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## On Bounded Solutions of One-Dimensional Compressible Navier-Stokes Equations.

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**SUMMARY** - It is shown that if a solution  $(u, v)$  of one-dimensional compressible Navier-Stokes equations in Lagrangian mass coordinates is bounded for  $t \in R$  (i.e.  $u \in L^\infty(R; L^2(0, 1))$  and  $0 < \alpha \leq v(x, t) \leq \beta < \infty$ , where  $u$  is the velocity and  $v$  the specific volume of a fluid), then  $u = 0$ ,  $v = v(x)$  (i.e.  $(u, v) = (0, v)$  is a stationary solution).

### 1. - Introduction.

In the present work we study bounded solutions of one-dimensional compressible Navier-Stokes equations in the Lagrangian form:

$$(1.1) \quad u_t + p(v)_x - \mu \left( \frac{u_x}{v} \right)_x = f \left( \int_0^x v(\xi, t) d\xi \right),$$

$$(1.2) \quad v_t = u_x, \quad (x, t) \in Q \equiv (0, 1) \times R,$$

$$(1.3) \quad u(0, t) = u(1, t) = 0,$$

$$(1.4) \quad \int_0^1 v(x, t) dx = 1, \quad t \in R,$$

$$(1.5) \quad p = p(v), \quad (p \in C^1((0, \infty)), p'(\xi) < 0, \text{ for } \xi > 0).$$

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Here  $u$  is the velocity,  $v$  the specific volume,  $\mu = \text{const} > 0$  the viscosity of the fluid,  $f = f(x)$ ,  $f \in C([0, 1])$  a forcing term, which is supposed to be independent of time.

Global existence theorems for the equations (1.1)-(1.5) with Cauchy data at  $t = t_0$  for  $t \geq t_0$  have been given in [4], [7].

Having global existence of solutions for the problem (1.1)-(1.5) we would like to know some qualitative properties of the solutions. In the system (1.1), (1.2) the equation (1.1) has a dissipative term  $-\mu(u_x/v)_x$  which is known to cause a dissipation of energy due to viscosity. This suggests a question: Does the system (1.1)-(1.5) have the usual properties of dissipative equations? To be more specific, we would like to know e.g. whether the following assertions are true:

- (a) If  $(u, v)$  is a bounded solution (see Definitions 2.3, 2.7) then  $(u, v)$  is stationary;
- (b) any solution converges to a stationary one (with a precise rate of convergence) as  $t \rightarrow \infty$ .

Some results close to these assertions have already been proved. See [4] for  $f \equiv 0$ , [5], [6] for  $f = f(x, t)$ ,  $p(v)$  special. These results are reviewed in [7], where (for  $f = f(x)$ ) we have proved that if there exists a sufficiently smooth solution  $(u, v)$  of the system (1.1)-(1.5) on  $[t_0, \infty)$  which is bounded in the sense that

$$(1.6) \quad 0 < \alpha \leq v(x, t) \leq \beta < \infty,$$

then it converges to the stationary solution in  $H^1(0, 1)$  as  $t \rightarrow \infty$ . The result has been obtained by the help of a series of a priori estimates for the solution and its derivatives in  $L^2$ -spaces on  $[t_0, \infty)$  which allow us to choose a subsequence  $u(\cdot, t_n)$ ,  $v(\cdot, t_n)$  converging in  $H^1(0, 1)$  to a limit  $(\bar{u}, \bar{v})$  of the form  $\bar{u} = 0$ ,  $\bar{v} = \bar{v}(x)$ . This limit is shown to be a stationary solution to (1.1)-(1.5).

In view of the results of Shelukhin [6], who, as mentioned above, under some special assumptions proved the existence of a bounded solution for  $t \in R$ , we tried to prove it for more general  $p$ . But our result in [7] implies that (for  $f = f(x)$ ) a bounded solution cannot exist on  $[t_0, \infty)$  unless there exists a stationary one. This suggests the hypotheses that if the solution satisfies (1.6) on  $R$  and

$$(1.7) \quad \sup \left\{ \int_0^1 u(x, t)^2 dx; t \in R \right\} < \infty$$

then  $u = 0$ ,  $v = \bar{v}(x)$  is stationary. This is exactly what we prove in this paper. The condition (1.7) must be added since for  $t \rightarrow -\infty$  the boundedness of  $\int_0^1 u(x, t)^2 dx$  does not follow from (1.6) and in general need not hold. One has to remark however that the stationary solution was proved to exist if and only if  $f$  and  $p(v)$  satisfy suitable compatibility condition (see [1], Theorem 5.3, rewritten here as Lemma 2.9). Hence our result says that a  $(u, v)$ -bounded solution on  $R$  cannot exist if  $f$  and  $p(v)$  do not satisfy such a condition.

