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On Outer Automorphisms of Černikov $p$-Groups.

Orazio Puglisi (*)

0. Introduction.

As is well known, every finite $p$-group that is not cyclic of order $p$, has non inner $p$-automorphisms. This theorem, proved by Gaschütz in [1], was later made more precise by Schmid and then extended by Menegazzo and Stonehewer. In [2] in fact, Schmid proves that, apart from some exceptions, Out $G$ has a normal $p$-subgroup (always in the hypothesis that $G$ is a finite $p$-group) while in [3] Menegazzo and Stonehewer prove an analogous theorem to that one of Gaschütz in the case of infinite nilpotent $p$-groups. Even in the case that $G$ is infinite the normal $p$-subgroups of Out $G$ have been studied and in [4], Marconi has reached an analogous result to the one obtained by Schmid. In this paper the problem of the existence of outer $p$-automorphisms is studied in the hypothesis that $G$ is an infinite Černikov $p$-group, obtaining an affirmative answer for a certain class of such groups. To be more precise, if $G$ is a Černikov $p$-group, indicating with $G_0$ its finite residual and with Fit $G$ its Fitting subgroup, we have the following

Theorem. — Let $G$ be a non nilpotent Černikov $p$-group. If Fit $G > G_0$ and $G_0 \cap Z(G)$ is divisible then $G$ has outer $p$-automorphisms.

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Even the case $\text{Fit } G = G_0$ is examined obtaining

**Theorem.** – Let $G$ be a non nilpotent Černikov $p$-group and assume $\text{Fit } G = G_0$ and $Z(G)$ divisible. Then $G$ has non inner $p$-automorphisms or $H^1(G/G_0, G_0) = 0$ and the natural image of $G/G_0$ in $\text{Aut } G_0$ is a Sylow $p$-subgroup of $\text{Aut } G_0$.

The last section of this work is devoted to the construction of some examples which show what can happen if $\text{Fit } G = G_0$, $Z(G)$ is divisible and the image of $G/G_0$ in $\text{Aut } G_0$ is a Sylow $p$-subgroup.

1. Preliminaries.

If $G$ is a Černikov $p$-group we shall indicate from now on with $G_0$ its finite residual that is an artinian divisible abelian group and with $\text{Fit } G$ the Fitting subgroup of $G$. It is worth while remembering that $G/G_0$ is a finite group so that $|G| < \aleph_0$, while $\text{Fit } G = C_o(G_0)$ is nilpotent and its centralizer in $G$ coincides with $Z(\text{Fit } G)$. In the proof of theorem 2.1, we shall use the results about nilpotent $p$-groups cited in the introduction, which are here below listed for the readers’ use.

**Theorem 1.1** (Gaschütz [1]). If $G$ is a finite $p$-group that is not cyclic of order $p$, then $G$ has a non inner $p$-automorphism.

**Theorem 1.2** (Schmid [2]). Let $G$ be a finite non abelian $p$-group. Then $p$ divides the order of $C_{\text{Out } G}(Z(G))$.

**Theorem 1.3** (Menegazzo-Stonehewer [3]). Let $G$ be a nilpotent $p$-group. If $G$ is neither cyclic of order $p$ nor isomorphic to a direct product of $k$ quasi-cyclic $p$-groups with $k < p - 1$, then $G$ has an outer automorphism of order $p$.

**Theorem 1.4** (Marconi [4]). Let $H$ be an infinite nilpotent $p$-group. Then $O_p(\text{Out } H) = 1$ if and only if one of the following conditions holds:

i) $H$ is elementary abelian

ii) $H$ is divisible and $p$ is odd
iii) $H$ is the central product of $\Omega_4(G)$ and of a quasi cyclic $p$-group with $\Omega_4(G)$ extra special of exponent $p \neq 2$.

First of all we want to prove that a Černikov $p$-group always has outer automorphisms, a fact which comes easily from the following theorem.

**Theorem 1.5** (Pettet [5]). Let $G$ be periodic and $H \triangleleft G$ a Černikov group such that $|G: N_G(H)|$ is finite. If $C_{\text{Aut}_{G_{\sigma}}}(H)$ is finite or countable and $C_{\text{Inn}_{G_{\sigma}}}(H)$ is Černikov, then $G$ is Černikov and $G_0 = H_0^\sigma$.

