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Groups in which n -maximal subgroups are dualpronormal.

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1. – Introduction.

In the theory of finite soluble groups, embedding properties of subgroups, as pronormality, normal embedding, etc., have a very important rôle.

The concept of pronormality is one of the most important, it was first introduced by P. Hall and then investigated by Gaschütz, Fischer, Mann and many others. Pronormality plays a part in the theory of F -projectors, F a formation, and in this sense it can be dualized.

Let G be a group, U a subgroup of G , U is said to be dualpronormal, in short U dpn G , if the Fitting subgroup of $\langle U, U^g \rangle$ is contained in U , for every g in G . In a previous work [1] it was proved that if every maximal subgroup is dualpronormal then the group is nilpotent. Huppert [5], Mann [6] and Schmidt [7] have investigated groups whose n -maximal subgroups, $n > 1$, are respectively normal, subnormal, modular. The purpose of this paper is to study groups whose n -maximal subgroups, $n > 1$, are dualpronormal.

Groups in which 2- or 3-maximal subgroups are dualpronormal are completely described; it is proved, in particular, that they are «almost all» supersoluble and, as for $n = 1$, they are exactly the groups in which 2- or 3-maximal subgroups are normal.

Some results, holding for $n = 2$ and $n = 3$, hold for every natural number n .

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Throughout this paper «group» is used to mean «finite soluble group». The notation will be mostly standard.

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2. – Let G be a group, n -maximal subgroups of G are defined by induction: if U is a maximal subgroup of G , U is said to be 1-maximal in G ; let $n > 1$, a subgroup U is said to be n -maximal in G , if U is $(n - 1)$ -maximal in a maximal subgroup M of G .

Observe that a subgroup of a nilpotent group is dualpronormal if and only if it is normal. Many properties of dualpronormal subgroups can be founded in [1], the next will be frequently used in the following.

LEMMA 1 ([1] prop. 2.4). *Let G be a group, V an N -injector of G . Then:*

- (a) V is dualpronormal in G .
- (b) Every nilpotent dualpronormal subgroup of G is contained in a conjugate of V .

The following theorem describes the groups in which every 2-maximal subgroup is dualpronormal.

THEOREM 1. *If every 2-maximal subgroup of a group G is dualpronormal, then either:*

- (a) G is nilpotent, or
- (b) G is minimal non nilpotent, $|G| = pq^{\beta}$, $G_p \trianglelefteq G$, $q/p - 1$.

In particular G is supersoluble and every 2-maximal subgroup is normal in G .

PROOF. Maximal subgroups of G are nilpotent ([1] teorema B), so either G is nilpotent or minimal non nilpotent. Let G be minimal non nilpotent, then $|G| = p^{\alpha}q^{\beta}$, $G_p \trianglelefteq G$, G_q cyclic. The Fitting subgroup M of G is a maximal subgroup, it is the unique N -injector of G and $[G:M] = q$. Let U be a maximal subgroup of G such that $G_q \leq U$,

then $F(G)/U \cap F(G)$ is a chief factor of G , this means $[G:U] = p = |F(G)/U \cap F(G)|$. Suppose $G_\alpha < U$, then there exists a subgroup X , 2-maximal in G , such that $G_\alpha \leq X < U$; X is nilpotent and dualpronormal, hence, by lemma 1, X , and G_α , must be contained in M , a contradiction. It follows $G_\alpha = U$ and $\alpha = 1$. Finally it is clear that such a group is supersoluble and every 2-maximal subgroup is normal in it.

Our aim now is to study groups in which every 3-maximal subgroup is dualpronormal.

THEOREM 2. *The following statements are equivalent:*

- (a) *Every 3-maximal subgroup of a group G is dualpronormal.*
- (b) *Every 3-maximal subgroup of a group G is normal.*

PROOF. First suppose (a) holds. Maximal subgroups of G satisfy the hypothesis of theorem 1, so they are nilpotent or minimal non nilpotent with the described structure. Let M_3 be 3-maximal in G , then there exists a chain $M_3 < M_2 < M_1 < G$. Every 2-maximal subgroup of M_1 is dualpronormal in M_1 , so, by theorem 1, normal, it follows $M_3 \trianglelefteq M_1$. Suppose $N_G(M_3) = M_1$ and let g be an element of $G - M_1$. If $\langle M_3, M_3^g \rangle$ is nilpotent, then

