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On $M$-Sequences Associated to Filtrations.

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0. Introduction.

Generalizations of the notion of regular $M$-sequences have been studied both from an algebraic and a geometric point of view. Whereas the algebraic theory is related to properties of Cohen-Macaulay-, Gorenstein- or Buchsbaum-rings, the geometric considerations are usually related to the study of suitable generalizations of complete intersections. In this note we present a unifying theory dealing with generalized regular sequences associated to filtrations on modules; in this way we recapture the relative regular sequences introduced by M. Fiorentini [5], as well as the homogeneous regular sequences of P. Bouchara [3] a.o. In a first part of this paper we compare regularity with respect to a filtration $FM$ on $M$ to the projective regularity with respect to the associated graded module $G(M)$ and to projective regularity with respect to the associated Rees module $\overline{M}$. In the second part we focus on filtrations in the sense of Gabriel topologies. The final theorem links the existence of a weakly $\pi$-regular $M$-sequence contained in an ideal $I$ of $R$ to properties of $\text{Ext}_R^i(N, M)$ for all finitely generated $R$-modules $N$ having support $\text{Supp}(N)$ contained in the closed set $V(I)$ determined by the ideal $I$.


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1. Preliminaries.

Let $R$ be a commutative ring with filtration $FR = \{F_n R, n \in \mathbb{Z}\}$, i.e. an ascending chain of additive subgroups $F_n R$ of $R$ satisfying:

$$F_n R F_m R \subset F_{n+m} R \quad \text{for} \quad n, m \in \mathbb{Z}, \quad \bigcup_{n \in \mathbb{Z}} F_n R = R \quad \text{and} \quad 1 \in F_0 R.$$

An $R$-module $M$ is said to be a filtered $R$-module if it has a filtration by additive subgroups $FM = \{F_n M, n \in \mathbb{Z}\}$ such that

$$F_n R F_m M \subset F_{n+m} M \quad \text{for} \quad n, m \in \mathbb{Z}, \quad \bigcup_{n \in \mathbb{Z}} F_n M = M.$$

For full detail on filtered rings and modules we refer to [15]. We write

$$G(R) = \bigoplus_{n \in \mathbb{Z}} G(R)_n = \bigoplus_{n \in \mathbb{Z}} F_n R / F_{n-1} R \quad \text{and} \quad G(M) = \bigoplus_{n \in \mathbb{Z}} F_n M / F_{n-1} M$$

for the associated graded ring of $FR$, resp. the associated graded module of $FM$. If $x \in F_n M - F_{n-1} M$ then $\sigma(x)$ is the image of $x$ in $G(M)_n$.

The Rees ring of $FR$, $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} F_n R$, is a graded ring in the obvious way and similarly the Rees module $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} F_n M$ is a graded $\mathcal{R}$-module. Viewing $1 \in F_1 R$ as an element of $\mathcal{R}$, we obtain an element $X \in \mathcal{R}$, that is a regular element of $\mathcal{R}$ such that:

$$\mathcal{R} / X \mathcal{R} \cong G(R), \quad \mathcal{R} / (1 - X) \mathcal{R} \cong R, \quad \mathcal{R}_X \cong R[X, X^{-1}]$$

and

$$\mathcal{M} / X \mathcal{M} \cong G(M), \quad \mathcal{M} / (1 - X) \mathcal{M} \cong M, \quad \mathcal{M}_X \cong M[X, X^{-1}].$$

The category of the filtered $R$-modules, $R$-filt, is equivalent to the category of $X$-torsionfree graded $\mathcal{R}$-modules. Recall from [2] or [10]:

i) If $M \in R$-filt then $M$ is filt-free, filt-projective, if and only if $\mathcal{M}$ is gr-free, projective, resp.
ii) The filtration $F_M$ is good, i.e. there exist $m_1, \ldots, m_t \in M$ and $d_1, \ldots, d_t \in \mathbb{Z}$ such that $F_n M = \sum_{i=1}^{t} F_{n-d_i} R \cdot m_i$ for all $n \in \mathbb{Z}$, if and only if $\tilde{M}$ is a finitely generated $\tilde{R}$-module.

iii) $FM$ is separated, i.e.

$$\bigcap_{n \in \mathbb{Z}} F_n M = 0, \quad \text{if and only if} \quad \bigcap_{n \in \mathbb{Z}} X^n \tilde{M} = 0.$$ 

A filtered ring is said to be a Zariski ring if $\tilde{R}$ is Noetherian and $F_{-1} R \subset J(F_0 R)$, where $J(-)$ denotes the Jacobson radical, cf. [10]. Typical good properties of Zariski rings are: good filtrations induce good filtrations on arbitrary submodules and filtrations equivalent to good filtrations are good too, moreover all good filtrations are automatically separated.

A subset $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a regular $M$-sequence with respect to $FM$ if for $i = 0, \ldots, n - 1$, we have the following for some $m \in \mathbb{Z},$

$$F_m M \cap [(a_1, \ldots, a_i) F_m M : a_{i+1}] \subset (a_1, \ldots, a_i) M,$$

where we put $(a_1, \ldots, a_i) = 0$ if $i = 0$, and $(a_1, \ldots, a_i)$ is the $R$-ideal generated by $\{a_1, \ldots, a_i\}$. $[X:a] = \{m \in M, am \in X\}$ for $a \in R$. When for all $d < 0$ we have for $i = 0, \ldots, n - 1$, that

$$F_{d-1} M \cap [(a_1, \ldots, a_i) F_{d-1} M : a_{i+1}] \subset (a_1, \ldots, a_i) F_d M,$$

we say that $\{a_1, \ldots, a_n\}$ is a filtered regular $M$-sequence. When the latter holds for all $d \in \mathbb{Z}$ then we say that $\{a_1, \ldots, a_n\}$ is a completely filtered regular $M$-sequence.