**Corollary 1.1.** Let $G$ be an infinite Černikov $p$-group. Then $|\text{Aut } G| > \aleph_0$. In particular $\text{Out } G \neq 1$.

**Proof.** If $|\text{Aut } G| = \aleph_0$ then, with the same notations of Theorem 1.5 let $H = 1$. $H$ and $G$ satisfy the hypotheses of Theorem 1.5 so $G_0 = H_0^\sigma = 1$, a contradiction. So $|\text{Aut } G| > \aleph_0$ and, therefore, $\text{Out } G \neq 1$.

The proof of Theorem 2.1 is based in great part on the following fact concerning the cohomology groups of $G/\text{Fit } G$.

**Lemma 1.1.** Let $G$ be a Černikov $p$-group, $G_0$ its finite residual, $F = \text{Fit } G$. Suppose $G_0 \cap Z(G)$ divisible. If $H^1(G/F, Z(F)) = 0$ then $H^n(G/F, Z(F)) = 0$ for $m > 0$.

**Proof.** Let $K = G/F$ and $A = Z(F)$. $F$ is nilpotent so $A \supseteq G_0$ and therefore we can write $A = G_0 \oplus L_1$ where $L_1$ is finite. Also $Z(G) = D \oplus L_2$ with $D$ divisible and $L_2$ finite. Let $p^n = \max \{|L_1|, |K|\}$ and consider the following short exact sequence in $G$-$\text{Mod}$ (and therefore in $K$-$\text{Mod}$)

$$0 \rightarrow A[p^n] \rightarrow A \xrightarrow{j} G_0 \rightarrow 0$$

where $j$ is the multiplication by $p^n$. We have also the related long exact sequence

$$0 \rightarrow H^0(K, A[p^n]) \rightarrow H^0(K, A) \rightarrow H^0(K, G_0) \rightarrow$$

$$\rightarrow H^1(K, A[p^n]) \rightarrow \ldots \rightarrow H^n(K, A[p^n]) \rightarrow H^n(K, A) \rightarrow \ldots .$$
For every $K$-module we have $H^0(K, M) = \{ m \in M : m^x = m \ \forall x \in K \}$, so that we can rewrite this sequence as follows

$$0 \to A[p^n] \cap Z(G) \to Z(G) \to G_0 \cap Z(G) \to H^1(K, A[p^n]) \to$$

$$0 \to H^1(K, G_0) \to H^2(K, A[p^n]) \to \ldots \to H^n(K, A[p^n]) \to$$

$$H^n(K, A) \to H^{n+1}(K, G_0) \to H^{n+1}(K, A[p^n]) \to H^{n+1}(K, A) \to \ldots$$

because $H^1(G/F, Z(F)) = 0$. Now $\theta$ is surjective and $G_0 \cap Z(G)$ is divisible, so $H^1(K, A[p^n]) = 0$ because it is a finite group. Then, by $[1]$, $H^m(K, A[p^n]) = 0$ $\forall m > 0$ so that, as it is easy to see, $H^1(K, G_0) = 0$ and $H^m(K, A)$ is isomorphic to $H^m(K, G_0)$ $\forall m > 0$. Now consider the exact sequence

$$0 \to G_0[p^n] \to G_0 \xrightarrow{j} G_0 \to 0$$

where $j$ is the multiplication by $p^n$, and the related cohomology sequence

$$0 \to G_0[p^n] \cap Z(G) \to G_0 \cap Z(G) \to G_0 \cap Z(G) \to H^1(K, G_0[p^n]) \to$$

$$H^1(K, G_0) \xrightarrow{j} H^1(K, F_0) \to H^2(K, G_0[p^n]) \to \ldots \to H^n(K, G_0[p^n]) \to$$

$$H^n(K, G_0) \xrightarrow{j} H^n(K, G_0) \to H^{n+1}(K, G_0[p^n]) \to H^{n+1}(K, G_0) \to \ldots .$$

As before we can see that $H^m(K, G_0[p^n]) = 0$ $\forall m > 0$ so that, $\forall m > 1$, we have

$$0 \to H^m(K, G_0) \xrightarrow{j} H^m(K, G_0) \to 0 .$$

But $j$ is the trivial morphism because the exponent of $H^m(K, G_0)$ divides $|K|$ and therefore $H^m(K, G_0) = 0 = H^m(K, A)$ as claimed.

