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B. TORRECILLAS
F. VAN OYSTAEYEN

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On M-Sequences Associated to Filtrations.

B. TORRECILLAS - F. VAN OYSTAEYEN (*)

0. Introduction.

Generalizations of the notion of regular M-sequences have been studied both from an algebraic and a geometric point of view. Whereas the algebraic theory is related to properties of Cohen-Macaulay-, Gorenstein- or Buchsbaum-rings, the geometric considerations are usually related to the study of suitable generalizations of complete intersections. In this note we present a unifying theory dealing with generalized regular sequences associated to filtrations on modules; in this way we recapture the relative regular sequences introduced by M. Fiorentini [5], as well as the homogeneous regular sequences of P. Bouchard [3] a.o. In a first part of this paper we compare regularity with respect to a filtration FM on M to the projective regularity with respect to the associated graded module $G(M)$ and to projective regularity with respect to the associated Rees module $\tilde{M}$. In the second part we focus on filtrations in the sense of Gabriel topologies. The final theorem links the existence of a weakly $\varkappa$-regular M-sequence contained in an ideal I of R to properties of $\mathrm{Ext}^i_R(N, M)$ for all finitely generated R-modules N having support Supp (N) contained in the closed set $V(I)$ determined by the ideal I.

(*) Indirizzo degli AA.: B. Torrecillas: Dpto. de Algebra y Fundamentos, Universidad de Granada, Spain; F. Van Oystaeyen: Dept. of Mathematics. University of Antwerp, UIA, Belgium.

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1. Preliminaries.

Let R be a commutative ring with filtration $FR = \{F_nR, n \in \mathbb{Z}\}$, i.e. an ascending chain of additive subgroups F_nR of R satisfying:

$$F_nRF_mR \subset F_{n+m}R \quad \text{for } n, m \in \mathbb{Z}, \ \bigcup_{n\in\mathbf{Z}} F_nR = R \text{ and } 1 \in F_0R .$$

An R-module M is said to be a filtered R-module M if it has a filtration by additive subgroups $FM = \{F_nM, n \in \mathbb{Z}\}$ such that

$$F_nRF_mM \subset F_{n+m}M \quad \text{for } n, m \in \mathbb{Z}, \ \bigcup_{n\in\mathbf{Z}} F_nM = M .$$

For full detail on filtered rings and modules we refer to [15]. We write

$$G(R) = \bigoplus_{n\in\mathbf{Z}} G(R)_n = \bigoplus_{n\in\mathbf{Z}} F_nR/F_{n-1}R \text{ and } G(M) = \bigoplus_{n\in\mathbf{Z}} F_nM/F_{n-1}M$$

for the associated graded ring of FR, resp. the associated graded module of FM. If $x \in F_nM - F_{n-1}M$ then $\sigma(x)$ is the image of x in $G(M)_n$.

The Rees ring of FR, $\tilde{R} = \bigoplus_{n\in\mathbf{Z}} F_nR$, is a graded ring in the obvious way and similarly the Rees module $\tilde{M} = \bigoplus_{n\in\mathbf{Z}} F_nM$ is a graded $\tilde{R}$-module. Viewing $1 \in F_1R$ as an element of $\tilde{R}_1$ we obtain an element $X \in \tilde{R}_1$, that is a regular element of $\tilde{R}$ such that:

$$\tilde{R}/X\tilde{R} \cong G(R), \quad \tilde{R}/(1-X)\tilde{R} \cong R, \quad \tilde{R}_X \cong R[X, X^{-1}]$$

and

$$\tilde{M}/X\tilde{M} \cong G(M), \quad \tilde{M}/(1-X)\tilde{M} \cong M, \quad \tilde{M}_X = M[X, X^{-1}] .$$

The category of the filtered R-modules, R-filt, is equivalent to the category of X-torsionfree graded $\tilde{R}$-modules. Recall from [2] or [10]:

i) If $M \in R$-filt then M is filt-free, filt-projective, if and only if $\tilde{M}$ is gr-free, projective, resp.

ii) The filtration FM is good, i.e. there exist $m_1, \ldots, m_t \in M$ and $d_1, \ldots, d_t \in \mathbb{Z}$ such that $F_n M = \sum_{i=1}^{t} F_{n-d_i} R \cdot m_i$ for all $n \in \mathbb{Z}$, if and only if $\tilde{M}$ is a finitely generated $\tilde{R}$-module.

iii) FM is separated, i.e.

$$\bigcap_{n \in \mathbb{Z}} F_n M = 0 , \quad \text{if and only if } \bigcap_{n \in \mathbb{Z}} X^n \tilde{M} = 0 .$$

A filtered ring is said to be a Zariski ring if $\tilde{R}$ is Noetherian and $F_{-1}R \subset J(F_0 R)$, where $J(-)$ denotes the Jacobson radical, cf. [10]. Typical good properties of Zariski rings are: good filtrations induce good filtrations on arbitrary submodules and filtrations equivalent to good filtrations are good too, moreover all good filtrations are automatically separated.

A subset $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a *regular M-sequence with respect to FM if for* $i = 0, \ldots, n-1$, we have the following for some $m \in \mathbb{Z}$,

$$F_m M \cap [(a_1, \ldots, a_i) F_m M : a_{i+1}] \subset (a_1, \ldots, a_i) M ,$$

where we put $(a_1, \ldots, a_i) = 0$ if $i = 0$, and $(a_1, \ldots, a_i)$ is the R-ideal generated by $\{a_1, \ldots, a_i\}$, $[X : a] = \{m \in M, am \in X\}$ for $a \in R$. When for all $d \leqslant 0$ we have for $i = 0, \ldots, n-1$, that

$$F_{d-1} M \cap [(a_1, \ldots, a_i) F_{d-1} M : a_{i+1}] \subset (a_1, \ldots, a_i) F_d M ,$$

we say that $\{a_1, \ldots, a_n\}$ is a *filtered regular M-sequence.* When the latter holds for all $d \in \mathbb{Z}$ then we say that $\{a_1, \ldots, a_n\}$ is a completely filtered regular M-sequence.