After this manuscript has been finished the authors acquainted with the most recent results of Beirão de Veiga contained in two preprints [2], [3]. It is proved that any stationary solution is exponentially stable under small perturbations and that there is a positive threshold for the norms of initial conditions  $u_0, v_0$  and the  $L^\infty$ -norm of the right hand side  $f$  under which the solution satisfies estimate (2.1) below for  $J = [0, \infty)$ . These results improve substantially understanding the large time behavior of the solution especially in the physically significant case  $p(v) = Av^{-\gamma}$ ,  $\gamma \in (1, 2)$ .

In what follows we adopt the usual notation, namely  $C_0^\infty$  for spaces of smooth functions with a compact support,  $W^{k,p}$  for the Sobolev spaces, in particular  $H^s = W^{s,2}$ ,  $H^0 = L^2$ ,  $\dot{H}^s = \dot{W}^{s,2}$ ,  $|\cdot|_s = \|\cdot\|_{H^s}$ ,  $(\dot{H}^s)' = H^{-s}$ ; or  $H^s(J; B)$  for  $H^s$ -functions with values in a Banach space  $B$  and the like.

## 2. - Basic notions and results.

Let the problem (1.1)-(1.5) be given with  $p = p(v)$  and  $f = f(x)$  satisfying the following assumptions:

- (i)  $p \in C^1((0, \infty))$ ,  $p'(s) < 0$ ,  $s \in (0, \infty)$ ;
- (ii)  $f \in L^\infty(0, 1)$ .

For  $w \in L^1(0, 1)$  denote  $Iw(x) = \int_0^x w(\xi) d\xi$ .

**2.1 DEFINITION.** By a solution of the problem (1.1)-(1.5) on an interval  $J \subseteq R$  we mean a couple of functions  $(u, v)$  such that

$$u \in C(J; H^0(0, 1)) \cap L_{loc}^2(J; \dot{H}^1(0, 1)), \quad u_t \in L_{loc}^2(J; H^{-1}(0, 1)),$$

$$v \in C(J; H^1(0, 1)), \quad v_t \in L_{loc}^2(J; H^0(0, 1)), \quad v(x, t) > 0$$

for any  $t \in J$  everywhere on  $[0, 1]$ ,  $u, v$  satisfy (1.1) in the sense of  $H^{-1}(0, 1)$ , (1.2) in the sense of  $H^0(0, 1)$  and (1.4) for  $t \in J$ .

**2.2 THEOREM.** Let the functions  $p$  and  $f$  satisfy the assumptions (i), (ii). Then for any interval  $J \subset R$  bounded from below there exists, a solution to (1.1)-(1.5) (having assigned the initial value  $u(\cdot, t_0) \in \dot{H}^1(0, 1)$ ,  $v(\cdot, t_0) \in H^1(0, 1)$  at  $t = t_0$  a unique one).

The proof for initial data in  $H^1(0, 1)$  can be found in [7]. For  $u(\cdot, t_0) \in H^0(0, 1)$ ,  $v(\cdot, t_0) \in H^1(0, 1)$  it is shown in [2], Theorem 4.1, that a solution can be constructed as a limit of solutions with smooth initial data. Uniqueness of such a solution is proved in [2], Theorem 5.3 under the assumption that  $p'$  is locally Lipschitz continuous and  $f \in L^2(J; L^2(0, 1)) \cap L^1(J; C^{(0)+1}([0, 1]))$ .

**2.3 DEFINITION.** By a  $v$ -bounded solution of (1.1)-(1.5) on  $J$  we mean a solution  $(u, v)$  of (1.1)-(1.5) on  $J$  such that

$$(2.1) \quad 0 < \alpha \leq v(x, t) \leq \beta < \infty \quad \text{for all } t \in J, \quad x \in [0, 1],$$

where  $\alpha, \beta$  are constants.

A proof of the following theorem can again be found in [7].

