if $1 < p \leq 2$:

$$\text{j) } |a(y, \xi_1) - a(y, \xi_2)| \leq \beta |\xi_1 - \xi_2|^{p-1},$$

$$\text{jj) } (a(y, \xi_1) - a(y, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^2 (|\xi_1| + |\xi_2|)^{p-2}, \quad \alpha > 0,$$

$$H_3) \quad a(y, 0) \in L_n^{p'}(Y).$$

Set

$$V_*^p = \{\varphi \in H^{1,p}(\Omega_\varepsilon): \varphi = 0 \text{ on } \partial\Omega\},$$

a variational solution of problem $(\mathcal{P}_\varepsilon)$ is a function $u_\varepsilon \in V_\varepsilon^p$ such that:

$$(1.2) \quad \int_{\Omega_\varepsilon} a(x/\varepsilon, Du_\varepsilon) \cdot D\varphi \, dx = \int_{\Omega_\varepsilon} f\varphi \, dx + \varepsilon \int_{\partial T_\varepsilon} g_\varepsilon \varphi \, d\sigma$$

for any $\varphi \in V_\varepsilon^p$.

REMARK 1.1. It is well known that, under the above hypotheses, the problem $(\mathcal{P}_\varepsilon)$ has a unique solution $u_\varepsilon \in V_\varepsilon^p$. ■

Finally, let us consider the problem:

$$(1.3) \quad \begin{cases} \int_{Y^*} a(y, Dv(y)) \cdot D\varphi(y) \, dy = 0 & \forall \varphi \in H_{\text{per}}^{1,p}(Y^*), \\ v \in \xi \cdot y + H_{\text{per}}^{1,p}(Y^*), \end{cases}$$

where $\xi \in \mathbb{R}^n$ and

$$\begin{aligned} H_{\text{per}}^{1,p}(Y^*) &= \\ &= \{u(y) \in H^{1,p}(Y^*): u \text{ has the same trace on the opposite faces of } Y\}. \end{aligned}$$

In the § 3 we will prove the convergence of the solutions $u_\varepsilon \in V_\varepsilon^p$ of $(\mathcal{P}_\varepsilon)$ to the solution of the « homogenized » problem:

$$(\mathcal{P}_0) \quad \begin{cases} -\operatorname{div} b(Du) = f\theta + \mu_\sigma(x) & \text{in } \Omega, \\ u \in H_0^{1,p}(\Omega), \end{cases}$$

where

$$(1.4) \quad \mu_\sigma(x) = (1/|Y|) \int_{\partial T} g(x, y) \, d\sigma_y$$

and, for any $\xi \in \mathbb{R}^n$, if $v(y)$ is the solution of (1.3), b is defined by

$$(1.5) \quad b(\xi) = (1/|Y|) \int_{Y^*} a(y, Dv) \, dy.$$

2. – Preliminary results.

We recall some lemmas about the spaces introduced in § 1 and the existence of a family of extension-operators.

LEMMA 2.1. *If $q(y)$ is an element of $L_{n, \text{per}}^{p'}(Y)$, then it can be extended by periodicity to an element of $L_{n, \text{loc}}^{p'}(\mathbb{R}^n)$, still denoted by q , such that*

$$\operatorname{div}_x q = 0. \quad \blacksquare$$

LEMMA 2.2. *Let f be an Y -periodic function of $L_{\text{loc}}^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$ and set*

$$f_\varepsilon(x) = f(x/\varepsilon) \quad \text{for any } x \text{ a.e. in } \mathbb{R}^n,$$

then as $\varepsilon \rightarrow 0$:

$$\begin{aligned} f_\varepsilon &\rightarrow \langle f \rangle && \text{in } w\text{-}L_{\text{loc}}^p(\mathbb{R}^n) \text{ if } p < +\infty \\ f_\varepsilon &\rightarrow \langle f \rangle && \text{in } w^*\text{-}L^\infty(\mathbb{R}^n) \text{ if } p = +\infty. \quad \blacksquare \end{aligned}$$

For a proof of the previous lemmas one may see [16] annexe 2.

