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On *-Modules Over Valuation Rings.

PAOLO ZANARDO (*)

The problem of investigating *-modules over valuation rings was proposed to the author by C. Menini. We recall the definition of *-module, given by D’Este in [3]. Let \( R \) be a ring, \( _RM \) a left \( R \)-module and \( _RE \) an injective cogenerator of the category of all \( R \)-modules; let \( S = \text{End}_R(M) \) and \( H = \text{Hom}_R(_RM, _RE) \) and denote by \( \text{Gen}(R^M) \) the category of all left \( R \)-modules generated by \( _RM \) and by \( \text{Cog}(sH) \) the category of all left \( S \)-modules cogenerated by \( H \). In this situation, \( _RM \) is said to be a *-module if there exists an equivalence of categories

\[
\text{Gen}(R^M) \cong \text{Cog}(sH)
\]

such that the functor \( F \) is naturally isomorphic to \( \text{Hom}_R(_RM, -) \) and the functor \( G \) is naturally isomorphic to \( M_S \otimes - \) (we shall write \( F \approx \text{Hom}_R(_RM, -), G \approx M_S \otimes - \)).

The main motivation for the study of *-modules is the following result by Menini and Orsatti ([8], Theorem 3.1): let \( R, S \) be rings; if \( S \) is a full subcategory of \( R\text{-Mod} \) closed under direct sums and factor modules, \( D \) is a full subcategory of \( S\text{-Mod} \) containing \( sS \) and closed under submodules, and \( S \cong D \) is any equivalence with \( F \) and \( G \) additive functors, then there exists a module \( _RM \) such that: \( S = \text{End}_R(_RM), S = \text{Gen}(R^M), D = \text{Cog}(sH) \) (where \( sH \) is as above), \( F \approx \text{Hom}_R(_RM, -) \) and \( G \approx M_S \otimes - \).

Recent results on *-modules have been obtained by D’Este [3], D’Este and Happel [4], Colpi [1], Colpi and Menini [2].

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In the present paper we characterize finitely generated *-modules over a valuation ring \( R \). Using a theorem by Colpi ([1], Prop. 4.3) and some results in [9] (see also [5], Ch. IX), we prove that a finitely generated module \( X \) over a valuation ring \( R \) is a *-module if and only if \( X \cong (R/A)^n \), for suitable \( n > 0 \) and \( A \) ideal of \( R \) (Theorem 3). Note that a module of the form \( (R/A)^n \) is a *-module for any ring \( R \), as a consequence of the above mentioned result by Colpi. Hence our Theorems 3 shows that the class of finitely generated *-modules over a valuation ring is, in a certain sense, as small as possible.

Note that, at present, there are no examples of rings which admit *-modules not finitely generated; Colpi and Menini in [2] proved that *-modules over artinian rings or noetherian domains with Krull dimension one are necessarily finitely generated. The author feels that the same is true for *-modules over valuation rings. Our final Remark 4 gives a contribution in this direction.

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1. - In the sequel, \( R \) will always denote a valuation ring, i.e. a commutative ring, not necessarily a domain, whose ideals are linearly ordered by inclusion; the maximal ideal of \( R \) is denoted by \( P \). For general terminology and results on modules over valuation rings we refer to the book by Fuchs and Salce [5]; the results we need on finitely generated modules can be found in [9] or in Ch. IX of [5].

In the proof of Theorem 2 we shall need the following facts (see [9] or [5], Ch. IX): let \( X \) be a finitely generated \( R \)-module; then there exists a submodule \( B \) of \( X \) such that:

i) \( B \) is a direct sum of cyclic submodules;

ii) \( B \) is pure in \( X \);

iii) \( B \) is essential in \( X \);

such a \( B \) is said to be basic in \( X \); the basic submodules of \( X \) are all isomorphic. Moreover, given a basic submodule \( B \) of \( X \), there exists a minimal set of generators \( \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\} \) of \( X \) such that:

a) \( B = \langle x_1, \ldots, x_k \rangle = \bigoplus_{i=1}^{k} \langle x_i \rangle \);
b) if $A_j = \text{Ann}(x_i + \langle x_1, \ldots, x_{j-1} \rangle)$ for all $j > k$, we have $A_{k+1} \leq \cdots \leq A_n$.

c) for all $r \in A_{k+1}$ we have the relation

$$(1) \quad rx_{k+1} = r \sum_{i=1}^{k} a_i^r x_i, \quad \text{for suitable units } a_i^r \in R.$$ 

The construction of $x$ needs some explanation: we start with $B = \bigoplus_{i=1}^{k} \langle x_i \rangle$ basic in $X$ and consider $X/B$; if $\{x_{k+1} + B, \ldots, x_n + B\}$ is a minimal set of generators of $X/B$, from the purity of $B$ it follows that $x = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ is a minimal set of generators of $X$; in view of Lemma 1.1 of [9], we can permute the indexes $k + 1, \ldots, n$ to obtain property $b)$. Since $B$ is pure in $X$, certainly, for all $r \in A_{k+1}$, the relation $(1)$ holds for suitable elements $a_i^r \in R$, not necessarily units. However, if $r \in \text{Ann} x_i$ for some $i < k$, obviously we can replace $a_i^r$ with $1$; moreover, if there exist $i < k$ and $s \in A_{k+1}\setminus \text{Ann} x_i$ such that $a_i^s \in P$, then for all $r \in A_{k+1}\setminus \text{Ann} x_i$ we have $a_i^r \in P$: in fact, if $r$ divides $s$, from $(1)$ we get $s(a_i^r - a_i^s)x_i = 0$, hence $a_i^r \in P$ implies $a_i^s \in P$; analogously, if $s$ divides $r$, $r(a_i^r - a_i^s)x_i = 0$ implies $a_i^r \in P$.