$$F(\langle M_3, M_3^g \rangle) = \langle M_3, M_3^g \rangle = M_3,$$

a contradiction. If $\langle M_3, M_3^g \rangle$ is not nilpotent, then it is either maximal in G and minimal non nilpotent as in theorem 1, or it is the whole group G . Suppose $\langle M_3, M_3^g \rangle$ is maximal in G . Then $F(\langle M_3, M_3^g \rangle) < \langle M_3, M_3^g \rangle$, this means $M_3 = F(\langle M_3, M_3^g \rangle) \trianglelefteq \langle M_3, M_3^g \rangle = M_1$, a contradiction. Finally suppose $\langle M_3, M_3^g \rangle = G$ and so $F(G) \leq M_3$. Let M be a maximal subgroup of G such that $M \trianglelefteq G$. M is not nilpotent so it is minimal non nilpotent as in theorem 1. Also $F(M) \leq F(G) \leq M_3$ and $[G:M_3]/[G:F(M)] = rt$ (r, t primes), a contradiction.

The last contradiction proves the first part of the theorem. The converse is trivial.

In [7] (Satz 3) R. Schmidt proved that the same is true for modularity: more precisely he describes non supersoluble groups in which every 3-maximal subgroup is modular discovering that such

groups are exactly groups in which every 3-maximal is normal, while this equivalence is not true for $n = 1, 2$.

Using the previous theorem and Schmidt's result we can immediately get the following result.

COROLLARY 1. *Let G be a group in which every 3-maximal subgroup is dualpronormal, then one of the following holds:*

- (a) G is supersoluble;
- (b) $|G| = p^2q$, p and q primes or
- (c) G is the semidirect product of the quaternion group of order 8 and the cyclic group of order 3.

Viceversa the groups in b) and c) have even normal 3-maximal subgroups.

We have seen that the groups in which every 3-maximal subgroup is dualpronormal (normal) are «almost all» supersoluble, next theorem describes supersoluble such groups.

THEOREM 3. *Let G be a supersoluble group. If every 3-maximal subgroup of G is dualpronormal then one of the following holds:*

- (a) G nilpotent with normal 3-maximal subgroups.
- (b) G minimal non nilpotent, $|G| = pq^\beta$, $G_p \trianglelefteq G$.
- (c) $|G| = rpq^\beta$, $r \neq p \neq q$ primes, $G_r G_q$ and $G_p G_q$ minimal non nilpotent, $G_i \trianglelefteq G_i G_q$, if $\beta > 1$ G_p, G_q are normal in G .
- (d) $|G| = p^2 q^\beta$, G_q cyclic, $\Phi(G_q) \leq Z(G)$, if $\beta > 1$ the subgroups of G_p are normal in G .
- (e) $|G| = pq^{\beta+1}$, $G_p \trianglelefteq G$ and one of the following holds:
 - (i) G_q cyclic, $[G_q : C_{G_q}(G_p)] = q^2$;
 - (ii) G_q abelian of type (q, q^β) , $C_{G_q}(G_p)$ abelian of type $(q, q^{\beta-1})$;
 - (iii) $G_q \cong \langle a, b/a^q = b^q = 1, a^b = a^{1+a^{\beta-1}} \rangle$, $\beta \geq 4$, minimal non abelian, $C_{G_q}(G_p) = \langle a^q, b \rangle$;
 - (iv) $G_q \cong Q_8$, $|C_{G_q}(G_p)| = 4$.

Conversely the groups in (a), (b), (c), (d) and (e) have normal 3-maximal subgroups.

PROOF. Maximal subgroups of G satisfy the hypothesis of theorem 1, so they are nilpotent or minimal non nilpotent with the described structure, moreover, by theorem 2, every 3-maximal subgroup of G is normal in G . Let G be minimal non nilpotent, then $F(G)$ (= unique N -injector of G) is maximal in G . Let $|G| = p^\alpha q^\beta$, $G_p \trianglelefteq G$. Since G_q is not normal in G , $[G:G_q] \leq p^2$, and $|G| \leq p^2 q^\beta$. Suppose G_p elementary abelian. The whole group G is supersoluble and minimal non nilpotent, so G_q centralizes a subgroup P of order p ; by Mascke's theorem P has a complement Q in G_p , G_q -invariant. Since G is minimal non nilpotent Q , and G_p is centralized by G_q , a contradiction. Therefore G_p is cyclic and G minimal non abelian, this means (cfr. [4] pag. 285) $|G_p| = p$.

Suppose G has a maximal subgroup, say U , which is not nilpotent, this means U minimal non nilpotent, $|U| = pq^\beta$, $U_p \trianglelefteq U$ and $[G:U] = r$, r a prime. There are three possibilities:

(i) $r \neq p, q$. In this case $|G| = rpq^\beta$, $G_q = U_q$ is cyclic and $\Phi(G_q)$, which centralizes G_r and G_p , is contained in $Z(G)$. Let $\beta > 1$ and $H \leq \Phi(G_q)$, $|H| = q^{\beta-2}$, it results $[G:G_p H] = rq^2$ and $[G:G_r H] = pq^2$, so $G_p H$ and $G_r H$ are 3-maximal, nilpotent and, by hypothesis, normal subgroups of G , it follows $G_p, G_r \trianglelefteq G$.