1.1. Lemma. If $\{a_1, \ldots, a_n\} \subset R$ is a completely filtered regular M-sequence then it is also an M-sequence.

Proof. Take an $x \in M$ such that $a_{i+1} x \in (a_1, \ldots, a_i) M$, say $a_{i+1} x = \sum_{j=1}^{i} a_j \lambda_j$ with $\lambda_j \in M$, $j = 1, \ldots, i$. If we choose $h \in \mathbb{Z}$ large enough such that $\lambda_j \in F_h M$ for $i = 1, \ldots, i$, then we have $a_{i+1} x \in (a_1, \ldots, a_i) \cdot$

$\cdot F_h M$. Now take $h' > h$ such that $x \in F_{h'-1}M$ and $a_{i+1}x \in F_{h'}M$, then

$$x \in [(a_1, \ldots, a_i)F_{h'-1}M : a_{i+1}] \cap F_{h'-1}M$$

and thus $x \in (a_1, \ldots, a_i)F_{h'}M \subset (a_1, \ldots, a_i)M$. □

The converse of the lemma need not hold in general.

1.2. EXAMPLES. 1) Let $(a_1, \ldots, a_n)$ be an M-sequence in R and put $I = (a_1, \ldots, a_n)$. Define $F_{-n}R = I^n$, $F_n R = R$ for $n \geqslant 0$, and also $F_{I,-n}M = I^n M$, $F_{I,n}M = M$ for $n \geqslant 0$. In [8] it has been shown that for $M = R$ we have:

$$I^{p+1} \cap [(a_1, \ldots, a_i)I^{p+1} : a_{i+1}] = (a_1, \ldots, a_i)I^p .$$

In [12] this has been extended to the situation we consider here:

$$I^{p+1}M \cap [(a_1, \ldots, a_i)I^{p+1}M : a_{i+1}] = (a_1, \ldots, a_i)I^p M .$$

The latter relation reduces exactly to our definition of a completely filtered regular M-sequence with respect to the I-adic filtration.

2) Let N be any R-submodule of M. We say that $\{a_1, \ldots, a_n\} \subset$ $\subset R - \{0\}$ is a *regular M-sequence relative to N* if for all $i = 0, \ldots$ $\ldots, n-1$, we have:

$$[(a_1, \ldots, a_i)N : a_{i+1}] \cap N \subset (a_1, \ldots, a_i)M .$$

It is clear that this property still holds if we replace N by any smaller submodule. Moreover, $\{a_1, \ldots, a_n\}$ is a regular M-sequence relative to N if and only if for all prime ideals P of R it is a regular M_P-sequence relative to N_P. A regular M-sequence relative to N appears as a regular M-sequence with respect to FM if we filter R trivially, $F_{-n}R = 0$ for $n > 0$ and $F_m R = R$ for $m \geqslant 0$, and put $F_m M = M$ for $m \geqslant 0$, $F_{-1}M = N$ and $F_{-n}M = 0$ for $n > 1$. For this particular filtration the sequence will be a completely filtered regular M-sequence exactly when it is a regular M-sequence.

3) Let R be positively graded and M is a graded R-module. A set $\{a_1, \ldots, a_n\} \subset h(R) - \{0\}$ is *a projective M-regular sequence* if it is a regular M-sequence relative to the submodule $M_{\geqslant k} = \bigoplus_{n \geqslant k} M_n$ for

some $k \in \mathbb{N}$. If we define $F_{-t}R = R_{\geqslant t}$, $F_{-l}M = M_{\geqslant l}$ for $t, l \in \mathbb{Z}$, then a projective M-regular sequence is just a regular M-sequence with respect to FM consisting of homogeneous elements of R.

Foregoing examples show that previous definitions, e.g. regular M-sequences relative to a submodule, [12], [5], projective regular M-sequences, [3], as well as relative sequences [8], are captured by our definition of regular M-sequence with respect to a filtration FM. Alternatively we may view these definitions from the point of view of Gabriel topologies or torsion theories. Let $\mathfrak{L}(\varkappa)$ be the Gabriel filter of an idempotent kernel functor in the sense of [6], [7]. We say that $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a *$\varkappa$-regular M-sequence* if it is a regular M-sequence relative to a submodule N of M that is $\varkappa$-dense in M, i.e. $\varkappa(M/N) = M/N$. When M is finitely generated then one may replace N by JM for some $J \in \mathfrak{L}(\varkappa)$; in that case $\{a_1, \ldots, a_n\}$ will be a $\varkappa$-regular M-sequence exactly when it is a regular M-sequence with respect to the J-adic filtration on M for some $J \in \mathfrak{L}(\varkappa)$. Let us (re-)consider some examples first.