LEMMA 2.3 (see [7]). *There exists a linear continuous extension-operator $P \in \mathcal{L}(H^{1,p}(Y^*), H^{1,p}(Y))$ such that:*

$$(2.1) \quad \|D(P\varphi)\|_{L_n^p(r)} \leq \|D\varphi\|_{L_n^p(r^*)}$$

for any $\varphi \in H^{1,p}(Y^)$. $\blacksquare$*

LEMMA 2.4 (see [7]). *There exists a family P_ε of linear continuous extension-operators, $P_\varepsilon \in \mathcal{L}(V_\varepsilon^p, H_0^{1,p}(\Omega))$ verifying the following condition:*

$$(2.2) \quad \|D(P_\varepsilon\varphi)\|_{L_n^p(\Omega)} \leq \|D\varphi\|_{L_n^p(\Omega_\varepsilon)}$$

for any $\varphi \in V_\varepsilon^p$, where c is a constant independent of ε . $\blacksquare$

The previous lemmas are proved in the case $p = 2$ in [7]. The same argument can be used in the general case.

If $\gamma \in W^{1-1/p', p'}(\partial T)$ let us consider the linear form on $H^{1,p}(\Omega)$, $1/p + 1/p' = 1$:

$$\langle \mu_\gamma^\varepsilon, \varphi \rangle = \varepsilon \int_{\partial T_\varepsilon} \gamma(x/\varepsilon) \varphi \, d\sigma$$

and

$$\mu_\gamma = (1/|Y|) \int_{\partial T} \gamma(x) \, d\sigma_y.$$

PROPOSITION 2.5 (see [6], [12]). *If $\gamma \in W^{1-1/p', p'}(\partial T)$, $1 < p' < +\infty$, there exists a unique solution $\psi_\gamma \in W^{2,p'}(Y^*)$ of the problem:*

$$(2.3) \quad \begin{cases} -\Delta \psi_\gamma = -(|Y|/|Y^*|) \mu_\gamma & \text{in } Y^*, \\ \partial \psi_\gamma / \partial \nu = \gamma & \text{on } \partial T, \\ \psi_\gamma \text{ } Y\text{-periodic}, \\ \langle \psi_\gamma \rangle_{Y^*} = 0. \end{cases}$$

In particular if γ is a constant function, then $\psi_\gamma \in W^{1,\infty}(Y^)$. ■*

REMARK 2.6. It is easy to verify that the solution of problem (2.3) can be extended by periodicity to $\mathbb{R}^n \setminus \tau(\bar{T})$ and the function

$$\psi_\gamma^\varepsilon(x) = \psi_\gamma^\varepsilon(x/\varepsilon) \quad \text{a.e. } x \in \Omega_\varepsilon$$

verify:

$$(2.4) \quad \begin{cases} -\Delta \psi_\gamma^\varepsilon = -\varepsilon^{-2}(|Y|/|Y^*|) \mu_\gamma & \text{in } \mathbb{R}^n \setminus \tau(\varepsilon \bar{T}), \\ \partial \psi_\gamma^\varepsilon / \partial \nu = \varepsilon^{-1} \gamma(x/\varepsilon) & \text{on } \partial \tau(\varepsilon \bar{T}). \end{cases} \quad \blacksquare$$

Now, we can prove the following lemma that we'll use in the sequel:

LEMMA 2.7. *Let Q be an interval of $\mathbb{R}^n$, $S_\varepsilon = \tau(\varepsilon \bar{T}) \cap Q$ and $Q_\varepsilon = Q \setminus \bar{S}_\varepsilon$. If $\bar{S}_\varepsilon \cap \partial Q = \varphi$ then:*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\partial S_\varepsilon} \gamma(x/\varepsilon) \varphi_\varepsilon(x) \, d\sigma = \mu_\gamma \int_Q \varphi(x) \, dx$$

for any sequence $\{\varphi_\varepsilon\}$ of $H^{1,p}(\Omega)$ such that $\varphi_\varepsilon \rightarrow \varphi$ in $w\text{-}H^{1,p}(\Omega)$, $1 < p < +\infty$.