(ii) $r = p$. In this case $|G| = p^2 q^\beta$ and $G_q = U_q$ is cyclic. Moreover, being $p > q$ and $[G:\Phi(G_q)] = p^2 q$, G_p and $\Phi(G_q)$ are normal in G , in particular $\Phi(G_q) \leq Z(G)$. If $\beta > 1$, as in case (i), one can easily prove that every subgroup of G_p is normal in G .

(iii) $r = q$. In this case $|G| = pq^{\beta+1}$ and, being $p > q$, G_p is normal in G . G is supposed to be not nilpotent, G_q is a Carter subgroup of G , therefore if M is maximal in G and nilpotent, then it must be normal. This means G has at most one nilpotent maximal subgroup, apart from q -Sylow subgroups, and it is the Fitting subgroup of G ; since $\Phi(U_q) \leq C_{G_q}(G_p)$ and $|\Phi(U_q)| = q^{\beta-1}$, it results $[G_q:C_{G_q}(G_p)] \leq q^2$. First it will be proved that G_q is cyclic if and only if $[G_q:C_{G_q}(G_p)] = q^2$, in particular $q^2/(p-1)$. If G_q is cyclic then $\Phi(G_q) = U_q$ does not centralize G_p , this means $[G_q:C_{G_q}(G_p)] = q^2$. Vice versa let $[G_q:C_{G_q}(G_p)] = q^2$ and let H be a maximal subgroup of $G_q:G_p H$ is not nilpotent and maximal in G so $G_p H \cong U$ and H is cyclic. Every maximal subgroup of G_q is cyclic so either G_q is cyclic or $G_q \cong Q_8$. Suppose $G_q \cong Q_8$, then $G_q/C_{G_q}(G_p) \leq \text{Aut } G_p$ is cyclic, this means $C_{G_q}(G_p)$ maximal in G_q , a contradiction. Therefore let G_q be not cyclic and $[G_q:C_{G_q}(G_p)] = q$. $O_q(G)$ is a maximal subgroup of G_q ,

every other maximal subgroup of G_q , say M , does not centralize G_p so $G_p M \cong U$ and M is cyclic. There are two possibilities:

(1) G_q abelian, then it can be easily seen that G_q must be of type (q, q^β) and $C_{G_q}(G_p)$ of type $(q, q^{\beta-1})$.

(2) G_q not abelian: G_q has at least $q + 1 \geq 3$ maximal subgroups, so at least two, say M and N cyclic: it can be easily seen that $\Phi(G_q) = M \cap N = Z(G_q)$, therefore G_q is minimal non abelian. Then (cfr. [4] pag. 309) either

(a) $G_q \cong Q_8$, or

(b) $G_q = \langle a, b/a^{q^\alpha} = b^{q^\beta} = c^q = 1, c = [a, b] \rangle$, which is not verified because, $|G_q| = q^{\alpha+\beta+1}$, and $M = \langle a, b^q, c \rangle$ and $N = \langle a^q, b, c \rangle$ are different maximal subgroups not cyclic, or

(c) $G_q \cong \langle a, b/a^{q^\alpha} = b^{q^\mu} = 1, a^b = a^{1+q^{\beta-1}} \rangle$, $\beta \geq 2$, $\mu \geq 1$, $|G_q| = q^{\beta+\mu}$. If $\mu \geq 2$, then $\langle a^q, b^q \rangle \leq \Phi(G_q)$ is not cyclic. So $\mu = 1$ and $G_q \cong \langle a, b/a^{q^\beta} = b^q = 1, a^b = a^{1+q^{\beta-1}} \rangle$. If $\beta = 2$ or 3 then $\langle b \rangle$ or $G_p \langle b \rangle$ are 3-maximal in G and not normal, so it must be $\beta \geq 4$. This group has $q + 1$ maximal subgroups, one $\langle a^q, b \rangle$ is not cyclic. It will be proved that the q cyclic subgroups $\langle a \rangle, \langle ab \rangle, \dots, \langle ab^{q-1} \rangle$ are maximal. Consider $G'_q = \langle [a, b] \rangle^q = \langle a^{q^{\beta-1}} \rangle$; for every $t \in \{1, \dots, q-1\}$ a and b^t are permutable with $[a, b^{-t}]$, it follows $(ab^t)^q = a^q [a, b^{-t}]^q$. If $q \neq 2$, then $q \mid \binom{q}{2}$ so $[a, b^{-t}]^q = 1$. If $q = 2$, then