**2.4 THEOREM.** Under the assumptions (i), (ii) for any bounded interval  $J \subset R$  there exists a  $v$ -bounded solution of (1.1)-(1.5) on  $J$  (having assigned the initial values  $u(\cdot, t_0) \in \dot{H}^1(0, 1)$ ,  $v(\cdot, t_0) \in H^1(0, 1)$  at  $t = t_0$  a unique one).

Denote

$$(2.2) \quad P(\eta) = \int_1^\eta [p(1) - p(\zeta)] d\zeta,$$

$$(2.3) \quad F(y) = \int_0^y f(\eta) d\eta$$

and

$$(2.4) \quad E(t) = \int_0^1 \left[ \frac{1}{2} u(t)^2 + P(v(t)) + \lambda - F(Iv(t)) \right] dx$$

for any solution of (1.1)-(1.5). Here  $\lambda \geq 0$  is an arbitrary constant to

be chosen in what follows so that  $\lambda - F(Iv)$  is non-negative. ( $Iv(t)$  stands for  $\int_0^t v(\xi, t) d\xi$ .) Note that  $P(\eta) \geq 0$  for  $\eta > 0$ .

**2.5 LEMMA.** For any  $v$ -bounded solution of (1.1)-(1.5) on an interval  $J \subseteq R$  the function  $E(t)$  is non-increasing on  $J$  and for any  $t, s \in J, t < s$  the inequalities

$$(2.5) \quad \frac{\mu}{\beta} \int_t^s |u_x(\tau)|_0^2 d\tau \leq E(t) - E(s) \leq \frac{\mu}{\alpha} \int_t^s |u_x(\tau)|_0^2 d\tau$$

hold.

**PROOF.** We can prove analogously as in [7], Lemma 2.2 that

$$(2.6) \quad \dot{E}(t) = - \mu \int_0^1 \frac{u_x(t)^2}{v(t)} ds \leq 0.$$

The only distinction is that, handling with the equation (1.1), instead of pairing in  $H^0(0, 1)$  the pairing between  $\dot{H}^1(0, 1)$  and  $H^{-1}(0, 1)$  must be used. According to (2.6) the inequalities (2.5) follow from (2.1) by integration. ■

**2.6 LEMMA.** Let  $(u, v)$  be a  $v$ -bounded solution of (1.1)-(1.5) on  $J, \lambda > 0$  in (2.4) sufficiently large. Then the following assertions are equivalent:

- 1°  $E(t) \leq \text{const} < \infty, \quad t \in J;$
- 2°  $|u(t)|_0^2 \leq \text{const} < \infty, \quad t \in J;$
- 3°  $u \in L^2(J; \dot{H}^1(0, 1)).$

**PROOF.** 1°  $\Rightarrow$  2°: By (2.4), (2.1) we have  $\frac{1}{2} |u(t)|_0^2 \leq E(t)$  for sufficiently large  $\lambda > 0$  (but fixed). The implications 1°  $\Rightarrow$  3° and 3°  $\Rightarrow$  1° are easy consequences of (2.5), (2.6). Finally, 2°  $\Rightarrow$  1° follows from (2.4) and (2.1). ■

**2.7 DEFINITION.** A  $v$ -bounded solution of (1.1)-(1.5) is called  $(u, v)$ -bounded if one of the conditions 1°-3° from Lemma 2.6 holds.

2.8 LEMMA. A couple  $(u, v)$  is a stationary  $v$ -bounded solution of (1.1)-(1.5) if and only if  $u(t) \equiv 0$  and  $v(t) \equiv v$ , where  $w = Iv$  is the (unique) solution of the problem

$$(2.7) \quad w \in W^{2,\infty}([0, 1]),$$

$$(2.8) \quad p(w')' - f(w) = 0,$$

$$(2.9) \quad w(0) = 0, \quad w(1) = 1,$$

$$(2.10) \quad 0 < \alpha \leq w'(x) \leq \beta < \infty$$

with some constants  $\alpha, \beta$ .

