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DIETER HAPPEL

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Representable Equivalences are Represented by Tilting Modules.

GABRIELLA D'ESTE - DIETER HAPPEL (*)

Let K be a field, let A be a finite-dimensional K -algebra and let ${}_A T$ be a faithful and finite-dimensional left A -module. Under these hypotheses, we will show that, if ${}_A T$ induces an equivalence satisfying the requirements of Menini-Orsatti's Representation Theorem [5], then ${}_A T$ is a tilting module.

The proof of this fact makes use, on the one hand, of the results on torsion theories induced by tilting modules obtained by Hoshino [4], Assem [1] and Smalø [8], and, on the other hand, of the new results obtained by Colpi [2].

In this way, we solve an open problem of [3].

Before we do this, we recall some definitions.

Let ${}_A T$ be a finite dimensional A -module. Then ${}_A T$ is called a tilting module, if the following conditions are satisfied:

- (i) The projective dimension of ${}_A T$ is less than or equal to 1.
- (ii) $\text{Ext}_A^1({}_A T, {}_A T) = 0$.
- (iii) There is an exact sequence of the form

$$0 \rightarrow {}_A A \rightarrow T' \rightarrow T'' \rightarrow 0$$

with T' and T'' in $\text{add } T$, where $\text{add } T$ denotes the additive category whose objects are direct sums of direct summands of ${}_A T$.

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Finally, let R and S be two rings, let $\mathfrak{G}$ be a full subcategory of left R -modules closed under direct sums and factor modules, let $\mathfrak{D}$ be a full subcategory of left S -modules containing ${}_S S$ and closed under submodules, and let

$$\mathfrak{G} \overset{F}{\underset{G}{\rightleftarrows}} \mathfrak{D}$$

be an equivalence with F and G additive functors. Then Menini-Orsatti's theorem ([5] Theorem 3.1) asserts that there is a module ${}_R M$, with endomorphism ring S , such that $F \approx \text{Hom}_R({}_R M, -)$ and $\mathfrak{G}$ is the category of all R -modules generated by ${}_R M$, while $G \approx M_S \otimes -$ and $\mathfrak{D}$ is the category of all S -modules cogenerated by $\text{Hom}_R({}_R M, {}_R Q)$, where ${}_R Q$ is an injective cogenerator of the category of all R -modules.

In the following, according to [2] and [3], such a module ${}_R M$ is called a $*$ -module. Using this terminology, we deduce from ([5] Theorem 4.3) that any tilting module is a $*$ -module.

The next statement shows that the relationship between $*$ -modules and tilting modules is as strong as might be expected. In the proof of the next theorem all modules will be finite-dimensional.

THEOREM 1. Let A be a finite-dimensional K -algebra, let ${}_A T$ be a finite-dimensional faithful $*$ -module. Then ${}_A T$ is a tilting module.

PROOF. Let $\mathfrak{G}$ be the category of all modules generated by ${}_A T$. We claim that

$$\mathfrak{G} = \{ {}_A X \mid \text{Ext}_A^1({}_A T, {}_A X) = 0 \}.$$

To see this, take any module ${}_A X$ and let ${}_A I$ be an injective module such that ${}_A X \leqslant {}_A I$. Since ${}_A T$ is faithful, $\mathfrak{G}$ contains any injective module; hence ${}_A I \in \mathfrak{G}$. Moreover, applying $\text{Hom}_A({}_A T, -)$ to the short exact sequence

$$0 \rightarrow {}_A X \rightarrow {}_A I \xrightarrow{\pi} {}_A I / {}_A X \rightarrow 0,$$

we get the exact sequence

$$(*) \quad \text{Hom}_A({}_A T, {}_A I) \xrightarrow{\text{Hom}_A({}_A T, \pi)} \text{Hom}_A({}_A T, {}_A I / {}_A X) \rightarrow \text{Ext}_A^1({}_A T, {}_A X) \rightarrow 0.$$

Suppose first that ${}_A X \in \mathfrak{G}$. Then, by ([2] Corollary 4.2), $\text{Hom}_A({}_A T, \pi)$ is surjective. Hence, by (*), we have $\text{Ext}_A^1({}_A T, {}_A X) = 0$.