1.3. Examples. 1) Let R be a positively graded ring generated as an R_0-module by finitely many (homogeneous) elements and assume that R_0 is a Noetherian ring. Let M be a graded R-module. The sequence $\{a_1, \ldots, a_n\} \subset h(R) - \{0\}$ is a projective regular M-sequence if and only if it is $\varkappa_+$-regular where $\mathfrak{L}(\varkappa_+) = \{H$ ideal of R, $H \supset R_+^p$ for some $p \in \mathbb{N}\}$. Indeed, R is a Noetherian ring and therefore $\mathfrak{L}(\varkappa_+)$ does define an idempotent kernel functor on R-mod. Furthermore if h is at least the maximum of the degrees of the R_0-algebra generators of R then $R_{\geqslant hn} \subset R_+^n \subset R_{\geqslant n}$ and since in 1.2 2) we may replace N by a smaller submodule, the statements above are easily checked.

2) Let I be an ideal of a Noetherian ring R and M a finitely generated R-module. Let $\varkappa_I$ be given by the Gabriel filter: $\mathfrak{L}(\varkappa_I) = \{H$ ideal of R, $H \supset I^p$ for some $p \in \mathbb{N}\}$. Then, $\{a_1, \ldots, a_n\}$ is a $\varkappa_I$-regular M-sequence if and only if it is a regular M-sequence relative to the I-adic filtration.

3) For an R-submodule N of M let $\varkappa(M/N)$ be the « smallest » torsion theory for which M/N is a torsion-module, cf. [6], then a regular M-sequence relative to N is also a $\varkappa(M/N)$-regular sequence.

Let us point out that the property of being a regular M-sequence relative to FM, is a *topological* property in the sense that the property is being conserved if we replace FM by an equivalent filtration $F'M$. On the other hand, the property of being a filtered regular M-sequence is not insensitive to such a change of filtration; we may consider this property as a more « algebraic » one that is closer related to properties of the associated gradation.

2. Sequences and Zariskian filtrations.

For any graded ring R and (finitely generated) graded R-module M we extend the definition given in 1.2 (3) and define a *projective M-regular sequence* to be a set $\{a_1, \ldots, a_n\}$ in $h(R) - \{0\}$ that is a regular M-sequence relative to $M_{\geqslant k}$ for some $k \in \mathbb{Z}$.

2.1. PROPOSITION. Let $\{a_1, \ldots, a_r\} \subset R - \{0\}$. If $\{X, \tilde{a}_1, \ldots, \tilde{a}_r\}$ is a projective regular $\tilde{M}$-sequence in $\tilde{R}$ then $\{a_1, \ldots, a_r\}$ is a regular M-sequence with respect to FM.

PROOF. We write $\tilde{a}$ for an $a \in F_n M - F_{n-1} M$ viewed in $\tilde{M}_n$, hence we may identify $\tilde{a}$ and aX^n if we identify $\tilde{M}$ with $\sum_{n \in \mathbb{Z}} F_n M \cdot X^n$. We have to show that there is an $m \in \mathbb{Z}$ such that

$$[(a_1, \ldots, a_i) F_m M : a_{i+1}] \cap F_m M \subset (a_1, \ldots, a_i) M , \quad i = 0, \ldots, r-1 .$$

If $y \in F_m M$ is such that $a_{i+1} y \in (a_1, \ldots, a_i) F_m M$ and $y \in F_t M - F_{t-1} M$ for some $t \leqslant m$, then $y a_{i+1} = \sum_{j=1} \lambda_j a_j m_j$ with $\lambda_j \in R$, $m_j \in F_m M$. If $\tilde{y} = yX^t$, $\tilde{a}_{i+1} = a_{i+1} X^s$ then $(y a_{i+1})^{\sim} = y a_{i+1} X^k$ with $k \leqslant t + s$, hence $\tilde{y} \cdot \tilde{a}_{i+1} = y a_{i+1} X^{t+s} = (y a_{i+1})^{\sim} X^{t+s-k}$ and so we arrive at

$$\tilde{y} \tilde{a}_{i+1} = \sum_{j=1}^{i} \bar{\lambda}_j \tilde{a}_j \tilde{m}_j X^{\beta_j} \quad \text{for certain } \beta_j \in \mathbb{Z} .$$

Taking β large enough we arrive at $(\tilde{y} X^{\beta}) \tilde{a}_{i+1} \in (\tilde{a}_1, \ldots, \tilde{a}_n) \tilde{M}_{\geqslant m} \subset (X, \tilde{a}_1, \ldots, \tilde{a}_n) \tilde{M}_{\geqslant m}$ and therefore we may choose m such that the projectivity of $(X, \tilde{a}_1, \ldots, \tilde{a}_r)$ yields $\tilde{y} X^{\beta} \in (X, \tilde{a}_1, \ldots, \tilde{a}_i) \tilde{M}$. Applying $\tilde{M} \to \tilde{M}/(X-1)\tilde{M} = M$, we obtain that $y \in (a_1, \ldots, a_i) M$ as desired.

2.2. PROPOSITION. Let $\{a_1, \dots, a_r\} \subset R - \{0\}$. If $\{\sigma(a_1), \dots, \sigma(a_r)\}$ is a projective $G(M)$-regular sequence then $\{X, \tilde{a}_1, \dots, \tilde{a}_r\}$ is a projective $\tilde{M}$-regular sequence in $\tilde{R}$.

PROOF. Clearly X is not a zero divisor on $\tilde{M}$. Since $G(M) = \tilde{M}/X\tilde{M}$ and $\tilde{a}_i \bmod \tilde{R}X = \sigma(a_i)$, it follows from the hypothesis on $\{\sigma(a_1), \dots, \sigma(a_r)\}$ that $\{X, \tilde{a}_1, \dots, \tilde{a}_r\}$ is as claimed.

2.3. REMARK. In general, the fact that $\{\sigma(a_1), \dots, \sigma(a_r)\}$ is a regular sequence does not imply that $\{a_1, \dots, a_r\}$ is an R-sequence, see Example 2.5 of [13].