PROOF. By (2.6), for the stationary  $v$ -bounded solution we have  $u \equiv 0$ ,  $v \in H^1(0, 1)$ ,  $0 < \alpha \leq v(x) \leq \beta < \infty$ . Hence

$$w \equiv Iv \in H^2(0, 1), \quad w(0) = 0, \quad w(1) = 1$$

and (2.10) holds. Equation (1.1) can be interpreted in  $L^2(0, 1)$  since  $f \in L^\infty(0, 1)$ . By the assumptions (i) and (ii), (2.7) easily follows from (2.8) and (2.10). The reverse implication is trivial. ■

Define as in [1] (see also [7])

$$(2.11) \quad \pi(\xi) = - \int_1^\xi s^{-3} p' \left( \frac{1}{s} \right) ds, \quad \xi > 0,$$

$$-\infty \leq a \equiv \lim_{\xi \rightarrow 0^+} \pi(\xi) < 0, \quad 0 < b \equiv \lim_{\xi \rightarrow \infty} \pi(\xi) \leq \infty,$$

$$(2.12) \quad \Phi = \pi^{-1}, \quad m_0 = \min_{y \in [0, 1]} F, \quad M_0 = \max_{y \in [0, 1]} F.$$

2.9 LEMMA ([1], Theorem 5.3). The problem (2.7)-(2.10) has a (unique) solution if and only if

$$(2.13) \quad a - m_0 < b - M_0,$$

and

$$(2.14) \quad \int_0^1 \Phi(a - m_0 + F(y)) dy < 1 < \int_0^1 \Phi(b - M_0 + F(y)) dy.$$

**3. – Main results.**

In this section we shall prove some properties of  $v$ -bounded and  $(u, v)$ -bounded solutions. To this purpose we need some auxiliary results.

3.1 LEMMA. Let  $f \in C^0([0, 1])$  and  $\{v_n\}_{n=1}^\infty \subset H^0(0, 1)$  be such that  $0 < \alpha \leq v_n \leq \beta < \infty$ ,  $\int_0^1 v_n(x) dx = 1$  and

$$(3.1) \quad \left| \int_0^1 [p(v_n)\psi' + f(Iv_n)\psi] dx \right| \leq \delta_n |\psi'|_0$$

for  $n = 1, 2, \dots$ ,  $\psi \in C_0^\infty(0, 1)$  and some  $\delta_n \downarrow 0$ . Then there exists a solution  $w$  of (2.7)-(2.10) and  $v_n \rightarrow w'$  in  $H^0(0, 1)$ .

PROOF. From (3.1) we have

$$\left| \int_0^1 [p(v_n) - If(Iv_n)]\psi' dx \right| \leq \delta_n |\psi'|_0,$$

so that

$$\left| \int_0^1 g_n \phi dx \right| \leq \delta_n |\phi|_0$$

for  $\phi \in H^0(0, 1)$ ,  $M(\phi) \equiv \int_0^1 \phi dx = 0$ , where we have denoted  $g_n = p(v_n) - If(Iv_n)$ . If  $h \in H^0(0, 1)$  is arbitrary then

$$\begin{aligned} \left| \int_0^1 [g_n - M(g_n)]h dx \right| &= \left| \int_0^1 [g_n - M(g_n)][h - M(h)] dx \right| = \\ &= \left| \int_0^1 g_n[h - M(h)] dx \right| \leq \delta_n |h - M(h)|_0 \leq \delta_n |h|_0. \end{aligned}$$

This implies that  $g_n - M(g_n) \rightarrow 0$  in  $H^0(0, 1)$ . As  $\{M(g_n)\}_{n=1}^\infty$  is a

bounded sequence, we can extract a subsequence of  $g_n$  (denoted again by  $g_n$ ) such that  $M(g_n)$  converge to some constant  $k$ . Hence we have  $p(v_n) - If(Iv_n) \rightarrow k$  in  $H^0(0, 1)$ . Since  $If(Iv_n)$  is compact in  $H^0(0, 1)$ , there exists  $z \in H^0(0, 1)$  such that  $p(v_n) \rightarrow z$  in  $H^0(0, 1)$ , and  $p(\beta) < z < p(\alpha)$ . Put  $v = p^{-1}(z)$ . Then

$$|p(v_n) - p(v)| = \int_0^1 [-p'(\theta v_n + (1 - \theta)v)] d\theta |v_n - v| \geq \min_{s \in [\alpha, \beta]} [-p'(s)] |v_n - v|,$$

from where we get  $v_n \rightarrow v$  in  $H^0(0, 1)$ . Thus  $p(v) - If(Iv) = k$ . Differentiating the last equality for  $w = Iv$  we get the equation (2.8). The conditions (2.7), (2.9), (2.10) are now easy to verify. Since by Lemma 2.9 the solution of the problem (2.7)-(2.10) is unique we find that not only some subsequence of  $\{v_n\}_{n=1}^\infty$  but all the sequence  $v_n$  converges to  $v$  in  $H^0(0, 1)$ . ■