PROOF. Let ψ_γ be the solution of (2.3), then by remark 2.6 we have:

$$(2.5) \quad \varepsilon \int_{\partial S_\varepsilon} \gamma(x/\varepsilon) \varphi_\varepsilon(x) d\sigma = \\ = \varepsilon^2 \int_{Q_\varepsilon} D_x \psi_\gamma^\varepsilon \cdot D_x \varphi_\varepsilon dx + (|Y|/Y^*) \mu_\gamma \int_{Q_\varepsilon} \varphi_\varepsilon dx - \varepsilon^2 \int_{\partial Q} D_x \psi \varphi_\gamma^\varepsilon \cdot \nu_i d\sigma.$$

Then by lemma 2.2 and by observing that $D_x \psi_\gamma^\varepsilon(x) = (1/\varepsilon) D_\nu \psi_\gamma(x/\varepsilon)$, passing to the limit as $\varepsilon \rightarrow 0$ we get the result. ■

REMARK 2.8. If h is a constant function, by using similar arguments as in the previous lemma, it can be found that

$$\varepsilon \int_{\partial S_\varepsilon} |\varphi_\varepsilon| d\sigma \leq c$$

(with c independent of ε) for any sequence $\{\varphi_\varepsilon\}$ bounded in $H^{1,1}(Q)$. ■

Let $g: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ verifying (1.1)₁ ÷ (1.1)₃ and

$$\langle \mu_\sigma^\varepsilon, \varphi \rangle = \varepsilon \int_{\partial T_\varepsilon} g(x, x/\varepsilon) \varphi d\sigma \quad \forall \varphi \in H_0^{1,p}(\Omega), \quad 1 < p < +\infty.$$

LEMMA 2.9. The measures $\mu_\sigma^\varepsilon, \mu_\sigma$ are in $H^{-1,p'}(\Omega)$ and

$$\mu_\sigma^\varepsilon \rightarrow \mu_\sigma \quad \text{in } s\text{-}H^{-1,p'}(\Omega),$$

where μ_σ is given by (1.4).

PROOF. It is enough to prove that

$$(2.6) \quad \langle \mu_\sigma^\varepsilon, \varphi_\varepsilon \rangle \rightarrow \langle \mu_\sigma, \varphi \rangle$$

if $\varphi_\varepsilon, \varphi \in H_0^{1,p}(\Omega)$ and $\varphi_\varepsilon \rightarrow \varphi$ in $w\text{-}H_0^{1,p}(\Omega)$.

Let us consider $\forall \nu \in \mathbb{N}$ a partition of $\mathbb{R}^n$ by intervals $Q_{i\nu}$ of side $2^{-\nu} l_i$. Since the holes T_ε do not intersect $\partial\Omega$, we can assume that $\forall \varepsilon > 0$, T_ε does not intersect $\partial Q_{i\nu}$.

Let us denote by $x_{i\nu}$ and $\chi_{i\nu}$ respectively the center and the characteristic function of $Q_{i\nu}$. Set

$$g_\nu(x, y) = \sum_i \chi_{i\nu}(x) g(x_{i\nu}, y) .$$

We have

$$\varepsilon \int_{\partial T_\varepsilon} [g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)] \varphi_\varepsilon d\sigma = \varepsilon \int_{\partial T_\varepsilon} [g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)] \varphi_\varepsilon d\sigma + \varepsilon \int_{\partial T_\varepsilon} g_\nu(x, x/\varepsilon) \varphi_\varepsilon d\sigma .$$

We observe that by (1.1)₃ $\mu_\sigma \in C^0(\bar{\Omega})$, then by lemma 2.7 we have:

$$\begin{aligned} (2.7) \quad \lim_{\nu \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\partial T_\varepsilon} g_\nu(x, x/\varepsilon) \varphi_\varepsilon(x) d\sigma &= \lim_{\nu \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \sum_i \varepsilon \int_{Q_{i\nu} \cap \partial T_\varepsilon} g(x_{i\nu}, x/\varepsilon) \varphi_\varepsilon(x) d\sigma = \\ &= \lim_{\nu} \sum_i \int_{Q_{i\nu} \cap \Omega} \mu_\sigma(x_{i\nu}) \varphi(x) dx = \lim_{\nu} \int_{\Omega} \chi_{i\nu} \mu_\sigma(x_{i\nu}) \varphi(x) dx = \\ &= \int_{\Omega} \mu_\sigma(x) \varphi(x) dx . \end{aligned}$$