2.4. PROPOSITION. Let $\{a_1, \dots, a_r\} \subset R - \{0\}$. If $\{\tilde{a}_1, \dots, \tilde{a}_r\} \subset \tilde{R}$ is a projective $\tilde{M}$-regular sequence then $\{\sigma(a_1), \dots, \sigma(a_r)\}$ is a projective $G(M)$-regular sequence.

PROOF. Take

$$\sigma(y) \in G(M)_{\geqslant k} \cap \left[\big(\sigma(a_1), \dots, \sigma(a_i)\big)\, G(M)_{\geqslant k} : \sigma(a_{i+1})\right]$$

then $\sigma(y)$ is in the kernel of the mapping

$$G(M)_{\geqslant k}/\big(\sigma(a_1), \dots, \sigma(a_i)\big)\, G(M)_{\geqslant k} \to G(M)/\big(\sigma(a_1), \dots, \sigma(a_i)\big)\, G(M)_{\geqslant k}$$

induced by multiplication by $\sigma(a_{i+1})$. We have an exact sequence

$$0 \to \operatorname{Ker} \Psi \to \tilde{M}_{\geqslant k}/(\tilde{a}_1, \dots, \tilde{a}_i)\tilde{M}_{\geqslant k} \xrightarrow{\Psi} \tilde{M}/(\tilde{a}_1, \dots, \tilde{a}_i)\tilde{M}_{\geqslant k} .$$

We view Ψ as the restriction of an $\tilde{R}$-linear map Ψ' defined on

$$[\tilde{M}/(\tilde{a}_1, \dots, \tilde{a}_i)\tilde{M}_{\geqslant k}] .$$

Applying $- \bigotimes_{\tilde{R}} \tilde{R}/X\tilde{R}$ to Ψ' and restricting to $\tilde{M}_{\geqslant k}/(\tilde{a}_1, \dots, \tilde{a}_i)\tilde{M}_{\geqslant k}$ again we obtain:

$$0 \to G(\operatorname{Ker} \Psi) \to G(M)_{\geqslant k}/\big(\sigma(a_1), \dots, \sigma(a_i)\big)\, G(M)_{\geqslant k} \xrightarrow[\bar{\Psi}]{} \to G(M)/\big(\sigma(a_1), \dots, \sigma(a_i)\big)\, G(M)_{\geqslant k}$$

where $\overline{\Psi}$ is again determined by multiplication by $\sigma(a_{i+1})$. Therefore $\sigma(y) \in G(\operatorname{Ker} \Psi)$ and we may select a $\tilde{z} \in \operatorname{Ker} \Psi'$ such that $\sigma(y) = \sigma(z) = \tilde{z} \bmod X\tilde{M}/(\tilde{a}_1, \ldots, \tilde{a}_i)\tilde{M}_{\geqslant k}$ and note that we may actually find such a $\tilde{z}$ even in $\operatorname{Ker} \Psi$ because

$$\tilde{M}/(\tilde{a}_1, \ldots, \tilde{a}_i)\tilde{M}_{\geqslant k} \to GM/\big(\sigma(a_1), \ldots, \sigma(a_i)\big)G(M)_{\geqslant k}$$

is a graded morphism. The hypothesis on $\{\tilde{a}_1, \ldots, \tilde{a}_r\}$ implies that $\tilde{z} \in (\tilde{a}_1, \ldots, \tilde{a}_i)\tilde{M}$ if we assume we were looking at the suitable $k \in \mathbb{Z}$ from the beginning. Now $\sigma(y) = \sigma(z) \in \big(\sigma(a_1), \ldots, \sigma(a_i)\big)G(M)$ follows. □

2.5. Proposition. Let $\{a_1, \ldots, a_r\} \subset R - \{0\}$; if $\{\sigma(a_1), \ldots, \sigma(a_r)\}$ is a projective $G(M)$-regular sequence then $\{a_1, \ldots, a_r\}$ is a regular M-sequence with respect to F.

Proof. Combine Proposition 2.2 and Proposition 2.1. □

2.6. Lemma. Let R be a ring with a Zariskian filtration. Then the following statements are equivalent for an ideal $I = (x_1, \ldots, x_s)$

1) $G(I)$ is generated by $\{\sigma(x_1), \ldots, \sigma(x_s)\}$;

2) $F_nR \cap I = \sum_{i=1}^{s} F_{n-n_i}Rx_i$ where $\nu(x_i) = \deg \sigma(x_i) = n_i$ for $i = 1, 2, \ldots, s$.

Proof. 1) $\Rightarrow$ 2) See proof of implication 8) $\Rightarrow$ 1) in III.4.4 [9].
2) $\Rightarrow$ 1) It is clear that

$$G(I)_n = \frac{(F_nR \cap I) + F_{n-1}R}{F_{n-1}R}$$

and

$$\big(\sigma(x_1), \ldots, \sigma(x_s)\big)_n = \Big(\sum_{i=1}^{s} F_{n-n_i}Rx_i + F_{n-1}R\Big)/F_{n-1}R$$

therefore if

$$F_nR \cap I = \sum_{1}^{s} F_{n-n_i}Rx_i$$

we have that $G(I) = \big(\sigma(x_1), \ldots, \sigma(x_s)\big)$. □

2.7. PROPOSITION. Let R be a ring with a Zariskian filtration. Assume that $I = (x_1, \ldots, x_s)$, if $\sigma(x_1), \ldots, \sigma(x_s)$ is a $G(R)$-sequence then $G(I) = (\sigma(x_1), \ldots, \sigma(x_s))$.