**3.2 LEMMA.** Let  $V, H$  be Hilbert spaces  $V \hookrightarrow H \hookrightarrow V'$ ,  $V$  dense in  $H$ ,  $|u|_H \leq c|u|_V$  for  $u \in V$  with some constant  $c$ . Then for any  $a, b \in \mathbb{R}$ ,  $a < b$  we have

$$\mathcal{H} \equiv H^0(a, b; V) \cap H^1(a, b; V') \hookrightarrow C([a, b]; H)$$

and

$$(3.2) \quad |u(t)|_H^2 \leq \left( \int_a^b |u(s)|_V^2 ds \right)^{\frac{1}{2}} \left[ 2 \left( \int_a^b |u'(s)|_{V'}^2 ds \right)^{\frac{1}{2}} + \frac{c^2}{b-a} \left( \int_a^b |u(s)|_V^2 ds \right)^{\frac{1}{2}} \right]$$

for any  $u \in \mathcal{H}$ .

**PROOF.** The result is well-known. Let us just prove (3.2). For  $u \in \mathcal{H}$  we have

$$\frac{d}{dt} |u(t)|_H^2 = \frac{d}{dt} (u(t), u(t))_H = \frac{d}{dt} \langle u(t), u(t) \rangle_{V, V'} = 2 \langle u(t), u'(t) \rangle_{V, V'}.$$

Since there exists a  $t^* \in [a, b]$  such that

$$|u(t^*)|_H^2 = \frac{1}{b-a} \int_a^b |u(t)|_H^2 dt,$$

we get

$$\begin{aligned}
 |u(t)|_{\mathbb{R}}^2 &= \frac{1}{b-a} \int_a^b |u(t)|_{\mathbb{R}}^2 dt + \int_{t^*}^t 2 \langle u(\tau), u'(\tau) \rangle_{v, v'} d\tau < \\
 &< \frac{1}{b-a} \int_a^b |u(t)|_{\mathbb{R}}^2 dt + 2 \int_a^b |u(t)|_v |u'(t)|_{v'} dt < \\
 &< \frac{c^2}{b-a} \int_a^b |u(t)|_v^2 dt + 2 \left( \int_a^b |u(t)|_v^2 dt \right)^{\frac{1}{2}} \left( \int_a^b |u'(t)|_{v'}^2 dt \right)^{\frac{1}{2}} = \\
 &= \left( \int_a^b |u(t)|_v^2 dt \right)^{\frac{1}{2}} \left[ 2 \left( \int_a^b |u'(t)|_{v'}^2 dt \right)^{\frac{1}{2}} + \frac{c^2}{b-a} \left( \int_a^b |u(t)|_v dt \right)^{\frac{1}{2}} \right]. \quad \blacksquare
 \end{aligned}$$

**3.3 REMARK.** If  $H = H^0(0, 1)$ ,  $V = \dot{H}^1(0, 1)$  with

$$|u|_v^2 = \int_0^1 u_x^2 dx$$

then the best  $c$  is  $c = \pi^{-1}$ .

The following theorem has been proved in [7], Remark 3.19 for smoother solutions by the method of apriori estimates.

**3.4 THEOREM.** Let  $(u, v)$  be a  $v$ -bounded solution of (1.1)-(1.5) on  $(a, \infty)$ ,  $(a \in \mathbb{R})$ ,  $p \in C^1((0, \infty))$ ,  $f \in C([0, 1])$ . Then  $u(t) \rightarrow 0$  in  $H^0(0, 1)$  as  $t \rightarrow \infty$ , there exists a solution  $w$  of the problem (2.7)-(2.10) and  $v(t) \rightarrow w'$  in  $H^0(0, 1)$  as  $t \rightarrow \infty$ .

Since the proof of Theorem 3.4 is quite analogous to the proof of the subsequent Theorem 3.5 we do not present it in this place.

**3.5 THEOREM.** Let  $(u, v)$  be a  $(u, v)$ -bounded solution of (1.1)-(1.5) on  $(-\infty, b)$  ( $b \in \mathbb{R}$ ),  $p \in C^1((0, \infty))$ ,  $f \in C([0, 1])$ . Then  $u(t) \rightarrow 0$  in  $H^0(0, 1)$  as  $t \rightarrow -\infty$ , there exists a solution  $w$  of the problem (2.7)-(2.10) and  $v(t) \rightarrow w'$  in  $H^0(0, 1)$  as  $t \rightarrow -\infty$ .