Assume now that $\text{Ext}_A^1({}_A T, {}_A X) = 0$. Then we deduce from $(*)$ that $\text{Hom}_A({}_A T, \pi)$ is surjective. Consequently, by ([2] Proposition 4.3), ${}_A X \in \mathcal{T}$ and so $\mathcal{T}$ satisfies our claim. It immediately follows that $\mathcal{T}$ is closed under extensions. Now let ${}_A U$ be the direct sum of a complete set of representatives of the isomorphism classes of the indecomposable modules $U_i \in \mathcal{T}$ which are Ext-projective in $\mathcal{T}$, (see [1] and [8]), that is with the property that $\text{Ext}_A^1(U_i, {}_A X) = 0$ for any ${}_A X \in \mathcal{T}$. Then we know from ([8] Theorem) that ${}_A U$ is a tilting module. Hence, to prove the theorem, it suffices to check that $\text{add } {}_A T = \text{add } {}_A U$. To this end, we first note that our hypotheses on ${}_A T$ and the above characterization of $\mathcal{T}$ imply that $\text{add } {}_A T \subseteq \text{add } {}_A U$. Now let ${}_A V$ be an indecomposable summand of ${}_A U$. Then, by ([2] Theorem 4.1), there is an exact sequence in $\mathcal{T}$ of the form

$$0 \rightarrow {}_A W \rightarrow \bigoplus {}_A T \rightarrow {}_A V \rightarrow 0.$$

Since ${}_A V$ is Ext-projective in $\mathcal{T}$, it follows that ${}_A V \oplus {}_A W \cong \bigoplus {}_A T$. Hence using the Krull-Schmidt theorem we infer that $\text{add } {}_A U = \text{add } {}_A T$. Therefore ${}_A T$ is a tilting module, and the proof is complete.

As an immediate consequence of Theorem 1 and ([3] Lemma 1), we obtain the following corollary.

COROLLARY 2. Let A be a finite-dimensional K -algebra, let ${}_A M$ be a finite-dimensional module and let $\bar{A} = A/\text{ann } {}_A M$. Then ${}_A M$ is a $*$ -module if and only if ${}_{\bar{A}} M$ is a tilting module.

The next remark points out another application of Theorem 1.

REMARK 3. Let A be a finite-dimensional K -algebra and let ${}_A T$ be an ω -tilting module in the sense of [5]. Then ${}_A T$ is a tilting module. In fact, the definition of an ω -tilting module implies that ${}_A T$ is faithful and finite-dimensional, while ([5] Theorem 4.3) guarantees that ${}_A T$ is a $*$ -module.

The following observation gives a partial answer to the question whether or not $*$ -modules over finite-dimensional algebras are actually finitely generated.

REMARK 4. Let A be a finite-dimensional K -algebra, and let ${}_A M$ be a $*$ -module. If A is representation finite [6], then ${}_A M$ is finitely

generated. Indeed, our hypothesis on A and ([7] Corollary 4.4) guarantee that ${}_A M$ is of the form $\bigoplus_i N_i$, where the N_i 's are indecomposable modules of finite dimension over K . On the other hand, by ([2] Theorem 4.1), ${}_A M$ is self-small, i.e. for any morphism $f: {}_A M \rightarrow \bigoplus_{j \in J} M_j$, with $M_j \cong {}_A M$ for any $j \in J$, there exists a finite subset $F \subseteq J$ such that $f({}_A M) \subseteq \bigoplus_{j \in F} M_j$. Putting things together, we conclude that ${}_A M = \bigoplus_i N_i$ is the direct sum of finitely many N_i 's. Therefore ${}_A M$ is finitely generated, as claimed.

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