1.1. Lemma. If $\{a_1, \ldots, a_n\} \subset R$ is a completely filtered regular $M$-sequence then it is also an $M$-sequence.

Proof. Take an $x \in M$ such that $a_{i+1} x \in (a_1, \ldots, a_i) M$, say $a_{i+1} x = \sum_{j=1}^{i} a_j \lambda_j$ with $\lambda_j \in M$, $j = 1, \ldots, i$. If we choose $h \in \mathbb{Z}$ large enough such that $\lambda_j \in F_h M$ for $i = 1, \ldots, i$, then we have $a_{i+1} x \in (a_1, \ldots, a_i)$.
Now take $h' > h$ such that $x \in F_{h'-1}M$ and $a_{i+1}x \in F_{h'}M$, then
\[ x \in [(a_1, \ldots, a_i) F_{h'-1}M : a_{i+1}] \cap F_{h'-1}M \]
and thus $x \in (a_1, \ldots, a_i) F_hM \subset (a_1, \ldots, a_i) M$. \qed

The converse of the lemma need not hold in general.

1.2. EXAMPLES. 1) Let $(a_1, \ldots, a_n)$ be an $M$-sequence in $R$ and put $I = (a_1, \ldots, a_n)$. Define $F_{-n}R = I^n$, $F_nR = R$ for $n > 0$, and also $F_{i-n}M = I^nM$, $F_{i,n}M = M$ for $n > 0$. In [8] it has been shown that for $M = R$ we have:
\[ I^{r+1} \cap [(a_1, \ldots, a_i) I^{r+1} : a_{i+1}] = (a_1, \ldots, a_i) I^r. \]
In [12] this has been extended to the situation we consider here:
\[ I^{r+1}M \cap [(a_1, \ldots, a_i) I^{r+1}M : a_{i+1}] = (a_1, \ldots, a_i) I^rM. \]
The latter relation reduces exactly to our definition of a completely filtered regular $M$-sequence with respect to the $I$-adic filtration.

2) Let $N$ be any $R$-submodule of $M$. We say that $(a_1, \ldots, a_n) \subset R - \{0\}$ is a regular $M$-sequence relative to $N$ if for all $i = 0, \ldots, n - 1$, we have:
\[ [(a_1, \ldots, a_i) N : a_{i+1}] \cap N \subset (a_1, \ldots, a_i) M. \]
It is clear that this property still holds if we replace $N$ by any smaller submodule. Moreover, $(a_1, \ldots, a_n)$ is a regular $M$-sequence relative to $N$ if and only if for all prime ideals $P$ of $R$ it is a regular $M_P$-sequence relative to $N_P$. A regular $M$-sequence relative to $N$ appears as a regular $M$-sequence with respect to $FM$ if we filter $R$ trivially, $F_{-n}R = 0$ for $n > 0$ and $F_nR = R$ for $m > 0$, and put $F_nM = M$ for $m > 0$, $F_{-1}M = N$ and $F_{-n}M = 0$ for $n > 1$. For this particular filtration the sequence will be a completely filtered regular $M$-sequence exactly when it is a regular $M$-sequence.

3) Let $R$ be positively graded and $M$ is a graded $R$-module. A set $(a_1, \ldots, a_n) \subset h(R) - \{0\}$ is a projective $M$-regular sequence if it is a regular $M$-sequence relative to the submodule $M_{\geq k} = \bigoplus_{n \geq k} M_n$ for
some $k \in \mathbb{N}$. If we define $F_{-t}R = R_{\geq t}$, $F_{-t}M = M_{\geq t}$ for $t, l \in \mathbb{Z}$, then a projective $M$-regular sequence is just a regular $M$-sequence with respect to $FM$ consisting of homogeneous elements of $R$.

Foregoing examples show that previous definitions, e.g. regular $M$-sequences relative to a submodule, [12], [5], projective regular $M$-sequences, [3], as well as relative sequences [8], are captured by our definition of regular $M$-sequence with respect to a filtration $FM$. Alternatively we may view these definitions from the point of view of Gabriel topologies or torsion theories. Let $\mathcal{L}(\kappa)$ be the Gabriel filter of an idempotent kernel functor in the sense of [6], [7]. We say that $\{a_1, ..., a_n\} \subset R - \{0\}$ is a $\kappa$-regular $M$-sequence if it is a regular $M$-sequence relative to a submodule $N$ of $M$ that is $\kappa$-dense in $M$, i.e. $\kappa(M/N) = M/N$. When $M$ is finitely generated then one may replace $N$ by $JM$ for some $J \in \mathcal{L}(\kappa)$; in that case $\{a_1, ..., a_n\}$ will be a $\kappa$-regular $M$-sequence exactly when it is a regular $M$-sequence with respect to the $J$-adic filtration on $M$ for some $J \in \mathcal{L}(\kappa)$. Let us (re-)consider some examples first.

1.3. EXAMPLES. 1) Let $R$ be a positively graded ring generated as an $R_0$-module by finitely many (homogeneous) elements and assume that $R_0$ is a Noetherian ring. Let $M$ be a graded $R$-module. The sequence $\{a_1, ..., a_n\} \subset h(R) - \{0\}$ is a projective regular $M$-sequence if and only if it is $\kappa_+$-regular where $\mathcal{L}(\kappa_+) = \{H$ ideal of $R$, $H \supset R_p^*$ for some $p \in \mathbb{N}\}$. Indeed, $R$ is a Noetherian ring and therefore $\mathcal{L}(\kappa_+)$ does define an idempotent kernel functor on $R$-mod. Furthermore if $h$ is at least the maximum of the degrees of the $R_0$-algebra generators of $R$ then $R_{\geq nh} \subset R_+^n \subset R_{\geq n}$ and since in 1.2 2) we may replace $N$ by a smaller submodule, the statements above are easily checked.