**PROOF.** Let  $(u, v)$  be a  $(u, v)$ -bounded solution of (1.1)-(1.5) on  $(-\infty, b)$ . Then by (2.5) we have  $u \in L^2((-\infty, b); \dot{H}^1(0, 1))$  and if we

put  $\sigma(t) = \left( \int_{t-1}^t |u(s)|_1^2 ds \right)^{\frac{1}{2}}$  then

$$(3.3) \quad \lim_{t \rightarrow -\infty} \sigma(t) = 0.$$

From the equation (1.1) we find  $(|\cdot|_{V'} \leq |\cdot|_H)$ .

$$|u_t(t)|_{-1} \leq |p(v(t))|_0 + \frac{\mu}{\alpha} |u(t)|_1 + |f(Iv(t))|_0.$$

Hence clearly

$$(3.4) \quad \int_{t-1}^t |u_t(s)|_{-1}^2 ds \leq \gamma < \infty.$$

Lemma 3.2 and (3.4) yield

$$(3.5) \quad |u(t)|_0^2 \leq \sigma(t)(2\gamma^{\frac{1}{2}} + \pi^{-2}\sigma(t)),$$

from where by the help of (3.3) we get  $u(t) \rightarrow 0$  in  $H^0(0, 1)$  as  $t \rightarrow -\infty$ .

To prove the other part of the assertion we shall make use of Lemma 3.1. Obviously, it suffices to show that there exists a non-negative function  $\delta(t)$  with  $\lim_{t \rightarrow -\infty} \delta(t) = 0$  such that

$$(3.6) \quad \left| \int_0^1 [p(v(t))\psi' + f(Iv(t))\psi] \right| \leq \delta(t)|\psi|_0$$

for all  $\psi \in C_0^\infty(0, 1)$ .

Let  $\varphi \in C_0^\infty(-1, 0)$  be a fixed non-negative function with  $\int_{-1}^0 \varphi dt = 1$ .

By (1.1), integration by parts, (3.5), the Schwartz and the Poincaré inequality, (denoting  $\langle \cdot, \cdot \rangle$  pairing between  $\dot{H}^1(0, 1)$  and  $H^{-1}(0, 1)$ ), for any  $\psi \in C_0^\infty(0, 1)$  we have

$$\begin{aligned} & \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [p(v(\tau))\psi' + f(Iv(\tau))\psi] dx d\tau \right| = \\ & = \left| \int_{t-1}^t \varphi(\tau-t) \langle u_t(\tau) - \mu \left( \frac{u_x(\tau)}{v(\tau)} \right)_x, \psi \rangle d\tau \right| = \end{aligned}$$

$$\begin{aligned}
 &= \left| \int_{t-1}^t \left[ -\varphi'(\tau-t) \int_0^1 u(\tau) \psi \, dx + \mu \varphi(\tau-t) \int_0^1 \frac{u_x(\tau)}{v(\tau)} \psi' \, dx \right] d\tau \right| < \\
 &= \left( \int_{-1}^0 \varphi'(s)^2 \, ds \right)^{\frac{1}{2}} \left[ \int_{t-1}^t \left( \int_0^1 u(\tau) \psi \, dx \right)^2 d\tau \right]^{\frac{1}{2}} + \\
 &+ \mu \left( \int_{-1}^0 \varphi(\tau)^2 \, d\tau \right)^{\frac{1}{2}} \left[ \int_{t-1}^t \left( \int_0^1 \frac{u_x(\tau)}{v(\tau)} \psi' \, dx \right)^2 d\tau \right]^{\frac{1}{2}} < \\
 &< \left( \int_{-1}^0 \varphi'(s)^2 \, ds \right)^{\frac{1}{2}} \left[ \left( \int_{t-1}^t \int_0^1 u(\tau)^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_0^1 \psi^2 \, dx \right)^{\frac{1}{2}} + \right. \\
 &+ \left. \frac{\mu}{\alpha} \left( \int_{t-1}^t \int_0^1 u_x(\tau)^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_0^1 (\psi')^2 \, dx \right)^{\frac{1}{2}} \right] < \\
 &\qquad \qquad \qquad < (1 + \mu\alpha^{-1}) |\varphi'|_0 \cdot \sigma(t) \cdot |\psi'|_0 \equiv \eta(t) |\psi'|_0 .
 \end{aligned}$$