On the other hand we have:

$$\begin{aligned} (2.8) \quad \varepsilon \int_{\partial T_\varepsilon} [g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)] \varphi_\varepsilon(x) d\sigma &\leq \\ &\leq \varepsilon^{1/p'} \left[\sum_i \int_{Q_{i\nu} \cap \partial T_\varepsilon} |g(x, x/\varepsilon) - g(x_{i\nu}, x/\varepsilon)|^{p'} d\sigma \right]^{1/p'} \varepsilon^{1/p} \left[\sum_i \int_{Q_{i\nu} \cap \partial T_\varepsilon} |\varphi_\varepsilon(x)|^p d\sigma \right]^{1/p} . \end{aligned}$$

By remark 2.8 we get

$$(2.9) \quad \varepsilon^{1/p} \left[\sum_i \int_{Q_{i\nu} \cap \partial T_\varepsilon} |\varphi_\varepsilon(x)|^p d\sigma \right]^{1/p} \leq c .$$

Then by (2.8), (2.9) and (1.1)₁ ÷ (1.1)₃:

$$\begin{aligned} \varepsilon \int_{\partial T_\varepsilon} [g(x, x/\varepsilon) - g_\nu(x, x/\varepsilon)] \varphi_\varepsilon(x) d\sigma &\leq c \omega(1/2^\nu) \left[\varepsilon \int_{\partial T_\varepsilon} (1 + |\varphi(x/\varepsilon)|^{p'}) d\sigma \right]^{1/p'} \leq \\ &\leq c' \omega(1/2^\nu) \left[\varepsilon \int_{\partial T} (1 + |\varphi(y)|^{p'}) d\sigma \right]^{1/p'} . \end{aligned}$$

Then passing to the limit as $\varepsilon \rightarrow 0^+$ and $\nu \rightarrow +\infty$ we obtain (2.6).

The proof is completed by using standard arguments concerning the duality application between $H_0^{1,p}(\Omega)$ and its dual. ■

We state now some lemmas about the structure properties of b . Using the same arguments of [11] one can prove:

LEMMA 2.10. *For any $\xi \in \mathbb{R}^n$*

$$|b(\xi)| \leq c(1 + |\xi|)^{p-1}$$

where $c = c(\alpha, \beta, p, |Y|, \|a(y, 0)\|_{L^{p'}(Y)})$.

Further if v is the solution of problem (1.3) we have:

$$(2.10) \quad \int_{Y^*} |Dv(y)|^p dy \leq c(1 + |\xi|)^p$$

where $c = c(\alpha, \beta, p, |Y|, \|a(y, 0)\|_{L^{p'}(Y)})$. ■

LEMMA 2.1. *$b(\xi)$ is locally Holder (Lipschitz if $p = 2$).* ■

REMARK 2.12. We remark that the limit operator $b(\xi)$, as in the homogenization of Dirichlet problem (see [11]) may not verify the same structure conditions of $a(y, \xi)$.

In some special case, the Holder estimate on $b(\xi)$ can be improved (see [11]). ■

LEMMA 2.13. *For $\xi_1, \xi_2 \in \mathbb{R}^n$ we have*

$$(2.11) \quad (b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p, \quad \text{if } p > 2$$

$$(2.12) \quad (b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geq \alpha' |\xi_1 - \xi_2|^2 (1 + |\xi_1| + |\xi_2|)^{p-2},$$

$$\alpha' > 0, \quad \text{if } 1 < p \leq 2.$$

PROOF. Let us denote by v_1 and v_2 the solutions of problem (1.3) defining respectively $b(\xi_1)$ and $b(\xi_2)$.

Let us consider $u_i = v_i - \xi \cdot y$, $i = 1, 2$. Then $u_i(y)$ is an element of $H_{\text{per}}^{1,p}(Y^*)$ and so, by lemma 2.3, we can consider $Pu_i \in H_{\text{per}}^{1,p}(Y)$.

If we extend Pu_i by periodicity, the resulting function (still denoted by Pu_i) is in $H_{\text{loc}}^{1,p}(\mathbb{R}^n)$.