2) Let $I$ be an ideal of a Noetherian ring $R$ and $M$ a finitely generated $R$-module. Let $\kappa_I$ be given by the Gabriel filter: $\mathcal{L}(\kappa_I) = \{H$ ideal of $R$, $H \supset I^p$ for some $p \in \mathbb{N}\}$. Then, $\{a_1, ..., a_n\}$ is a $\kappa_I$-regular $M$-sequence if and only if it is a regular $M$-sequence relative to the $I$-adic filtration.

3) For an $R$-submodule $N$ of $M$ let $\kappa(M/N)$ be the "smallest" torsion theory for which $M/N$ is a torsion-module, cf. [6], then a regular $M$-sequence relative to $N$ is also a $\kappa(M/N)$-regular sequence.

Let us point out that the property of being a regular $M$-sequence
relative to $F_M$, is a topological property in the sense that the property is being conserved if we replace $F_M$ by an equivalent filtration $F'_M$. On the other hand, the property of being a filtered regular $M$-sequence is not insensitive to such a change of filtration; we may consider this property as a more «algebraic» one that is closer related to properties of the associated gradation.

2. Sequences and Zariskian filtrations.

For any graded ring $R$ and (finitely generated) graded $R$-module $M$ we extend the definition given in 1.2 (3) and define a projective $M$-regular sequence to be a set $\{a_1, \ldots, a_n\}$ in $h(R) - \{0\}$ that is a regular $M$-sequence relative to $M_{\geq k}$ for some $k \in \mathbb{Z}$.

2.1. Proposition. Let $\{a_1, \ldots, a_r\} \subset R - \{0\}$. If $\{X, a_1, \ldots, a_r\}$ is a projective regular $M$-sequence in $\hat{R}$ then $\{a_1, \ldots, a_r\}$ is a regular $M$-sequence with respect to $F_M$.

Proof. We write $\tilde{a}$ for an $a \in F_nM - F_{n-1}M$ viewed in $\tilde{M}_n$, hence we may identify $\tilde{a}$ and $aX^n$ if we identify $\tilde{M}$ with $\sum_{n \in \mathbb{Z}} F_nM \cdot X^n$. We have to show that there is an $m \in \mathbb{Z}$ such that

$$[(a_1, \ldots, a_i)F_mM : a_{i+1}] \cap F_mM \subset (a_1, \ldots, a_i)M, \quad i = 0, \ldots, r - 1.$$ 

If $y \in F_mM$ is such that $a_{i+1} \cdot y \in (a_1, \ldots, a_i)F_mM$ and $y \in F_tM - \sum_{n \geq m} F_nM$ for some $t < m$, then $ya_{i+1} = \sum_{j=1}^r \lambda_j a_j m_j$ with $\lambda_j \in R, m_j \in F_tM$. If $\tilde{y} = yX^t, \tilde{a}_{i+1} = a_{i+1}X^k$ then $(ya_{i+1})^\sim = ya_{i+1}X^k$ with $k < t + s$, hence $\tilde{y} \cdot \tilde{a}_{i+1} = ya_{i+1}X^{i+k} = (ya_{i+1})^\sim X^{i+s-k}$ and so we arrive at

$$\tilde{y}\tilde{a}_{i+1} = \sum_{i=1}^r \tilde{\lambda}_j \tilde{a}_j \tilde{m}_j X^{i\beta_1} \quad \text{for certain } \beta, \in \mathbb{Z}.$$ 

Taking $\beta$ large enough we arrive at $(\tilde{y}X^\beta)\tilde{a}_{i+1} \in (\tilde{a}_1, \ldots, \tilde{a}_n)\tilde{M}_{\geq m} \subset (X, a_1, \ldots, a_n)\tilde{M}_{\geq m}$ and therefore we may choose $m$ such that the projectivity of $(X, a_1, \ldots, a_r)$ yields $\tilde{y}X^\beta \in (X, a_1, \ldots, a_r)\tilde{M}$. Applying $\tilde{M} \to \tilde{M}/(X - 1)\tilde{M} = M$, we obtain that $y \in (a_1, \ldots, a_i)M$ as desired.
2.2. PROPOSITION. Let \{a_1, \ldots, a_r\} \subset R - \{0\}. If \{\sigma(a_1), \ldots, \sigma(a_r)\}

is a projective \(G(M)\)-regular sequence then \{X, \tilde{a}_1, \ldots, \tilde{a}_r\}

is a projective \(M\)-regular sequence in \(\tilde{R}\).

PROOF. Clearly \(X\) is not a zero divisor on \(\tilde{M}\). Since \(G(M) = \tilde{M}/X\tilde{M}\) and \(\tilde{a}_i \equiv \sigma(a_i) \mod \tilde{R}X = \sigma(a_i)\), it follows from the hypothesis on

\(\{\sigma(a_1), \ldots, \sigma(a_r)\}\) that \(\{X, \tilde{a}_1, \ldots, \tilde{a}_r\}\) is as claimed.

2.3. REMARK. In general, the fact that \(\{\sigma(a_1), \ldots, \sigma(a_r)\}\) is a regular sequence does not imply that \(\{a_1, \ldots, a_r\}\) is an \(R\)-sequence, see Example 2.5 of [13].