$$(ab)^2 = a^2[a, b] = a^2 a^{2^{\beta-1}} = a^{2(1+2^{\beta-1})}$$

and, being $\beta \geq 4$, it is $\langle a^{2(1+2^{\beta-1})} \rangle = \langle a^2 \rangle$. Therefore $\langle (ab^t)^q \rangle = \langle a^q \rangle$ and $\langle ab^t \rangle$ is maximal in G_q . Second-maximal subgroups of G_q coincide with the maximal subgroups of $\langle a^q, b \rangle$, namely they are $\langle a^{q^2}, b \rangle, \langle a^q, \langle a^2 b \rangle, \dots, \langle a^2 b^{q-1} \rangle$. At the same way, 3-maximal subgroups are $\langle a^{q^2}, b \rangle, \langle a^{q^2} \rangle, \langle a^{q^2} b \rangle, \dots, \langle a^{q^2} b^{q-1} \rangle$. These subgroups are normal in G_q since $G'_q = \langle a^{q^{\beta-1}} \rangle$ is contained in each of them, and centralize G_p , therefore they are normal in G . Using this fact it is easy to see that every 3-maximal subgroup of G is normal in G .

COROLLARY 2. *If every 3-maximal subgroup of a group G is dual-pronormal, then:*

- (a) $|\pi(G)| \geq 3 \Rightarrow G$ is supersoluble.
- (b) $|\pi(G)| \geq 4 \Rightarrow G$ is nilpotent.

COROLLARY 3. *If every n -maximal subgroup of a group G is dualpronormal, then:*

- (a) $|\pi(G)| \geq n \Rightarrow G$ is supersoluble.
- (b) $|\pi(G)| \geq n + 1 \Rightarrow G$ is nilpotent.

PROOF. (a) By induction on n . If $n = 1, 2, 3$ the corollary holds. Let $n > 3$ and U a maximal subgroup of G , $|\pi(U)| \geq n - 1$ and $(n - 1)$ -maximal subgroups of U are dualpronormal in U . By induction U is supersoluble, so either G is supersoluble or minimal non supersoluble. But if G is minimal non supersoluble then $|\pi(G)| \leq 3$, a contradiction. Hence G is supersoluble.

(b) By induction on n . If $n = 1, 2, 3$ the corollary holds. Let $n > 3$. As in (a) get easily, by induction, G nilpotent or minimal non nilpotent, but if G is minimal non nilpotent then $|\pi(G)| \leq 2$, a contradiction. Hence G is nilpotent.

LEMMA 2. *If U is dualpronormal in a group G and V is a subgroup of G containing $N_G(U)$ then V is dualpronormal in G . In particular $N_G(U)$ is dualpronormal in G .*

PROOF. By hypothesis

$$U \leq \langle V, V^g \rangle \leq N_G(\langle V, V^g \rangle).$$

By [1] prop. 2.2 $F(\langle V, V^g \rangle) \leq N_G(U) \leq V$, for every g in G . This means V is dualpronormal in G .

LEMMA 3. *Let H be a subgroup of $F(G)$, if H is dualpronormal in G then H is normal in G .*

PROOF. By [1] prop. 2.2 $H \cap F(G) = H$ is normal in G .

THEOREM 4. *Let G be a metanilpotent group. The following statements are equivalent:*

- (a) *Every n -maximal subgroup is dualpronormal.*
- (b) *Every n -maximal subgroup is normal.*

PROOF. (a) \rightarrow (b). If every n -maximal subgroup is dualpronormal, then, by [1] theorem B, they are nilpotent (in fact, $(n-1)$ -maximal subgroups have dualpronormal maximal subgroups, hence, by the theorem, they are nilpotent). Use induction on n . If $n = 1, 2$ or 3 the theorem holds. Let $n > 3$, and let H be n -maximal in G . By definition, H is $(n-1)$ -maximal in a maximal subgroup M ; every $(n-1)$ -maximal subgroup of M is dualpronormal in M , so, by induction, normal in M . It follows $H \trianglelefteq M$. Suppose $N_G(H) = M$. By lemma 4 M is dualpronormal in G , this means, being G metanilpotent and M maximal in G , M normal in G . So H is subnormal in G and $H \leq F(G)$. By lemma 4 H is normal in G , a contradiction. So $N_G(H) = G$ as required.

(b) \rightarrow (a). Trivial.

Probably this equivalence is always true, but at the moment as the proof as a counterexample seem to be hard to find.

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