Define

$$w_i^\varepsilon = \varepsilon Pu_i(x/\varepsilon) + \xi_i \cdot x, \quad i = 1, 2,$$

and for $\xi \in \mathbb{R}^n$

$$(2.13) \quad \tilde{a}(y, \xi) = \begin{cases} a(y, \xi) & \text{for } y \in Y^*, \\ 0 & \text{for } y \in \bar{T}. \end{cases}$$

It is easy to verify that

$$(2.14) \quad \begin{cases} w_i^\varepsilon \rightarrow \xi_i \cdot x & \text{in } w\text{-}H_{\text{loc}}^{1,p}(\mathbb{R}^n), \\ \tilde{a}(x/\varepsilon, Dw_i^\varepsilon) \rightarrow b(\xi_i) & \text{in } w\text{-}L_{n,\text{loc}}^{p'}(\mathbb{R}^n), \\ \operatorname{div} \tilde{a}(x/\varepsilon, Dw_i^\varepsilon) = 0, \end{cases}$$

where the last relation is proved by using lemma 2.1.

If $p \geq 2$ from ii) of H_2 , we get:

$$\alpha \int_{Y^*} \eta |Dw_1^\varepsilon - Dw_2^\varepsilon|^p dx \leq \int_Y \eta (\tilde{a}(x/\varepsilon, Dw_1^\varepsilon) - \tilde{a}(x/\varepsilon, Dw_2^\varepsilon), Dw_1^\varepsilon - Dw_2^\varepsilon) dx$$

where $\eta \in C_0^1(Y^*)$.

Then, passing to the limit as $\varepsilon \rightarrow 0$ and using (2.14), by the compensated compactness result of [12] we get

$$\alpha \int_{Y^*} \eta |\xi_1 - \xi_2|^p \leq \int_{Y^*} \eta (b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) dx.$$

Then from the arbitrariness of η , we deduce (2.11).

If $1 < p \leq 2$, from jj) of H_2 we have:

$$\begin{aligned} \sqrt{\alpha} \int_{Y^*} |Dw_1^\varepsilon - Dw_2^\varepsilon| dx &\leq \left(\int_Y \eta (\tilde{a}(x/\varepsilon, Dw_1^\varepsilon) - \tilde{a}(x/\varepsilon, Dw_2^\varepsilon), Dw_1^\varepsilon - Dw_2^\varepsilon) dx \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_Y \eta (|Dw_1^\varepsilon| + |Dw_2^\varepsilon|)^{2-p} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Then, passing to the limit as before, and remarking that by lemma 2.3

$$\int_Y |Dw_i^\varepsilon|^p dx \leq c \int_{Y^*} |Dv_i^\varepsilon|^p dy ,$$

we can argue in a similar way to prove (2.12). ■

3. – Homogenization results.

THEOREM 3.1. *If $a(x, \xi)$ verifies the structure conditions $H_1) \div H_3)$ and $g: \Omega \times Y \rightarrow \mathbb{R}$ satisfies $(1.1)_1 \div (1.1)_3$, then for any $f \in L^{p'}(\Omega)$ the sequence u_ε of the solutions of problem $(\mathcal{I}_\varepsilon)$ verifies:*

$$\begin{aligned} P_\varepsilon u_\varepsilon &\rightarrow u_0 && \text{in } w\text{-}H_0^{1,p}(\Omega) , \\ \tilde{a}(x/\varepsilon, Du_\varepsilon) &\rightarrow b(Du_0) && \text{in } w\text{-}L_n^{p'}(\Omega) , \end{aligned}$$

where u_0 is the solution of problem $(\mathcal{I}_0)$.

PROOF. We use the homogenization techniques of nonlinear operators introduced in [10], [11].

Let us denote by $P_\varepsilon u_\varepsilon$ the extension of u_ε given by lemma 2.4. By using lemma 2.9, it is very easy to verify that

$$\|P_\varepsilon u_\varepsilon\|_{H_0^{1,p}(\Omega)} \leq c$$

with c independent of ε .

Then by i) or j) of $H_2)$, we get also:

$$\|\tilde{a}(x/\varepsilon, Du_\varepsilon)\|_{L_n^{p'}} \leq c$$

with c independent of ε .