2.4. PROPOSITION. Let \(\{a_1, \ldots, a_r\} \subset R - \{0\}\). If \(\{\tilde{a}_1, \ldots, \tilde{a}_r\} \subset \tilde{R}\) is

a projective \(\tilde{M}\)-regular sequence then \(\{\sigma(a_1), \ldots, \sigma(a_r)\}\) is a projective \(G(M)\)-regular sequence.

PROOF. Take

\[\sigma(y) \in G(M)_{\geq k} \cap \left[ (\sigma(a_1), \ldots, \sigma(a_i)) G(M)_{\geq k} : \sigma(a_{i+1}) \right]\]

then \(\sigma(y)\) is in the kernel of the mapping

\[G(M)_{\geq k}/(\sigma(a_1), \ldots, \sigma(a_i)) G(M)_{\geq k} \to G(M)/(\sigma(a_1), \ldots, \sigma(a_i)) G(M)_{\geq k}\]

induced by multiplication by \(\sigma(a_{i+1})\). We have an exact sequence

\[0 \to \text{Ker } \Psi \to \tilde{M}_{\geq k}/(\tilde{a}_1, \ldots, \tilde{a}_i) \tilde{M}_{\geq k} \xrightarrow{\psi} \tilde{M}/(\tilde{a}_1, \ldots, \tilde{a}_i) \tilde{M}_{\geq k}.

We view \(\Psi\) as the restriction of an \(\tilde{R}\)-linear map \(\Psi'\) defined on

\[[\tilde{M}/(\tilde{a}_1, \ldots, \tilde{a}_i) \tilde{M}_{\geq k}]\).

Applying \(- \otimes \tilde{R}/X\tilde{R}\) to \(\Psi'\) and restricting to \(\tilde{M}_{\geq k}/(\tilde{a}_1, \ldots, \tilde{a}_i) \tilde{M}_{\geq k}\)

again we obtain:

\[0 \to G(\text{Ker } \Psi') \to G(M)_{\geq k}/(\sigma(a_1), \ldots, \sigma(a_i)) G(M)_{\geq k} \xrightarrow{\nu}\]

\[\to G(M)/(\sigma(a_1), \ldots, \sigma(a_i)) G(M)_{\geq k}\]
where $\bar{P}$ is again determined by multiplication by $\sigma(a_{i+1})$. Therefore $\sigma(y) \in G(Ker \bar{P})$ and we may select a $\tilde{z} \in Ker \bar{P}'$ such that $\sigma(y) = \sigma(z) \equiv \tilde{z} \mod X$ and note that we may actually find such a $\tilde{z}$ even in $Ker \bar{P}$ because

$$\bar{M}/(\tilde{a}_1, ..., \tilde{a}_i) \bar{M}_{\geq k} \rightarrow GM/(\sigma(a_1), ..., \sigma(a_i))G(M)_{\geq k}$$

is a graded morphism. The hypothesis on $\{\tilde{a}_1, ..., \tilde{a}_i\}$ implies that $\tilde{z} \in (\tilde{a}_1, ..., \tilde{a}_i)\bar{M}$ if we assume we were looking at the suitable $k \in \mathbb{Z}$ from the beginning. Now $\sigma(y) = \sigma(z) \in (\sigma(a_1), ..., \sigma(a_i))G(M)$ follows.

2.5. PROPOSITION. Let $\{a_1, ..., a_r\} \subset R - \{0\}$; if $\{\sigma(a_1), ..., \sigma(a_r)\}$ is a projective $G(M)$-regular sequence then $\{a_1, ..., a_r\}$ is a regular $M$-sequence with respect to $F$.

PROOF. Combine Proposition 2.2 and Proposition 2.1.

2.6. LEMMA. Let $R$ be a ring with a Zariskian filtration. Then the following statements are equivalent for an ideal $I = (x_1, ..., x_s)$

1) $G(I)$ is generated by $\{\sigma(x_1), ..., \sigma(x_s)\}$;

2) $F_nR \cap I = \sum_{i=1}^{s} F_{n-n_i}Rx_i$ where $\nu(x_i) = \deg \sigma(x_i) = n_i$ for $i = 1, 2, ..., s$.

PROOF. 1) $\Rightarrow$ 2) See proof of implication 8) $\Rightarrow$ 1) in III.4.4 [9].

2) $\Rightarrow$ 1) It is clear that

$$G(I)_n = \frac{(F_nR \cap I) + F_{n-1}R}{F_{n-1}R}$$

and

$$(\sigma(x_1), ..., \sigma(x_s))_n = \left(\sum_{i=1}^{s} F_{n-n_i}Rx_i + F_{n-1}R\right)/F_{n-1}R$$

therefore if

$$F_nR \cap I = \sum_{i=1}^{s} F_{n-n_i}Rx_i$$

we have that $G(I) = (\sigma(x_1), ..., \sigma(x_s))$. □
2.7. **Proposition.** Let \( R \) be a ring with a Zariskian filtration. Assume that \( I = (x_1, \ldots, x_s) \), if \( \sigma(x_1), \ldots, \sigma(x_s) \) is a \( G(R) \)-sequence then
\[
G(I) = (\sigma(x_1), \ldots, \sigma(x_s)).
\]

**Proof.** We apply induction on \( s \). The case \( s = 1 \) is easy. If \( \sigma(x) \)
is a non-zero divisor then for each \( y \in R \), \( \sigma(xy) = \sigma(x)\sigma(y) \). Thus
\[
G(I) = \bigoplus \frac{Rx \cap F_nR + F_{n-1}R}{F_{n-1}R} = G(R)\sigma(x).
\]