2. Main theorems.

By theorem 1.3 we can limit ourselves to the case in which $G$ is non nilpotent. The principal result obtained is the following

**Theorem 2.1.** Let $G$ be a non nilpotent Černikov $p$-group, $G_0$ its finite residual. If $\text{Fit.} G > G_0$ and $G_0 \cap Z(G)$ is divisible then $G$ has outer $p$-automorphisms.
PROOF. Consider the extension \( e: 1 \to F \to G \to K \to 1 \) where \( F = \text{Fit } G = C_o(G_o) \) and \( K = G/F \). \( F \) is characteristic in \( G \) so \( \text{Out } e = \text{Out } G \). The Wells sequence (Wells [6]) associated to \( e \) is

\[
0 \to H^1(K, Z(F)) \to \text{Out } G \to N_{\text{Out } F}(D)/D \to H^2(K, Z(F)) .
\]

Here \( D \) is the image of \( K \) in \( \text{Out } F \) obtained by the natural morphism \( \chi: K \to \text{Out } F \) associated to the extension \( e \). \( K \cong D \) because \( C_o(F) = Z(F) \leq F \). If \( H^1(K, Z(F)) \neq 0 \) then it is easy to construct a non inner \( p \)-automorphism of \( G \) choosing an outer derivation \( \delta: K \to Z(F) \) and setting \( x^\delta = x(xF)^\delta \). It is well known that \( \alpha \) is an outer \( p \)-automorphism of \( G \). Then we may assume \( H^1(K, Z(F)) = 0 \). By lemma 1.1 we have \( H^2(K, Z(F)) = 0 \) so that the Wells sequence becomes \( \text{Out } G \cong N_{\text{Out } F}(D)/D \). Our purpose is now to prove that \( N_{\text{Out } F}(D)/D \) has non trivial \( p \)-subgroups. The first step is to show that \( O_p(\text{Out } F) \neq 1 \) using Theorem 1.3. Surely \( F \) doesn’t satisfy conditions i) or ii) of that theorem. Furthermore, \( G \) being non nilpotent, \( \text{rg } G_o > p - 1 \) so that \( \text{rg } G_o > 1 \) and \( F \) doesn’t satisfy condition iii).

Two cases are to be examined:

a) \( O_p(\text{Out } F) < D \).

We can write \( F = BZ(F) \) with \( B \) a finite characteristic subgroup such that \( F/B \) divisible. If \( B \) is abelian so is \( F \).

\[
G = C_{\text{Aut } F}(F|G_o, G_o) \cong \text{Hom } (F|G_o, G_o) \neq 1
\]
is a normal \( p \)-subgroup of \( \text{Aut } F = \text{Out } F \) so it is contained in \( D \). But this is impossible because the only element in \( D \) centralizing \( G_o \) is 1. Then \( B \) cannot be abelian. By Theorem 1.2 there exist an outer \( p \)-automorphism \( \alpha \) of \( B \) centralizing \( Z(B) \supset B \cap Z(G) \). We can extend this automorphism \( \alpha \) to an automorphism \( \beta \) of \( F \) setting \( x^\beta = x^\alpha \) if \( x \in B \), \( x^\beta = x \) if \( x \in Z(G) \setminus B \). \( \beta \) is well defined, it is outer and has the same period of \( \alpha \). This implies that \( H = C_{\text{Out } F}(Z(F)) \) has non trivial \( p \)-subgroups. If \( \alpha \in C_{\text{Aut } F}(Z(F)) \) there exist an integer \( n \) such that \( \alpha^n \) is the identity on \( F/Z(F) \), that is \( \alpha^n \in C_{\text{Aut } F}(Z(F), F/Z(F)) \cong H^1(F/Z(F), Z(F)) \) that is a \( p \)-group of finite exponent. So \( C_{\text{Aut } F}(Z(F)) \) is periodic and therefore \( H \) is finite. \( D \) acts on \( H \) by conjugation, then it normalizes a non trivial \( p \)-Sylow subgroup of \( H \), say \( P \). \( D \) is strictly contained in \( PD \) because \( D \cap H = 1 \) and therefore \( N_{PD}(D) > D \). This implies that \( N_{\text{Out } F}(D)/D \) has non trivial \( p \)-subgroups.
b) $O_p(\text{Out } F) \not\subseteq D$.