Hence, up to a subsequence, we have

$$\begin{aligned} P_\varepsilon u_\varepsilon &\rightarrow u_0 && \text{in } w\text{-}H_0^{1,p}(\Omega) , \\ \tilde{a}(x/\varepsilon, Du_\varepsilon) &\rightarrow a_0(x) && \text{in } w\text{-}L_n^{p'}(\Omega) . \end{aligned}$$

The theorem will be proved if we show that:

$$(3.1) \quad a_0(x) = b(Du_0) \quad \text{a.e. in } \Omega.$$

Indeed, using lemma 2.9 and the fact that

$$(3.2) \quad \chi_{\Omega_\nu} \rightarrow \theta \quad \text{in } w^*-L^\infty(\Omega)$$

we obtain:

$$\int_{\Omega} a_0(x) D\varphi \, dx = \int_{\Omega} \theta f \varphi \, dx + \int_{\Omega} \mu_g \varphi \, dx \quad \forall \varphi \in H_0^{1,p}(\Omega).$$

Let us fix $\nu \in \mathbb{N}$ and denote by $\{Q_{i\nu}\}_i$ a partition of $\mathbb{R}^n$ as in the proof of lemma 2.7. Then we define $I_\nu = \{i: Q_{i\nu} \subset \Omega\}$, $\Omega_\nu = \bigcup_{i \in I_\nu} Q_{i\nu}$. For any i let us consider $\langle Du_0 \rangle_{i\nu} = \langle Du_0 \rangle_{Q_{i\nu}}$. Then if $\chi_{i\nu}$ is the characteristic function of $Q_{i\nu}$, by the continuity of b (see lemma 2.11) we have, if $\nu \rightarrow +\infty$, that:

$$(3.3) \quad \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle Du_0 \rangle_{i\nu}) \rightarrow b(Du_0(x)) \quad \text{a.e. in } \Omega.$$

Moreover, from lemma 2.10 we have for any measurable set $E \subset \Omega$:

$$\int_E \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle Du_0 \rangle_{i\nu}) \right|^{p'} dx \leq c \int_E \left(1 + \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) \langle Du_0 \rangle_{i\nu} \right|^p \right) dx.$$

So, from the equi-absolute continuity of the integral on the left-hand side and from (3.3) we deduce that:

$$(3.4) \quad \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle Du_0 \rangle_{i\nu}) \rightarrow b(Du_0(x)) \quad \text{in } L_n^{p'}(\Omega), \quad \text{as } \nu \rightarrow +\infty.$$

Let $v_{i\nu} \in \langle Du_0 \rangle_{i\nu} \cdot y + H_{\text{per}}^{1,p}(Y^*)$ the solution of problem (1.3) corresponding to $\langle Du_0 \rangle_{i\nu}$:

Then:

$$u_{i\nu} = (v_{i\nu} - \langle Du_0 \rangle_{i\nu} \cdot y) \in H_{\text{per}}^{1,p}(Y^*)$$

and, by lemma 2.3, $Pu_{i\nu} \in H_{\text{per}}^{1,p}(Y)$.

Set

$$w_{i\nu}(y) = Pu_{i\nu}(y) - \langle Du_0 \rangle_{i\nu} \cdot y$$

and

$$w_{i\nu}^\varepsilon(x) = \varepsilon w_{i\nu}(x/\varepsilon),$$

by arguing as in the proof of lemma 2.13 we obtain:

$$(3.5) \quad \begin{cases} w_{i\nu}^\varepsilon \rightarrow \langle Du_0 \rangle_{i\nu} \cdot x & \text{in } w\text{-}H_{\text{loc}}^{1,p}(\mathbb{R}^n), \\ \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon) \rightarrow b(\langle Du_0 \rangle_{i\nu}) & \text{in } w\text{-}L_{n,\text{loc}}^{p'}(\mathbb{R}^n), \\ \operatorname{div}_x \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon) = 0. \end{cases}$$

Using the periodicity of $Pu_{i\nu}$ and lemma 2.3, we have:

$$\begin{aligned} \sum_{i \in I_\nu} \int_{Q_{i\nu}} |Dw_{i\nu}^\varepsilon|^p dx &\leq \sum_{i \in I_\nu} 2^{-r_n} \varepsilon^n (1/\varepsilon + 2^n)^n \int_Y |Dw_{i\nu}^\varepsilon(y)|^p dy \leq \\ &\leq c \sum_{i \in I_\nu} 2^{-r_n} (1 + \varepsilon^n 2^{r_n}) \left(\int_{Y^*} |Dv_{i\nu}|^p dy + |\langle Du_0 \rangle_{i\nu}|^p \right)^p. \end{aligned}$$

The from (2.10), writing the last term as an integral over Ω_ν we have:

$$(3.6) \quad \sum_{i \in I_\nu} \int_{Q_{i\nu}} |Dw_{i\nu}^\varepsilon|^p dx \leq c(1 + \varepsilon^n 2^{r_n}) \int_{\Omega} (1 + |Du_0|)^p dx.$$

Let now $\eta \in C_0^1(Q_{i\nu})$, $0 \leq \eta \leq 1$, and extend it by periodicity to the whole $\mathbb{R}^n$.