Assume now that the result is true for \( s - 1 \) and we will prove it for \( s \).
Let \( a \in I \cap F_nR = F_nI \). As \( I = (x_1, \ldots, x_s) \) we will have
\[
a = a_1x_1 + \ldots + a_sx_s \quad \text{with } a_s \in F_tR.
\]

We can find a minimal \( t \) with this property, otherwise \( a \) will be in the closure of \( (x_1, \ldots, x_{s-1}) \) and since the filtration is Zariskian \( (x_1, \ldots, x_{s-1}) \) is closed and \( a \in (x_1, \ldots, x_{s-1}) \). The result then follows from the induction hypothesis. Now take \( t \) minimal such that \( a = x + \)
\[
+ a_sx_s \quad \text{with } x \in (x_1, \ldots, x_{s-1}), \quad a_sx_s = a - x \in [(x_1, \ldots, x_{s-1}) + F_nR] \cap \\
\cap F_{t+n_s}R \quad \text{with } v(x) = n_s.
\]
If \( t + n_s \leq n \),
\[
a \in (x_1, \ldots, x_{s-1}) \cap F_nR + F_{n-n_s}Rx_s = \sum_{i=1}^{n-1} F_{n-n_i}Rx_i + F_{n-n_s}Rx_s
\]
(by Lemma 2.6), and by applying the same lemma the claim follows.

Assume now that \( t + n_s > n \), \( F_nR \subset F_{t+n_s}R \) so
\[
a_sx_s \in [(x_1, \ldots, x_{s-1}) + F_nR] \cap \\
\cap F_{t+n_s} \subset [(x_1, \ldots, x_{s-1}) + F_{t+n_s-1}R] \cap F_{t+n_s}R.
\]

It follows that \( \sigma(a)\sigma(x_s) \in G((x_1, \ldots, x_{s-1})) = (\sigma(x_1), \ldots, \sigma(x_{s-1})) \) by
the induction hypothesis. Since \( \sigma(x_1), \ldots, \sigma(x_s) \) is a \( G(R) \)-sequence then
\( \sigma(a_s) \in G((x_1, \ldots, x_{s-1}) \). Then \( a_s \in [(x_1, \ldots, x_{s-1}) \cap F_tR] + F_{t-1}R \) and
hence \( a \in (x_1, \ldots, x_{s-1}) + F_{t-1}Rx_s \). This contradicts the choice of \( t \).
Hence the last case is not possible and the proof is completed. \( \Box \)
2.8. DEFINITION. A sequence \( x_1, \ldots, x_s \) in \( R \) is called super-regular if the sequence \( \sigma(x_1), \ldots, \sigma(x_s) \) is a regular sequence in \( G(R) \) (cf. [8]).

The following theorem gives a characterization of this property.

2.8. THEOREM. Let \( R \) be a ring with a Zariski filtration and let \( I = (x_1, \ldots, x_s) \). Then the following conditions are equivalent.

1) \((x_1, \ldots, x_s)\) is a super regular \( M \)-sequence.

2) \((x_1, \ldots, x_s)\) is a regular \( M \)-sequence and

\[
I_i \cap F_nR = \sum_{j=1}^{s} F_{n-n_j}Rx_j,
\]

where \( I_i = (x_1, \ldots, x_i) \), \( n_i = \nu(x_i) \), where \( \nu(x_i) = \deg \sigma(x_i) \), as usual.

PROOF. 1) \( \Rightarrow \) 2) If \( \sigma(x_1), \ldots, (x_s) \) is a \( G(R) \)-sequence by Proposition 2.7 it follows that \( G(I) = (\sigma(x_1), \ldots, \sigma(x_s)) \) and now by the Lemma 2.6 we obtain

\[
I_i \cap F_nR = \sum_{j=1}^{s} F_{n-n_j}Rx_j.
\]

We only have to show that \( I_{i+1}:x_i \subseteq I_{i+1} \). Let \( a \in I_{i+1}:x_i \) with \( \nu(a) = n \) then \( \sigma(a) \sigma(x_i) \in G(I_{i+1}) = (\sigma(x_1), \ldots, \sigma(x_{i+1})) \) but \( \sigma(x_1), \ldots, \sigma(x_s) \) is a \( G(R) \)-sequence, hence \( \sigma(a) \in G(I_{i+1}) \). Therefore \( a \in (I_{i+1} \cap F_nR) + F_{n-n_j}R \), \( a = x - a_{n+1} \), with \( x \in I_i \), \( a_{n+1} \in I_{i+1} \). Repeating the argument for \( a_{n+1} \) we will obtain that \( a \) is in the closure of \( I_{i+1} \) that equals \( I_{i+1} \) by the Zariski condition.

2) \( \Rightarrow \) 1) Let \( a \in R \) such that \( \sigma(a) \sigma(x_i) \in (\sigma(x_1), \ldots, \sigma(x_s)) \), \( \nu(a) = n \) then \( ax_i \in (x_1, \ldots, x_{i-1}) + F_{n+n_i-1}R \), i.e.

\[
a x_i = b + \sum_{j=1}^{i-1} a_j x_j \quad \text{where} \quad b \in F_{n+n_i-1}R.
\]

Thus \( b = ax_i - \sum_{j=1}^{i-1} a_j x_j \) belongs to \( I_{i+1} \cap F_{n+n_i-1}R \).

Now write \( b = \sum_{j=1}^{i} b_j x_j \) where \( b_j \in F_{n+n_i-1-n_j}R \).