PROOF. We apply induction on s. The case $s = 1$ is easy. If $\sigma(x)$ is a non-zero divisor then for each $y \in R$, $\sigma(xy) = \sigma(x)\sigma(y)$. Thus

$$G(I) = \oplus \frac{Rx \cap F_nR + F_{n-1}R}{F_{n-1}R} = G(R)\sigma(x) .$$

Assume now that the result is true for $s-1$ and we will prove it for s.

Let $a \in I \cap F_nR = F_nI$. As $I = (x_1, \ldots, x_s)$ we will have

$$a = a_1x_1 + \ldots + a_sx_s \qquad \text{with } a_s \in F_tR .$$

We can find a minimal t with this property, otherwise a will be in the closure of $(x_1, \ldots, x_{s-1})$ and since the filtration is Zariskian $(x_1, \ldots, x_{s-1})$ is closed and $a \in (x_1, \ldots, x_{s-1})$. The result then follows from the induction hypothesis. Now take t minimal such that $a = x + a_sx_s$ with $x \in (x_1, \ldots, x_{s-1})$, $a_sx_s = a - x \in [(x_1, \ldots, x_{s-1}) + F_nR] \cap F_{t+n_s}R$ with $\nu(x_s) = n_s$.

If $t + n_s \leqslant n$,

$$a \in (x_1, \ldots, x_{s-1}) \cap F_nR + F_{n-n_s}Rx_s = \sum_{i=1}^{n-1} F_{n-n_i}Rx_i + F_{n-\nu_s}Rx_s$$

(by Lemma 2.6), and by applying the same lemma the claim follows.

Assume now that $t + n_s > n$, $F_nR \subset F_{t+n_s}R$ so

$$a_sx_s \in [(x_1, \ldots, x_{s-1}) + F_nR] \cap \cap F_{t+n_s}R \subseteq [(x_1, \ldots, x_{s-1}) + F_{t+n_s-1}R] \cap F_{t+n_s}R .$$

It follows that $\sigma(a_s)\sigma(x_s) \in G((x_1, \ldots, x_{s-1})) = (\sigma(x_1), \ldots, \sigma(x_{s-1}))$ by the induction hypothesis. Since $\sigma(x_1), \ldots, \sigma(x_s)$ is a $G(R)$-sequence then $\sigma(a_s) \in G((x_1, \ldots, x_{s-1}))$. Then $a_s \in [(x_1, \ldots, x_{s-1}) \cap F_tR] + F_{t-1}R$ and hence $a \in (x_1, \ldots, x_{s-1}) + F_{t-1}Rx_s$. This contradicts the choice of t. Hence the last case is not possible and the proof is completed. □

2.8. Definition. A sequence $x_1, \ldots, x_s$ in R is called super-regular if the sequence $\sigma(x_1), \ldots, \sigma(x_s)$ is a regular sequence in $G(R)$ (cf. [8]).

The following theorem gives a characterization of this property.

2.8. Theorem. Let R be a ring with a Zariski filtration and let $I = (x_1, \ldots, x_s)$. Then the following conditions are equivalent.

1) $(x_1, \ldots, x_s)$ is a super regular M-sequence.

2) $(x_1, \ldots, x_s)$ is a regular M-sequence and

$$I_i \cap F_n R = \sum_{j=1}^{i} F_{n-n_j} R x_j$$

where $I_i = (x_1, \ldots, x_i)$, $n_i = \nu(x_i)$, where $\nu(x_i) = \deg \sigma(x_i)$, as usual.

Proof. 1) $\Rightarrow$ 2) If $\sigma(x_1), \ldots, (x_i)$ is a $G(R)$-sequence by Proposition 2.7 it follows that $G(I) = \big(\sigma(x_1), \ldots, \sigma(x_i)\big)$ and now by the Lemma 2.6 we obtain

$$I_i \cap F_n R = \sum_{j=1}^{i} F_{n-n_j} R x_j \,.$$

We only have to show that $I_{i-1} : x_i \subseteq I_{i-1}$. Let $a \in I_{i-1} : x_i$ with $\nu(a) = n$ then $\sigma(a)\sigma(x_i) \in G(I_{i-1}) = \big(\sigma(x_1), \ldots, \sigma(x_{i-1})\big)$ but $\sigma(x_1), \ldots, \sigma(x_s)$ is a $G(R)$-sequence, hence $\sigma(a) \in G(I_{i-1})$. Therefore $a \in (I_{i-1} \cap F_n R) + F_{n-1}R$, $a = x - a_{n-1}$, with $x \in I_{i-1}$, $a_{n-1} \in I_{i-1}$. Repeating the argu ment for a_{n-1} we will obtain that a is in the closure of I_{i-1} that equals I_{i-1} by the Zariski condition.

2) $\Rightarrow$ 1) Let $a \in R$ such that $\sigma(a)\sigma(x_i) \in \big(\sigma(x_1), \ldots, \sigma(x_1)\big)$, $\nu(a) = n$ then $a x_i \in (x_1, \ldots, x_{i-1}) + F_{n+n_i-1} R$, i.e.

$$a x_i = b + \sum_{j=1}^{i-1} a_j x_j \qquad \text{where } b \in F_{n+n_i-1} R \,.$$

Thus $b = a x_i - \sum_{j=1}^{i-1} a_j x_j$ belongs to $I_{i-1} \cap F_{n+n_i-1} R$.