Further, by easy calculations with help of (1.2), we find

$$\begin{aligned}
 (3.7) \quad & \left| \int_0^1 [p(v(t)) \psi' + f(Iv(t)) \psi] \, dx \right| = \\
 &= \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [p(v(t)) \psi' + f(Iv(t)) \psi] \, dx \, d\tau \right| < \\
 &< \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [p(v(\tau)) \psi' + f(Iv(\tau)) \psi] \, dx \, d\tau \right| + \\
 &+ \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [p(v(t)) - p(v(\tau))] \psi' \, dx \, d\tau \right| + \\
 &+ \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [f(Iv(t)) - f(Iv(\tau))] \psi \, dx \, d\tau \right| < \eta(t) |\psi'|_0 + \\
 &+ \left| \int_{t-1}^t \varphi(\tau-t) \int_{\tau}^1 p'(v(s)) v_i(s) \psi' \, dx \, ds \, d\tau \right| +
 \end{aligned}$$

$$\begin{aligned}
& + \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [f(Iv(t)) - f(Iv(\tau))] \psi \, dx \, d\tau \right| \leq \\
& \leq \left[ \eta(t) + \sup_{y \in [\alpha, \beta]} |p'(y)| \sigma(t) \right] |\psi'|_0 + \\
& \quad + \left| \int_{t-1}^t \varphi(\tau-t) \int_0^1 [f(Iv(t)) - f(Iv(\tau))] \psi \, dx \, d\tau \right|.
\end{aligned}$$

Since for  $t-1 \leq \tau \leq t$ , it is

$$\begin{aligned}
|Iv(t)(x) - Iv(\tau)(x)| & \leq \int_{\tau}^t \int_0^1 |v_i(s)| \, dx \, ds \leq \\
& \leq \int_{\tau}^t \left( \int_0^1 |v_i(s)|^2 \, dx \right)^{\frac{1}{2}} ds \leq \left( \int_{\tau}^t |v_i(s)|_0^2 \, ds \right)^{\frac{1}{2}} (t-\tau)^{\frac{1}{2}} \leq \sigma(t),
\end{aligned}$$

we have

$$|f(Iv(t)) - f(Iv(\tau))|_{L^\infty(0,1)} \leq \omega(\sigma(t)),$$

where  $\omega$  is a modulus of continuity of  $f$  on  $[0, 1]$ . Thus

$$\begin{aligned}
(3.8) \quad \left| \int_0^1 [f(Iv(t)) - f(Iv(\tau))] \psi \, dx \right| & \leq \omega(\sigma(t)) \int_0^1 |\psi| \, dx \leq \\
& \leq \omega(\sigma(t)) |\psi|_0 \leq \omega(\sigma(t)) |\psi'|_0.
\end{aligned}$$

By (3.7), (3.8) we get (3.6) with

$$\delta(t) = \eta(t) + \sup_{y \in [\alpha, \beta]} |p'(y)| \sigma(t) + \omega(\sigma(t)). \quad \blacksquare$$

**3.6 THEOREM.** Let  $p \in C^1((0, \infty))$ ,  $f \in C([0, 1])$ . Then any  $(u, v)$ -bounded solution of (1.1)-(1.5) on  $R$  is stationary.

**PROOF.** Let  $(u, v)$  be such a solution. Then by Theorems 3.4, 3.5  $u(t) \rightarrow 0$ ,  $v(t) \rightarrow w'$  in  $L^2(0, 1)$  as  $|t| \rightarrow \infty$ . As  $w$  is determined uniquely and (2.10) holds, by the Lebesgue theorem we have

$$E(\infty) \equiv \lim_{|t| \rightarrow \infty} E(t) = \int_0^1 [P(w') + \lambda - F(w)] \, dx.$$

Since  $E$  is nonincreasing by Lemma 2.5, it is constant. So by (2.6),  $u \equiv 0$  and (1.1) together with Lemma 2.8 yield  $v = w'$ . ■

An immediate consequence of Theorem 3.6 and Lemma 2.9 is the following

**3.7 THEOREM.** Under the assumption of Theorem 3.6 a  $(u, v)$ -bounded solution on  $R$  exists if and only if the conditions (2.13), (2.14) are satisfied.