Let $T = O_p(\text{Out } F)D$. $T$ is a Černikov $p$-group so $D$ is strictly contained in its normalizer and, for this reason, $N_{\text{Aut } F}(D)/D$ has non trivial $p$-subgroups. 

We are then left to examine the case in which $G_0 = \text{Fit } G$. In these hypotheses the existence of outer $p$-automorphisms in no longer certain. We have in fact

THEOREM 2.2. Let $G$ be a non nilpotent Černikov $p$-group, $G_0$ its finite residual and assume $\text{Fit } G = C_0(G_0) = G_0$, $H^1(K, Z(F)) = 0$ and $Z(G)$ divisible. Then $G$ has outer $p$-automorphisms if and only if the natural image of $G/G_0$ in $\text{Aut } G_0$ is not a Sylow $p$-subgroup of $\text{Aut } G_0$.

PROOF. - As in the proof of Theorem 2.1 we obtain $\text{Out } G \cong N_{\text{Aut } G_0}(D)/D$. If $D$ is not a Sylow $p$-subgroup of $\text{Aut } G_0$, then there exists a $p$-subgroup $P$ of $\text{Aut } G_0$ such that $D < P$. $P$ is finite so $D < N_p(D)$, hence $N_{\text{Aut } G_0}(D)/D$ has non trivial $p$-subgroups. On the other hand, if $G$ has an outer $p$-automorphism then $\exists \alpha \in N_{\text{Aut } G_0}(D)/D$ such that $\alpha^p = 1$, then the group $R = \langle \alpha \rangle D$ is a $p$-group, $R > D$ and, therefore, $D$ cannot be a Sylow $p$-subgroup of $\text{Aut } G_0$. 

COROLLARY 2.1. Let $G$ be a non nilpotent Černikov $p$-group. Suppose $C_0(G_0) = G_0$, $Z(G)$ divisible and that the image of $G/G_0$ in $\text{Aut } G_0$ is a Sylow $p$-subgroup of $\text{Aut } G_0$. Then $G$ has outer $p$-automorphisms if and only if $H^1(G/G_0, G_0) \neq 0$.

3. Examples.

Corollary 2.1, though establishing a necessary and sufficient condition for the existence of outer $p$-automorphisms, doesn't allow to establish the existence of Černikov $p$-groups for which this condition is verified. In this section we shall construct some examples which prove how, if a group satisfies the hypotheses of corollary 2.1, we can have either $H^1(G/G_0, G_0) = 0$ or $H^1(G/G_0, G_0) \neq 0$. From here onwards we shall indicate with $R_p$ and $Q_p$ respectively the ring of $p$-adic integer and its field of fractions. Let also remember that if $G_0 = (\mathbb{Z}(p^\infty))^*$, then $\text{Aut } G \cong GL(n, R_p)$. The results about the struc-
ture of Sylow $p$-subgroups of $GL(n, \mathbb{Q}_p)$ we shall use, have been proved by Vol'vacev in [7].

**Remark.** If $p = 2$ there are no Černikov 2-groups satisfying the hypotheses of Corollary 2.1. In fact, if $\alpha$ is the element of $\text{Aut} G_0$ sending every element $a$ of $G_0$ in its inverse $a^{-1}$, $\alpha$ belongs to the centre of $\text{Aut} G_0$ so, if $D$ (the image of $G/G_0$ in $\text{Aut} G_0$) is a Sylow 2-subgroup of $\text{Aut} G_0$ then it contains $\alpha$. Hence there is an element $g$ of $G$ such that $a^\alpha = a^{-1}$ $\forall a \in G_0$. Then $Z(G)$ cannot be divisible because

$$Z(G) \leq C_{G_0}(g) = \Omega_1(G_0).$$

**Example 1.** Let $p \geq 3$. Let $C$ be the companion matrix of the polynomial $1 + t + t^2 + \ldots + t^{p-1}$ and set $A = (1 \, 1 \, 0 \, \ldots \, 0)$.