Now write $b = \sum_{j=1}^{i} b_j x_j$ where $b_j \in F_{n+n_i-1-n_j} R$.
Then

$$(a - b_i) x_i = \Big(b + \sum_{j=1}^{i-1} a_j x_j\Big) - \Big(b - \sum_{j=1}^{i-1} b_j x_j\Big) \in I_{i-1} \,.$$

We obtain $(a - b_i) \in (I_{i-1} : x) \subseteq I_{i-1}$ and that implies $a \in I_{i-1} + F_{n-1}R$. Therefore $\sigma(a) \in G(I_{i-1})$. □

Note that the second statement of 2) does not follow from the Zariski hypothesis because we have fixed the generators $x_1, \dots, x_i$, and the goodness of $I_i \cap FR$ cannot necessarily be phrased in terms of prescribed generators! Let us conclude this section by a change of base result.

2.9. THEOREM. If $(a_1, \dots, a_n)$ is a regular M-sequence with respect to FM and let $f: R \to S$ be a filtered flat ring morphism then $f(a_1), \dots \dots, f(a_n)$ is a regular $S \bigotimes_R M$-sequence with respect to the tensor-product filtration $FS \bigotimes_R M$.

PROOF. An easy modification of the proof of Proposition 3 in [12] following the ideas of the proof of Proposition 2.4. □

2.10. COROLLARY. Let R be a ring with a Zariskian filtration FR and let S be a multiplicatively closed set in R such that $\sigma(S) = \{\sigma(s), s \in S\}$ is an Ore set of $G(R)$ consisting of regular elements. Then for a finitely generated filtered R-module M a regular M-sequence $\{a_1, \dots, a_n\}$ relative to FM is also a regular $Q^{\mu}_S(M)$-sequence relative to $FG^{\mu}_S(M)$, where $Q^{\mu}_S(M)$ is the microlocalization at S and $FQ^{\mu}_S(M)$ is the microlocalized filtration.

PROOF. Recall from [2] that $R \hookrightarrow Q^{\mu}_S(R)$ is a filtered flat ring morphism and apply the theorem, keeping in mind that $Q^{\mu}_S(M) = Q^{\mu}_S(R) \bigotimes_R M$ since M is finitely generated. □

3. Relative sequences.

DEFINITION. Let $\mathfrak{L}(\varkappa)$ be a filter of an idempotent kernel functor on R-mod, cf. [6], [7], then $\mathfrak{L}(\varkappa)$ generates a linear topology on R-mod. For an R-module M we say that $\{a_1, \dots, a_r\} \subset R\{0\}$ is a *$\varkappa$-regular M-sequence* if it is a regular M-sequence relative to a submodule N of M that is $\varkappa$-dense in M.

It is clear that if $\varkappa_1 \leqslant \varkappa_2$ then any $\varkappa_1$-regular M-sequence is a $\varkappa_2$-regular M-sequence. Also if $\{a_1, \dots, a\} \subset R - \{0\}$ is a $\varkappa_1$-regular

M-sequence and $\varkappa_2$-regular M-sequence then it is a $\varkappa_1 \vee \varkappa_2$-regular M-sequence.

Definition. For an R-module M we say that $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a *weak $\varkappa$-regular M-sequence* if

$$((a_1, \ldots, a_i)M : a_{i+1}) \subseteq \mathrm{Cl}_\varkappa^M((a_1, \ldots, a_i)M)$$

for $i = 0, \ldots, n-1$, where $\mathrm{Cl}_\varkappa^M(X) = \{m \in M, Im \subset X \text{ for some } I \in \mathfrak{L}(\varkappa)\}$. This is equivalent to exactness of the following sequence

$$0 \to Q_\varkappa\left(\frac{M}{(a_1, \ldots, a_i)M}\right) \xrightarrow{a_{i+1}} Q_\varkappa\left(\frac{M}{(a_1, \ldots, a_i)M}\right)$$

where $Q_\varkappa(-)$ is the localization functor associated with $\varkappa$.

3.1. Proposition. If $\{a_1, \ldots, a_n\} \subset R - \{0\}$ is a $\varkappa$-regular M-sequence then it is a weak $\varkappa$-regular M-sequence.

Proof. We have to show that

$$((a_1, \ldots, a_i)M : a_{i+1}) \subseteq \mathrm{Cl}_\varkappa^M((a_1, \ldots, a_i)M) .$$

Take $y \in M$ such that

$$a_{i+1}y \in (a_1, \ldots, a_i)M .$$

Then there exist $y_1, \ldots, y_i \in M$ such that $a_{i+1}y = a_1y_1 + \ldots + a_iy_i$.

Since $\{a_1, \ldots, a_n\}$ is a $\varkappa$-regular M-sequence wa can find a $\varkappa$-dense submodule N of M such that $\{a_1, \ldots, a_n\}$ is a regular M-sequence relative to N. We may select an ideal $J \in \mathfrak{L}(\varkappa)$ satisfying that $Jy \subseteq N$ and $Jy_j \subseteq N$ for $j = 1, \ldots, i$. Hence

$$Ja_{i+1}y \subset a_1Jy_1 + \ldots + a_iJy_i \subseteq (a_1, \ldots, a_i)N .$$

Thus as $Ja_{i+1}y \subseteq N$ and $\{a_1, \ldots, a_n\}$ is a regular M-sequence relative to N we can deduce that

$$Jy \subseteq (a_1, \ldots, a_i)M .$$

Therefore it follows that $y \in \mathrm{Cl}_\varkappa^M((a_1, \ldots, a_i)M)$. $\square$

3.2. PROPOSITION. Let $\{a_1, \ldots, a_n\} \subset R - \{0\}$ and let M be an R-module. Then the following conditions are equivalent

1) $\{a_1, \ldots, a_n\}$ is a weak $\varkappa$-regular M-sequence;

2) a_1 is weak $\varkappa$-regular and $\{a_2, \ldots, a_n\}$ is a weak $\varkappa$-regular $M/a_1 M$-sequence.