Then

\[
(a - b_i)x_i = \left( b + \sum_{j=1}^{i-1} a_j x_j \right) - \left( b - \sum_{j=1}^{i-1} b_j x_j \right) \in I_i.
\]
We obtain \((a - b_i) \in (I_{i-1}:x) \subseteq I_{i-1}\) and that implies \(a \in I_{i-1} + F_{n-1}R\). Therefore \(\sigma(a) \in G(I_{i-1})\). □

Note that the second statement of 2) does not follow from the Zariski hypothesis because we have fixed the generators \(x_1, \ldots, x_t\), and the goodness of \(I_i \cap FR\) cannot necessarily be phrased in terms of prescribed generators! Let us conclude this section by a change of base result.

2.9. **Theorem.** If \((a_1, \ldots, a_n)\) is a regular \(M\)-sequence with respect to \(FM\) and let \(f: R \to S\) be a filtered flat ring morphism then \(f(a_1), \ldots, f(a_n)\) is a regular \(S \otimes_R M\)-sequence with respect to the tensor-product filtration \(FS \otimes_R M\).

**Proof.** An easy modification of the proof of Proposition 3 in [12] following the ideas of the proof of Proposition 2.4. □

2.10. **Corollary.** Let \(R\) be a ring with a Zariskian filtration \(FR\) and let \(S\) be a multiplicatively closed set in \(R\) such that \(\sigma(S) = \{s \in S\} \) is an Ore set of \(G(R)\) consisting of regular elements. Then for a finitely generated filtered \(R\)-module \(M\) a regular \(M\)-sequence \(\{a_1, \ldots, a_n\}\) relative to \(FM\) is also a regular \(Q^\mu_S(M)\)-sequence relative to \(FG^\mu_S(M)\), where \(Q^\mu_S(M)\) is the microlocalization at \(S\) and \(FG^\mu_S(M)\) is the microlocalized filtration.

**Proof.** Recall from [2] that \(R \hookrightarrow Q^\mu_S(R)\) is a filtered flat ring morphism and apply the theorem, keeping in mind that \(Q^\mu_S(M) = Q^\mu_S(R) \otimes_R M\) since \(M\) is finitely generated. □

3. **Relative sequences.**

**Definition.** Let \(F(\kappa)\) be a filter of an idempotent kernel functor on \(R\)-mod, cf. [6], [7], then \(F(\kappa)\) generates a linear topology on \(R\)-mod. For an \(R\)-module \(M\) we say that \(\{a_1, \ldots, a_t\} \subseteq R\{0\}\) is a \(\kappa\)-regular \(M\)-sequence if it is a regular \(M\)-sequence relative to a submodule \(N\) of \(M\) that is \(\kappa\)-dense in \(M\).

It is clear that if \(\kappa_1 < \kappa_2\) then any \(\kappa_1\)-regular \(M\)-sequence is a \(\kappa_2\)-regular \(M\)-sequence. Also if \(\{a_1, \ldots, a\} \subseteq R - \{0\}\) is a \(\kappa_1\)-regular...
Let $\bar{\mathcal{M}}$, $\mathcal{M}$-sequence and $\lambda_2$-regular $\mathcal{M}$-sequence then it is a $\lambda_1\cap\lambda_2$-regular $\mathcal{M}$-sequence.

**DEFINITION.** For an $R$-module $M$ we say that $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a weak $\lambda$-regular $M$-sequence if

$$((a_1, \ldots, a_i) M : a_{i+1}) \subseteq \text{Cl}_\lambda^M ((a_1, \ldots, a_i) M)$$

for $i = 0, \ldots, n - 1$, where $\text{Cl}_\lambda^M (X) = \{m \in M, Im \subset X \text{ for some } I \in \mathcal{I}(\lambda)\}$. This is equivalent to exactness of the following sequence

$$0 \to Q_\lambda \left( \frac{M}{(a_1, \ldots, a_i) M} \right) \xrightarrow{a_{i+1}} Q_\lambda \left( \frac{M}{(a_1, \ldots, a_i) M} \right)$$

where $Q_\lambda(-)$ is the localization functor associated with $\lambda$.

3.1. PROPOSITION. If $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a $\lambda$-regular $M$-sequence then it is a weak $\lambda$-regular $M$-sequence.

**PROOF.** We have to show that

$$((a_1, \ldots, a_i) M : a_{i+1}) \subseteq \text{Cl}_\lambda^M ((a_1, \ldots, a_i) M) .$$

Take $y \in M$ such that

$$a_{i+1} y \in (a_1, \ldots, a_i) M .$$

Then there exist $y_1, \ldots, y_i \in M$ such that $a_{i+1} y = a_1 y_1 + \ldots + a_i y_i$.

Since $\{a_1, \ldots, a_n\}$ is a $\lambda$-regular $M$-sequence we can find a $\lambda$-dense submodule $N$ of $M$ such that $\{a_1, \ldots, a_n\}$ is a regular $M$-sequence relative to $N$. We may select an ideal $J \in \mathcal{I}(\lambda)$ satisfying that $J y \subseteq N$ and $J y_j \subseteq N$ for $j = 1, \ldots, i$. Hence

$$J a_{i+1} y \subseteq a_1 J y_1 + \ldots + a_i J y_i \subseteq (a_1, \ldots, a_i) N .$$

Thus as $J a_{i+1} y \subseteq N$ and $\{a_1, \ldots, a_n\}$ is a regular $M$-sequence relative to $N$ we can deduce that

$$J y \subseteq (a_1, \ldots, a_i) M .$$

Therefore it follows that $y \in \text{Cl}_\lambda^M ((a_1, \ldots, a_i) M). \quad \Box$
3.2. PROPOSITION. Let \( \{a_1, \ldots, a_n\} \subset R - \{0\} \) and let \( M \) be an \( R \)-module. Then the following conditions are equivalent

1) \( \{a_1, \ldots, a_n\} \) is a weak \( \kappa \)-regular \( M \)-sequence;

2) \( a_1 \) is weak \( \kappa \)-regular and \( \{a_2, \ldots, a_n\} \) is a weak \( \kappa \)-regular \( M/a_1M \)-sequence.