**3.8 REMARK.** If the assumptions of Theorem 3.6 are satisfied,  $(u, v)$  is a solution e.g. on  $Q_\infty = (0, 1) \times [0, \infty)$ ,

$$(3.9) \quad \liminf_{s \rightarrow 0^+} [-sp'(s)] > 0$$

(as, for instance, in the case  $p(s) = As^{-\gamma}$ ,  $A > 0$ ,  $\gamma > 0$ ) and

$$(3.10) \quad v(x, t) \leq \beta < \infty \quad \text{in } Q_\infty$$

then there exists an  $\alpha > 0$  such that

$$(3.11) \quad v(x, t) \geq \alpha \quad \text{in } Q_\infty.$$

Thus, in this case,  $v$ -boundedness follows just from boundedness of  $v$  from above. This leads to an obvious version of Theorem 3.4 which the reader can formulate himself easily. The above implication is proved in [2], Theorem 7.5. Before this fact came to our attention we found a proof which, we believe, still has a sense to present in this context. We multiply (1.1) by  $(\log v)_x$  and integrate with respect to  $x$  over  $(0, 1)$ . After standard calculations we get

$$(3.12) \quad \frac{\mu}{2} \frac{d}{dt} \int_0^1 [(\log v)_x]^2 dx = \frac{d}{dt} \int_0^1 u(\log v)_x dx + \int_0^1 \frac{u_x^2}{v} dx + \int_0^1 \frac{p'(v)}{v} v_x^2 dx - \int_0^1 f(Iv)(\log v)_x dx.$$

Put  $\varphi(t) = (\mu/2) \int_0^1 [(\log v)_x - u/\mu]^2 dx$ . From (2.4), (2.6) we get

$$(2.13) \quad \frac{1}{2} \varphi(t) - c_1 \leq \frac{\mu}{2} \int_0^1 (\log v)_x^2 dx \leq 2\varphi(t) + c_1$$

with some constant  $c_1$ . The identity (3.12) yields

$$\begin{aligned} \varphi'(t) + \inf[-vp'(v)] \int_0^1 (\log v)_x^2 dx &\leq \\ &\leq \int_0^1 \frac{u_x^2}{v} dx + \frac{1}{2\mu} \frac{d}{dt} \int_0^1 u^2 dx + \varepsilon \int_0^1 (\log v)_x^2 dx + c(\varepsilon) \sup |f|^2. \end{aligned}$$

Since  $\inf[-vp'(v)] > 0$  in virtue of (3.9), (3.10), choosing  $\varepsilon > 0$  sufficiently small and making use of (3.13) we find

$$(3.14) \quad \varphi'(t) + c_2 \varphi(t) \leq c_3 + \int_0^1 \frac{u_x^2}{v} dx + \frac{1}{2\mu} \frac{d}{dt} \int_0^1 u^2 dx$$

with some positive constants  $c_2, c_3$ . Multiplying (3.14) by  $e^{c_2 t}$ , integrating over  $(0, t)$ , multiplying the result by  $e^{-c_2 t}$  and integrating by parts the last term we get

$$\begin{aligned} (3.15) \quad \varphi(t) &\leq \varphi(0) e^{-c_2 t} + \frac{c_3}{c_2} (e^{c_2 t} - 1) e^{-c_2 t} + e^{-c_2 t} \int_0^t e^{c_2 s} \int_0^1 \frac{u_x(x, s)^2}{v(x, s)} dx ds + \\ &+ \frac{1}{2\mu} e^{-c_2 t} \int_0^t e^{c_2 s} \frac{d}{ds} \int_0^1 u(s)^2 dx ds \leq \varphi(0) + \frac{c_3}{c_2} + \int_0^t \int_0^1 \frac{u_x^2}{v} dx ds + \\ &+ \frac{1}{2\mu} \int_0^1 u(t)^2 dx - \frac{e^{-c_2 t}}{2\mu} \int_0^1 u(0)^2 dx - \frac{c_2 e^{-c_2 t}}{2\mu} \int_0^t e^{c_2 s} \int_0^1 u(s)^2 dx ds \leq c_4 < \infty. \end{aligned}$$

In the last inequality we have used (2.4), (2.6). Now, from (3.13),

(3.15) the estimate

$$(3.16) \quad \int_0^1 (\log v)_x^2 dx \leq c_5 < \infty$$

follows. From (3.16) the estimate (3.11) is obtained in a standard manner (see e.g. [7], proof of Lemma 2.3). The estimate (3.11) for generalized solutions in the sense of Definition 2.1 can be obtained via weak\* limit of smoother solutions.

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