PROOF. 1) $\Rightarrow$ 2) The first statement is clear. Assume now that $m + a_1 M \in M/a_1 M$ is such that $a_{i+1}(m + a_1 M) \in (a_2, \ldots, a_i)(M/a_1 M)$ then $a_{i+1} m \in (a_1, \ldots, a_i) M$. By 1) it follows that $m \in \mathrm{Cl}_\varkappa^M((a_1, \ldots, a_i)M)$. Thus $m + a_1 M \in \mathrm{Cl}_\varkappa^{M/a_1 M}((a_2, \ldots, a_i)(M/a_1 M))$ and $\{a_2, \ldots, a_n\}$ is a weak $\varkappa$-regular $M/a_1 M$-sequence.

2) $\Rightarrow$ 1) Let $m \in M$ be such that $a_{i+1} m \in (a_1, \ldots, a_i) M$ then $a_{i+1}(m + a_1 M) \in (a_2, \ldots, a_i)(M/a_1 M)$. By 2) we obtain $m + a_1 M \in \in \mathrm{Cl}_\varkappa^{M/a_1 M}((a_2, \ldots, a_i)(M/a_1 M))$.

Now, it is clear that $m \in \mathrm{Cl}_\varkappa^M((a_1, \ldots, a_i) M)$. □

3.3. LEMMA. Let R be a Noetherian ring and let M be an R-module. If $a \in R$ is a weak $\varkappa$-regular element on M then

$$Q_\varkappa(\mathrm{Ext}_R^q(N, \Psi)) : Q_\varkappa(\mathrm{Ext}_R^q(N, M)) \to Q_\varkappa(\mathrm{Ext}_R^q(N, M/K))$$

is an isomorphism for every finitely generated R-module N, where $K = \mathrm{Ker}\,\varphi$ and φ: $M \to M$ is the multiplication by a, ψ: $M \to M/K$ is the canonical homomorphism.

PROOF. Since a is a weak $\varkappa$-regular element on M then K is $\varkappa$-torsion. Hence $\mathrm{Hom}_R(N, K)$ is $\varkappa$-torsion for every finitely generated R-module N. Since R is Noetherian we can take a projective resolution of N

$$\to P_n \to \ldots \to P_1 \to P_0 \to N \to 0$$

where every projective R module P_i is finitely generated, then it is clear that $\mathrm{Ext}_R^i(N, K)$ is $\varkappa$-torsion for all i, since they are submodules of quotients of $\varkappa$-torsion modules.

The exact sequence

$$0 \to K \to M \to M/K \to 0$$

yields a long exat sequence

$$\ldots \to \mathrm{Ext}^i_R(N, K) \to \mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^i_R(N, M/K) \to \mathrm{Ext}^{i+1}_R(N, K)\,.$$

Applying the functor $Q_\varkappa$ we obtain the isomorphism

$$Q_\varkappa(\mathrm{Ext}^i_R(N, M)) \cong Q_\varkappa(\mathrm{Ext}^i_R(N, M/K))\,. \quad \square$$

If $p \in \mathrm{Spec}(R)$ then it is easy to see that p is either $\varkappa$-dense or $\varkappa$-closed; we will denote by $\mathrm{Spec}_\varkappa(R)$ the set of $\varkappa$-closed prime ideals.

3.4. Lemma. Let $I \subseteq R$ be an ideal and let M be an R-module. If $\mathrm{Hom}(R/I, M)$ is $\varkappa$-torsion then

$$[\mathrm{Ass}(M) \cap \mathrm{Spec}_\varkappa(R)] \cap V(I) = \emptyset\,.$$

Proof. Assume $p \in \mathrm{Ass}(M) \cap \mathrm{Spec}_\varkappa(R) \cap V(I)$. We have $I \subset p$ and we can obtain an homomorphism $R/p \to M$. So we obtain:

$$f\colon R/I \to R/p \to M\,,\; 1 + I \to 1 + p \to m\,.$$

Since f is $\varkappa$-torsion there exists a $J \in \mathfrak{L}(\varkappa)$ satisfying $Jf = 0$.

Then $Jm = J(f(1+I)) = (Jf)(1+I) = 0$. This $J \subseteq anm = p$ and $p \in \mathfrak{L}(\varkappa)$. This is a contradiction. $\square$

3.5. Proposition. Let R be $\varkappa$-Noetherian (a weak version of the Noetherian property, cf. [6]) and let M be a finitely generated R-module. If $\mathrm{Hom}(R/I, M)$ is $\varkappa$-torsion then there exists an element that is weakly $\varkappa$-regular on M and contained in I.

Proof. Since M is finitely generated, $\mathrm{Ass}(M)$ is a finite set of prime ideals. Then by the Proposition on p. 134 of [4] there exists a submodule $N \subseteq M$ satisfying

$$\mathrm{Ass}(N) = \mathrm{Ass}(M) \cap \mathrm{Spec}_\varkappa(R)\,,$$
$$\mathrm{Ass}(M/N) = \mathrm{Ass}(M) \cap \mathfrak{L}_\varkappa(R)\,,$$

then M/N is $\varkappa$-torsion and since $V(I) \cap \mathrm{Ass}(N) = \emptyset$ it follows that $I \not\subseteq \bigcup_{p \in \mathrm{Ass}(N)} p$. Therefore there exists an element $a \in I$ such that a is regular on N. This states that a is $\varkappa$-regular on M and therefore a will be weakly $\varkappa$-regular by Proposition 3.1. □

3.6. THEOREM. Let R be a Noetherian ring. Let M be an R-module that is finitely generated. Let I be an ideal and let r be a positive integer. The following conditions are equivalent.