PROOF. 1) \( \Rightarrow \) 2) The first statement is clear. Assume now that \( m + a_1M \in M/a_1M \) is such that \( a_{i+1}(m + a_1M) \in (a_2, \ldots, a_i)(M/a_1M) \) then \( a_{i+1}m \in (a_1, \ldots, a_i)M \). By 1) it follows that \( m \in \text{Cl}^M_\kappa ((a_1, \ldots, a_i)M) \). Thus \( m + a_1M \in \text{Cl}^M_\kappa ((a_2, \ldots, a_i)(M/a_1M)) \) and \( \{a_2, \ldots, a_n\} \) is a weak \( \kappa \)-regular \( M/a_1M \)-sequence.

2) \( \Rightarrow \) 1) Let \( m \in M \) be such that \( a_{i+1}m \in (a_1, \ldots, a_i)M \) then \( a_{i+1}(m + a_1M) \in (a_2, \ldots, a_i)(M/a_1M) \). By 2) we obtain \( m + a_1M \in \text{Cl}^M_\kappa ((a_2, \ldots, a_i)(M/a_1M)) \).

Now, it is clear that \( m \in \text{Cl}^M_\kappa ((a_1, \ldots, a_i)M) \). \( \square \)

3.3. LEMMA. Let \( R \) be a Noetherian ring and let \( M \) be an \( R \)-module. If \( a \in R \) is a weak \( \kappa \)-regular element on \( M \) then

\[
Q_\kappa(\text{Ext}_R^n(N, \mathcal{O})): Q_\kappa(\text{Ext}_R^n(N, M)) \to Q_\kappa(\text{Ext}_R^n(N, M/K))
\]

is an isomorphism for every finitely generated \( R \)-module \( N \), where \( K = \text{Ker} \varphi \) and \( \varphi: M \to M \) is the multiplication by \( a \), \( \psi: M \to M/K \) is the canonical homomorphism.

PROOF. Since \( a \) is a weak \( \kappa \)-regular element on \( M \) then \( K \) is \( \kappa \)-torsion. Hence \( \text{Hom}_R(N, K) \) is \( \kappa \)-torsion for every finitely generated \( R \)-module \( N \). Since \( R \) is Noetherian we can take a projective resolution of \( N \)

\[
\to P_n \to \ldots \to P_1 \to P_0 \to N \to 0
\]

where every projective \( R \) module \( P_i \) is finitely generated, then it is clear that \( \text{Ext}_R^i(N, K) \) is \( \kappa \)-torsion for all \( i \), since they are submodules of quotients of \( \kappa \)-torsion modules.

The exact sequence

\[
0 \to K \to M \to M/K \to 0
\]
yields a long exact sequence

\[ \cdots \to \text{Ext}^i_R(N, K) \to \text{Ext}^i_R(N, M) \to \text{Ext}^i_R(N, M/K) \to \text{Ext}^{i+1}_R(N, K). \]

Applying the functor \( Q_\kappa \) we obtain the isomorphism

\[ Q_\kappa(\text{Ext}^i_R(N, M)) \cong Q_\kappa(\text{Ext}^i_R(N, M/K)). \]

If \( p \in \text{Spec}(R) \) then it is easy to see that \( p \) is either \( \kappa \)-dense or \( \kappa \)-closed; we will denote by \( \text{Spec}_\kappa(R) \) the set of \( \kappa \)-closed prime ideals.

3.4. Lemma. Let \( I \subseteq R \) be an ideal and let \( M \) be an \( R \)-module. If \( \text{Hom}(R/I, M) \) is \( \kappa \)-torsion then

\[ [\text{Ass}(M) \cap \text{Spec}_\kappa(R)] \cap V(I) = \emptyset. \]

Proof. Assume \( p \in \text{Ass}(M) \cap \text{Spec}_\kappa(R) \cap V(I) \). We have \( I \subseteq p \) and we can obtain an homomorphism \( R/p \to M \). So we obtain:

\[ f: R/I \to R/p \to M, \quad 1 + I \to 1 + p \to m. \]

Since \( f \) is \( \kappa \)-torsion there exists a \( J \in \mathcal{L}(\kappa) \) satisfying \( Jf = 0 \).

Then \( Jm = J(f(1 + I)) = (Jf)(1 + I) = 0 \). This \( J \subseteq \text{ann} m = p \) and \( p \in \mathcal{L}(\kappa) \). This is a contradiction. \( \square \)

3.5. Proposition. Let \( R \) be \( \kappa \)-Noetherian (a weak version of the Noetherian property, cf. [6]) and let \( M \) be a finitely generated \( R \)-module. If \( \text{Hom}(R/I, M) \) is \( \kappa \)-torsion then there exists an element that is weakly \( \kappa \)-regular on \( M \) and contained in \( I \).

Proof. Since \( M \) is finitely generated, \( \text{Ass}(M) \) is a finite set of prime ideals. Then by the Proposition on p. 134 of [4] there exists a submodule \( N \subseteq M \) satisfying

\[ \text{Ass}(N) \subseteq \text{Ass}(M) \cap \text{Spec}_\kappa(R) \cap V(I), \]

then \( M/N \) is \( \kappa \)-torsion and since \( V(I) \cap \text{Ass}(N) = \emptyset \) it follows that \( I \not\subseteq \bigcup_{p \in \text{Ass}(N)} p \). Therefore there exists an element \( a \in I \) such that \( a \) is
regular on \( N \). This states that \( a \) is \( x \)-regular on \( M \) and therefore \( a \) will be weakly \( x \)-regular by Proposition 3.1. \( \square \)

3.6. **Theorem.** Let \( R \) be a Noetherian ring. Let \( M \) be an \( R \)-module that is finitely generated. Let \( I \) be an ideal and let \( r \) be a positive integer. The following conditions are equivalent.