Consider $X = \begin{pmatrix} C & 0 \\ A & 1 \end{pmatrix}$ where $0$ is a column of $p - 1$ zeroes. If $B_i = \sum_{i=0}^{p-1} C^i$ we have $X^i = \begin{pmatrix} C^i & 0 \\ AB_i & 1 \end{pmatrix}$. The Sylow $p$-subgroups of $GL(p, \mathbb{Q}_p)$ have order $p$ because $p \geq 3$, hence $\langle X \rangle$ is a Sylow $p$-subgroup of $GL(p, \mathbb{Q}_p)$. Consider the group $G = G_0 \langle x \rangle$ where $G_0 = (\mathbb{Z}(p^\infty))^p$ the direct sum of $p$ copies of $\mathbb{Z}(p^\infty)$ and $x$ is the automorphism represented by the matrix $X$. An easy calculation shows that $G$ satisfies the hypotheses of Corollary 2.1. We claim that $H^1(G/G_0, G_0) = 0$. Let $\sigma, \tau: G_0 \to G_0$ be the morphisms defined by

$$a^\sigma = [a, x] \quad \text{and} \quad a^\tau = \prod_{i=0}^{p-1} a^{x^i} \quad \forall a \in G_0.$$  

We know that

$$H^1(G/G_0, G_0) \cong \ker \tau/\text{Im} \, \sigma, \quad \text{Im} \, \sigma \cong G_0/Z(G) \cong (\mathbb{Z}(p^\infty))^{p-1}.$$  

More difficult is to find $\ker \tau$. The matrix associated to $\tau$ is $Y = 1 + X + X^2 + \ldots + X^{p-1}$ that is $Y = \begin{pmatrix} 0 & 0 \\ B & p \end{pmatrix}$ for some $B \in B_p^{p-1}$. We claim that the first element of $B$ is $p - 2$. In fact we have

$$B = A \left( \sum_{i=1}^{p-2} B_i \right) = \left( \sum_{i=1}^{p-1} \sum_{j=0}^{i-1} C^i \right) = A \left( \sum_{i=0}^{p-2} (p - i - 1) C^i \right).$$  

The elements of place $(1, 1)$ and $(2, 1)$ of the matrix $\sum_{i=0}^{p-2} (p - i - 1) C^i$ are, respectively, $p - 1$ and $-1$ so that the first element of $B$ is $p - 2$ as claimed.
Let $a = (a_1, \ldots, a_n)$ be an element of $G_0$, $a_i \in \mathbb{Z}(p^\infty)$. By a direct calculation we see that $a^\tau = (0, 0, \ldots, (p - 2)a_1 + \sum_{i=2}^{p-1} \lambda_i a_i + pa_{p+1})$. But $p - 2$ is a unit in $R_p$ so we have

$$\text{Ker} \tau = \left\{ (a_1, \ldots, a_n); a_1 = \frac{-1}{p - 2} \left[ \sum_{i=2}^{p-1} \lambda_i a_i + pa_{p+1} \right] \right\}.$$ 

Define

$$A_i = \left\{ \left( \frac{-\lambda_i}{p - 2} a, 0, \ldots, a, \ldots, 0 \right); a \in \mathbb{Z}(p^\infty) \right\}.$$ 

$A_i$ is, obviously, a divisible subgroup of $G_0$ of rank 1. Furthermore, $A_i \cap \sum_{j \neq i} A_j = 0$ so that $\text{Ker} \tau$ is the direct sum of the subgroups $A_i$ and, therefore, is divisible of rank $p - 1$. Hence $H^1(G/G_0, G_0) = 0$ and $G$ has no outer $p$-automorphisms.

**Example 2.** Let $p > 3$. With the same notations of example 1, let $E = \begin{pmatrix} C & 0 \\ A & 1 \end{pmatrix}$ and $X = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix}$. $X$ is an element of $GL(p + 1, R_p)$.

$\langle X \rangle$, as in example 1, is a Sylow $p$-subgroup of $GL(p + 1, R_p)$ so the group $G = G_0\langle x \rangle$ (where $G_0 = (\mathbb{Z}(p^\infty))^{p+1}$ and $x$ is the automorphism induced by $X$) satisfies the hypotheses of corollary 2.1. Using the same arguments of example 1 we can see that $\text{Im} \sigma$ is a divisible group of rank $p - 1$.