1) $\mathrm{Ext}_R^i(N, M)$ is $\varkappa$-torsion for all finitely generated R-modules N with $\mathrm{Supp}(N) \subseteq V(I)$ and all integers $i < r$.

2) $\mathrm{Ext}_R^i(R/I, M)$ is $\varkappa$-torsion for all $i < r$.

3) There exists a weakly $\varkappa$-regular M-sequence $\{a_1, \ldots, a_r\}$ in I.

PROOF. 3) $\Rightarrow$ 1) We will use induction on r. Let $\{a_1, \ldots, a_r\}$ be a weakly $\varkappa$-regular M-sequence contained in I. Since a_1 is $\varkappa$-regular (weak), we have the following exact sequence

$$0 \to K \to M \xrightarrow{a_1} M \to M/a_1M \to 0$$

where K is $\varkappa$-torsion. By Proposition 3.2, $\{a_2, \ldots, a_r\}$ is a weakly $\varkappa$-regular M/a_1M-sequence, then $\mathrm{Ext}_R^i(N, M)$ is $\varkappa$-torsion for $i < r-1$ and any N such that $\mathrm{Supp}(N) \subseteq V(I)$.

Applying Lemma 3.3 to the exact sequence

$$0 \to K \to M \to M/K$$

we obtain that $Q_\varkappa(\mathrm{Ext}_R^i(N, M)) \simeq Q_\varkappa(\mathrm{Ext}_R^i(N, M/K))$. The short exact sequence

$$0 \to M/K \to M \to M/a_1M \to 0$$

yields the long exact sequence

$$\to \mathrm{Ext}_R^{i-1}(N, M/a_1M) \to \mathrm{Ext}_R^i(N, M/K) \to \mathrm{Ext}_R^i(N, M) \to$$
$$\to \mathrm{Ext}_R^i(N, M/a, M) \to \mathrm{Ext}_R^{i-1}(N, M/K) \to \ldots .$$

Now by applying $Q_\varkappa$ we obtain:

$$0 \to Q_\varkappa(\mathrm{Ext}_R^i(N, M/K)) \to Q_\varkappa(\mathrm{Ext}_R^i(N, M)) .$$

From the commutative triangle

$$\begin{array}{ccc} M & \xrightarrow{a_1} & M \\ \downarrow & \nearrow_{a_1} & \\ M/K & & \\ \downarrow & & \\ 0 & & \end{array}$$

We will obtain the following triangle that is also commutative

$$\begin{array}{ccc} \mathrm{Ext}^i_R(N, M) & \xrightarrow{a_1} & \mathrm{Ext}^i_R(N, M) \\ \downarrow & \nearrow & \\ \mathrm{Ext}^i_R(N, M/K) & & \end{array}$$

Applying the functor $Q_\varkappa(-)$ we will have

$$\begin{array}{ccc} Q_\varkappa\big(\mathrm{Ext}_R(\mathrm{Ext}^i_R(N\ M))\big) & \xrightarrow{a_1} & Q_\varkappa(\mathrm{Ext}^i_R(N, M)) \\ \downarrow & \nearrow & \\ Q_\varkappa(\mathrm{Ext}^i_R(N, M/K)) & & \end{array}$$

Since $\mathrm{Supp}(N) \subset V(I)$ we have $I \subseteq \mathrm{rad}\big(\mathrm{Ann}(N)\big)$ and so $a_1^t N = 0$ for some t i.e. $N \xrightarrow{a_1} N$ is nilpotent and then we obtain that

$$\mathrm{Ext}^i_R(N, M) \xrightarrow{a_1} \mathrm{Ext}^i_R(N, M)$$

is also nilpotent. Hence $Q_\varkappa(\mathrm{Ext}^i_R(N, M)) \xrightarrow{a_1} Q_\varkappa(\mathrm{Ext}^i_R(N, M))$ is nilpotent, but it is also injective and therefore it will be zero. Finally we obtain that $\mathrm{Ext}^i_R(N, M)$ is $\varkappa$-torsion.

1) $\Rightarrow$ 2) Easy.

2) $\Rightarrow$ 3) If $\mathrm{Hom}_R(R/I, M)$ is $\varkappa$-torsion then there exists a weakly $\varkappa$-regular element $a \in I$, by Proposition 3.5.

The short exact sequence

$$0 \to M/K \to M \to M/a_1 M \to 0$$

yields a long exact sequence:

$$\ldots \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, M/a_1 M) \to \mathrm{Ext}^{i+1}_R(R/I, M/K) \to \ldots .$$

By Lemma 3.3 we have:

$$Q_\varkappa(\mathrm{Ext}_R^{i+1}(R/I, M/K)) \cong Q_\varkappa(\mathrm{Ext}_R^{i+1}(R/I, M)) .$$

Hence $\mathrm{Ext}_R^{i+1}(R/I, M/K)$ is $\varkappa$-torsion for all $i < r-1$.

By the hypothesis $\mathrm{Ext}_R^i(R/I, M)$ is $\varkappa$-torsion for all $i < r$.

Thus using the induction hypothesis on M/a_1M we will obtain a weakly $\varkappa$-regular M/a_1M-sequence $\{a_2, \ldots, a_n\}$ in I. Then by Proposition 3.2 we may infer that $\{a_1, \ldots, a_n\}$ is the weakly regular M-sequence that we are looking for. □

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