Case $p \geq 2$. If $\varphi \in C_n^0(\bar{\Omega})$, set $M_\varphi = \sup_{\Omega} |\varphi|$, from i) of H_2) we have:

$$\begin{aligned} (3.7) \quad &\left| \int_{\Omega} \tilde{a}(x/\varepsilon, Du_\varepsilon) \varphi \eta dx - \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon) \varphi \eta dx \right| \leq \\ &\leq c M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \left| \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} [a(x/\varepsilon, Du_\varepsilon) - a(x/\varepsilon, Dw_{i\nu}^\varepsilon)] \varphi \eta dx \right| \leq \\ &\leq c M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} M_\varphi \eta \{ (|Du_\varepsilon| + |Dw_{i\nu}^\varepsilon|)^{p-2} \cdot |Du_\varepsilon - Dw_{i\nu}^\varepsilon| \} dx \leq \\ &\leq c M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + c M_\varphi^{p/(p-1)} \delta^{p/(p-1)} (1 + \varepsilon^n 2^{r_n}) + \\ &+ \delta^{-p} \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^p dx \end{aligned}$$

where the last inequality is obtained by applying Young inequality with $\delta > 0$ and the estimate (3.5).

On the other hand from ii)

$$\begin{aligned} \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^p dx &\leq \\ &\leq \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (\tilde{a}(x/\varepsilon, Du_\varepsilon) - \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon), Du_0 - \langle Du_0 \rangle_{i\nu}) dx + \\ &+ \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (\tilde{a}(x/\varepsilon, Du_\varepsilon) - \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon), (DP_\varepsilon u_\varepsilon - Dw_{i\nu}^\varepsilon) - \\ &- (Du_0 - \langle Du_0 \rangle_{i\nu})) dx. \end{aligned}$$

Then, integrating by parts and using (3.5) and the fact that u_ε is the solution of $(\mathcal{F}_\varepsilon)$, we get:

$$\begin{aligned} (3.8) \quad \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^p dx &\leq \\ &\leq \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (\tilde{a}(x/\varepsilon, Du_\varepsilon) - \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon), Du_0 - \langle Du_0 \rangle_{i\nu}) dx + \\ &+ \sum_{i \in I_\nu} \int_{Q_{i\nu}} [\eta f \chi_{\Omega_\varepsilon} - D\eta (\tilde{a}(x/\varepsilon, Du_\varepsilon) - \tilde{a}(x/\varepsilon, Dw_{i\nu}^\varepsilon))] \cdot \\ &\cdot [(P_\varepsilon u_\varepsilon - u_0) - (w_{i\nu}^\varepsilon - \langle Du_0 \rangle_{i\nu} \cdot x)] dx + \\ &+ \langle \mu_\nu^\varepsilon, \eta [(P_\varepsilon u_\varepsilon - u_0) - (w_{i\nu}^\varepsilon - \langle Du_0 \rangle_{i\nu} \cdot x)] \rangle. \end{aligned}$$

Then, passing to the limit as $\varepsilon \rightarrow 0$ in (3.8), by (3.2), (3.5) and lemma 2.9, we obtain

$$\begin{aligned} (3.9) \quad \lim_{\varepsilon \rightarrow 0} \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |Du_\varepsilon - Dw_{i\nu}^\varepsilon|^p dx &\leq \\ &\leq \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (a_0(x) - b(\langle Du_0 \rangle_{i\nu}), Du_0 - \langle Du_0 \rangle_{i\nu}) dx. \end{aligned}$$

If we first pass to the limit in (3.7) as $\varepsilon \rightarrow 0$, then $\eta \rightarrow 1$, $\nu \rightarrow +\infty$, by (3.9) we get:

$$\left| \int_{\Omega} a_0(x) \varphi dx - \int_{\Omega} b(Du_0) \varphi dx \right| \leq c M_\varphi^{p/(p-1)} \delta^{p/(p-1)}$$

So, letting $\delta \rightarrow 0$, from the arbitrariness of φ we deduce (3.1).