1) \( \text{Ext}_n^i (N, M) \) is \( x \)-torsion for all finitely generated \( R \)-modules \( N \) with \( \text{Supp}(N) \subseteq V(I) \) and all integers \( i < r \).

2) \( \text{Ext}_n^i (R/I, M) \) is \( x \)-torsion for all \( i < r \).

3) There exists a weakly \( x \)-regular \( M \)-sequence \( \{a_1, \ldots, a_r\} \) in \( I \).

**Proof.** 3) \( \Rightarrow \) 1) We will use induction on \( r \). Let \( \{a_1, \ldots, a_r\} \) be a weakly \( x \)-regular \( M \)-sequence contained in \( I \). Since \( a_1 \) is \( x \)-regular (weak), we have the following exact sequence

\[
0 \to K \to M \xrightarrow{a_1} M \to M/a_1 M \to 0
\]

where \( K \) is \( x \)-torsion. By Proposition 3.2, \( \{a_2, \ldots, a_r\} \) is a weakly \( x \)-regular \( M/a_1 M \)-sequence, then \( \text{Ext}_n^i (N, M) \) is \( x \)-torsion for \( i < r - 1 \) and any \( N \) such that \( \text{Supp}(N) \subseteq V(I) \).

Applying Lemma 3.3 to the exact sequence

\[
0 \to K \to M \to M/K
\]

we obtain that \( Q_x(\text{Ext}_n^i (N, M)) \cong Q_x(\text{Ext}_n^i (N, M/K)) \). The short exact sequence

\[
0 \to M/K \to M \to M/a_1 M \to 0
\]

yields the long exact sequence

\[
to \text{Ext}_n^{i-1} (N, M/a_1 M) \to \text{Ext}_n^i (N, M/K) \to \text{Ext}_n^i (N, M) \to \\
\to \text{Ext}_n^i (N, M/a_1 M) \to \text{Ext}_n^{i-1} (N, M/K) \to \ldots .
\]

Now by applying \( Q_x \) we obtain:

\[
0 \to Q_x(\text{Ext}_n^i (N, M/K)) \to Q_x(\text{Ext}_n^i (N, M)) .
\]
From the commutative triangle

\[
\begin{array}{ccc}
M & \xrightarrow{a_1} & M \\
\downarrow & & \downarrow \\
M/K & \xrightarrow{a_1} & 0
\end{array}
\]

We will obtain the following triangle that is also commutative

\[
\begin{array}{ccc}
\text{Ext}^1_R(N, M) & \xrightarrow{a_1} & \text{Ext}^1_R(N, M) \\
\downarrow & & \downarrow \\
\text{Ext}^1_R(N, M/K)
\end{array}
\]

Applying the functor \(Q_\kappa(-)\) we will have

\[
\begin{array}{ccc}
Q_\kappa(\text{Ext}^1_R(N, M)) & \xrightarrow{a_1} & Q_\kappa(\text{Ext}^1_R(N, M)) \\
\downarrow & & \downarrow \\
Q_\kappa(\text{Ext}^1_R(N, M/K))
\end{array}
\]

Since \(\text{Supp}(N) \subseteq V(I)\) we have \(I \subseteq \text{rad}(\text{Ann}(N))\) and so \(a_1^tN = 0\) for some \(t\) i.e. \(N \xrightarrow{a_1} N\) is nilpotent and then we obtain that

\[
\text{Ext}^1_R(N, M) \xrightarrow{a_1} \text{Ext}^1_R(N, M)
\]

is also nilpotent. Hence \(Q_\kappa(\text{Ext}^1_R(N, M)) \xrightarrow{a_1} Q_\kappa(\text{Ext}^1_R(N, M))\) is nilpotent, but it is also injective and therefore it will be zero. Finally we obtain that \(\text{Ext}^1_R(N, M)\) is \(\kappa\)-torsion.

1) \(\Rightarrow\) 2) Easy.

2) \(\Rightarrow\) 3) If \(\text{Hom}_R(R/I, M)\) is \(\kappa\)-torsion then there exists a weakly \(\kappa\)-regular element \(a \in I\), by Proposition 3.5.

The short exact sequence

\[0 \rightarrow M/K \rightarrow M \rightarrow M/a_1 M \rightarrow 0\]

yields a long exact sequence:

\[\cdots \rightarrow \text{Ext}^1_R(R/I, M) \rightarrow \text{Ext}^1_R(R/I, M/a_1 M) \rightarrow \text{Ext}^1_R(R/I, M/K) \rightarrow \cdots .\]
By Lemma 3.3 we have:

$$Q_\alpha(\text{Ext}^{i+1}_R(R/I, M/K)) \cong Q_\alpha(\text{Ext}^{i+1}_R(R/I, M)).$$

Hence $\text{Ext}^{i+1}_R(R/I, M/K)$ is $\alpha$-torsion for all $i < r - 1$.

By the hypothesis $\text{Ext}^i_R(R/I, M)$ is $\alpha$-torsion for all $i < r$.

Thus using the induction hypothesis on $M/\alpha_i M$ we will obtain a weakly $\alpha$-regular $M/\alpha_i M$-sequence $\{a_1, ..., a_n\}$ in $I$. Then by Proposition 3.2 we may infer that $\{a_1, ..., a_n\}$ is the weakly regular $M$-sequence that we are looking for. \qed

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