If $a = (a_1, \ldots, a_{p+1}) \in G_0$, then

$$a^\tau = (0, 0, \ldots, (p - 2)a_1 + \sum_{i=1}^{p} \lambda_i a_i, pa_{p+1}).$$

So $\text{Ker} \tau = \left( \bigoplus_{i=2}^{p} A_i \right) \oplus B$ where $B$ is cyclic of order $p$. Then, in this case, $H^1(G/G_0, G_0) \neq 0$ and $G$ has non inner $p$-automorphisms.

**Example 3.** In this example we will construct a group $G$ such that the image of $G/G_0$ is a Sylow $p$-subgroup of $GL(n, R_p)$ but not of $GL(n, Q_p)$, as it was in the previous examples. Let $p = 3$ and

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X \in GL(4, Q_p) \text{ and } X^3 = I.$$
\( \langle X \rangle \) is not a Sylow 3-subgroup of \( GL(4, \mathbb{Q}_3) \) because they are elementary abelian of order 9. Suppose there exists \( Y \in GL(4, \mathbb{Q}_3) \) s.t. \( Y^2 = 1 \) and \( |\langle X, Y \rangle| = 9 \). Set \( G_0 = (\mathbb{Z}(3^\infty))^4 \). Let \( x \) and \( y \) be the automorphisms of \( G_0 \) induced by \( X \) and \( Y \). \( C_{G_0}(x) = \{(0,0,a,b) : a,b \in \mathbb{Z}(3^\infty)\} \). \( C_{G_0}(y) = C_{G_0}(x) \) and therefore \( Y \) has the form \( Y = \begin{pmatrix} L & 0 \\ M & N \end{pmatrix} \)

\( L, M, N \in M(2, \mathbb{Q}_3) \).

From this point on we set \( S = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \). Using the relation \( x^y = x \) we deduce that \( L^{-1}SL = S \) and a routine calculation proves that the only possibilities are \( L = I, S, S^2 \). If \( L = S^2 \) the first block of \( Y^2 \) is \( S \), so we can reduce our discussion to the cases \( L = I \) or \( L = S \). Note that \( N^3 = I \) and that \( x \) acts as the identity on the last two components of \( G_0 \) so we may assume \( N = S \) or \( N = I \).

Four cases are to be examined:

1) \( Y = \begin{pmatrix} S & 0 \\ M & S \end{pmatrix} \)

\( M = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \) \( xy = yx \Longleftrightarrow TS + M = MS + ST \Longleftrightarrow (m, n, r, s) \) is a solution, in \( \mathbb{Q}_3 \), of the equations

\[
\begin{align*}
    m + n &= 1 \\
    m - 2n &= 0 \\
    r + s &= 1 \\
    r - 2s &= 0
\end{align*}
\]

But these equations have no solutions in \( \mathbb{Q}_3 \).

2) \( Y = \begin{pmatrix} S & 0 \\ M & I \end{pmatrix} \)

\( xy = yx \Longleftrightarrow TS + M = MS + T \Longleftrightarrow M(S - I) = T(S - I) \Longleftrightarrow M = T \)

and this gives \( x = y \)

3) \( Y = \begin{pmatrix} I & 0 \\ M & S \end{pmatrix} \)

\( xy = yx \Longleftrightarrow T + M = MS + ST \Longleftrightarrow (m, n, r, s) \) is a solution of the following equations

\[
\begin{align*}
    m + n &= -1 \\
    m - 2n &= 1 \\
    r + s &= -1 \\
    r - 2s &= 1
\end{align*}
\]
But the solution of these equations is not in $R_a$.

$$4) \ Y = \begin{pmatrix} I & 0 \\ M & I \end{pmatrix}$$

$xy = yx \iff T + M = MS + T \iff M(S - I) = 0 \iff M = 0.$

This proves that $\langle X \rangle$ is a Sylow 3-subgroup of $GL(4, R_a)$. Now, as in example 2, we deduce that $H^1(G/G_o, G_o)$ is cyclic of order 3 so that $G$ has outer 3-automorphisms.

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