Case 1 $1 < p \leq 2$. In this case the proof is very similar to the previous case. Indeed, by using j) of H_2 , we have:

$$\begin{aligned} \left| \int_{\Omega} \tilde{a}(x/\varepsilon, Du_{\varepsilon}) \varphi \eta \, dx - \sum_{i \in I_{\nu}} \int_{Q_{i\nu}} \tilde{a}(x/\varepsilon, Dw_{i\nu}^{\varepsilon}) \varphi \eta \, dx \right| \leq \\ \leq c M_{\varphi} |\Omega \setminus \Omega_{\nu}|^{1/p} + \beta \sum_{i \in I_{\nu}} \int_{Q_{i\nu} \cap \Omega_{\varepsilon}} |Du_{\varepsilon} - Dw_{i\nu}^{\varepsilon}|^{p-1} |\varphi| \eta \, dx. \end{aligned}$$

Then, using jj) we can control the last term:

$$\begin{aligned} \sum_{i \in I_{\nu}} \int_{Q_{i\nu} \cap \Omega_{\varepsilon}} |Du_{\varepsilon} - Dw_{i\nu}^{\varepsilon}|^{p-1} |\varphi| \eta \, dx \leq \\ \leq c \delta^{2/(3-p)} \sum_{i \in I_{\nu}} \int_{Q_{i\nu} \cap \Omega_{\varepsilon}} (|Du_{\varepsilon}| + |Dw_{i\nu}^{\varepsilon}|)^{(2-p)(p-1)/(3-p)} \, dx + \\ + \delta^{-2/(3-p)} \sum_{i \in I_{\nu}} \int_{Q_{i\nu}} \eta (\tilde{a}(x/\varepsilon, Du_{\varepsilon}) - \tilde{a}(x/\varepsilon, Dw_{i\nu}^{\varepsilon}), DP_{\varepsilon} u_{\varepsilon} - Dw_{i\nu}^{\varepsilon}) \eta \, dx. \end{aligned}$$

Hence, arguing as in the previous case we obtain the result. $\blacksquare$

An easy consequence of theorem 3.1 is the following result:

COROLLARY 3.2. *Assume that $a(x, \xi)$ verifies the structure conditions $H_1) \div H_3)$ and is homogeneous of degree $p-1$ with respect to ξ .*

Then for any $f \in L^{p'}(\Omega)$ and $g(x, y)$ verifying $(1.1_1) \div (1.1)_3$ with $\mu_i \neq 0$, the function $b(\xi)$ is homogeneous of degree $p-1$ and the sequence v_{ε} of the solutions of the problem:

$$\begin{aligned} -\operatorname{div} a(x/\varepsilon, Dv_{\varepsilon}) &= f && \text{in } \Omega_{\varepsilon}, \\ v_{\varepsilon} &= 0 && \text{on } \partial\Omega, \\ a(x/\varepsilon, Dv_{\varepsilon}) \cdot \nu &= g_{\varepsilon} && \text{on } \partial T_{\varepsilon}, \end{aligned}$$

verifies:

$$\begin{aligned} \varepsilon^{1/(p-1)} P_{\varepsilon} v_{\varepsilon} &\rightarrow v_0 && \text{in } w\text{-}H_0^{1,p}(\Omega), \\ \varepsilon \tilde{a}(x/\varepsilon, Dv_{\varepsilon}) &\rightarrow b(Dv_0) && \text{in } w\text{-}L_n^{p'}(\Omega), \end{aligned}$$

where v_0 is the solution of the problem:

$$\begin{aligned} -\operatorname{div} b(Dv_0) &= \mu_0 \quad \text{in } \Omega_\varepsilon, \\ v_0 &\in H_0^{1,p}(\Omega), \end{aligned}$$

with $b(\xi)$ and μ_0 given respectively by (1.4), (1.5) and (1.3).

PROOF. The result follows by applying theorem 3.2 to the sequence $u_\varepsilon = \varepsilon^{1/(p-1)}v_\varepsilon$. ■

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