Let now $F = \{i < k: a_i^r \in P \text{ for all } r \in A_{k+1}\setminus \text{Ann} x_i\}$; if we replace $x_{k+1}$ with $x'_{k+1} = x_{k+1} + \sum_{i \in F} x_i$, we obtain that

$$\text{Ann}\left(x_{k+1}' + B\right) = \text{Ann} (x_{k+1} + B) = A_{k+1},$$ 

$$x' = \{x_1, \ldots, x_{k}, x_{k+1}', \ldots, x_n\}$$

is a minimal set of generators of $X$, and $(1)$ becomes

$$(1') \quad rx_{k+1}' = r \sum_{i=1}^{k} b_i^r x_i \quad \text{for all } r \in A_{k+1},$$

where $b_i^r = a_i^r$ if $i \notin F$ and $b_i^r = 1 + a_i^r$ if $i \in F$, so that $b_i^r$ is a unit for all $i < k$ and for all $r \in A_{k+1}$. We conclude that there exists a minimal set of generators $x$ of $X$ which satisfies properties $a), b), c)$, as desired.

Let us now recall Colpi's result (Prop. 4.3 of [1]).
**Theorem 1** (Colpi). Let $R$ be a ring, $_R M$ a left $R$-module. Then $_R M$ is a *-module if and only if the following conditions are satisfied:

i) $M$ is self-small;

ii) for each exact sequence

$$0 \to L \to N \to N/L \to 0$$

where $N$ is an object of $\text{Gen}(R M)$, the sequence

$$0 \to \text{Hom}_R (M, L) \to \text{Hom}_R (M, N) \to \text{Hom}_R (M, N/L) \to 0$$

is exact if and only if $L \in \text{Gen}(R M)$. \hfill ///

We can now prove our main result.

**Theorem 2.** Let $R$ be a valuation ring, let $X$ be a finitely generated $R$-module and let $\pi: X \to X/XP$ be the canonical homomorphism. If the map $\varphi: \text{End } X \to \text{Hom}_R (X, X/XP)$, $\varphi: f \mapsto \pi \circ f$ is surjective, then $X \cong (R/A)^n$ for suitable $n \geq 0$ and $A$ ideal of $R$.

**Proof.** In the following we assume $X \not\cong (R/A)^0 = \{0\}$, otherwise all is trivial. First of all, let us prove that $X$ is a direct sum of cyclic submodules. Let $B$ be a basic submodule of $X$; it is enough to verify that $B = X$. By contradiction, suppose that $B \subsetneq X$; let $x = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ be a minimal set of generators of $X$ which satisfies conditions a), b), c) above. Note that, since $B \subsetneq X$, we have $k \leq n$, hence condition c) and the relation (1) are not trivially satisfied. For all $j \leq n$, let $\bar{x}_j = x_j + PX$; we have $X/XP = \bigoplus_{j=1}^n \langle \bar{x}_j \rangle$. Let us now consider the homomorphism $g: X \to X/XP$ defined extending by linearity the assignments

$$g: x_j \mapsto 0 \quad \text{if } j \neq k + 1; \quad g: x_{k+1} \mapsto \bar{x}_{k+1}.$$ 

By hypothesis, there exists $f \in \text{End } X$ such that $g = \pi \circ f$. Hence, for $j \neq k + 1$, we will have

$$f(x_i) = p \sum_{h=1}^n a_{ih} x_h, \quad \text{with } p \in P, \quad a_{ih} \in R$$
(3) \[ f(x_{k+1}) = x_{k+1} + q \sum_{h=1}^{n} b_h x_h , \quad \text{with } q \in P , \ b_h \in R . \]

From (1), (2), (3), and the linearity of \( f \), it follows, for all \( r \in A_{k+1} \)

(4) \[ r(x_{k+1} + q \sum_{h=1}^{n} b_h x_h) = rp \sum_{i=1}^{k} a_i' \left( \sum_{h=1}^{n} a_{h,i} x_h \right) . \]

Since \( A_{k+1} \subset A_i \) for all \( t > k + 1 \), and \( B \) is pure, we deduce that, for all \( r \in A_{k+1} \)

(5) \[ rq \sum_{h=1}^{n} b_h x_h \in rqB \quad \text{and} \quad rp \sum_{i=1}^{k} a_i' \left( \sum_{h=1}^{n} a_{h,i} x_h \right) \in rpB . \]

Let \( \bar{p} \in P \) be a common divisor of \( p \) and \( q \); from (4) and (5) we get \( rx_{k+1} \in r\bar{p}B \) for all \( r \in A_{k+1} \), i.e.

(6) \[ rx_{k+1} = r\bar{p} \sum_{i=1}^{k} c_i' x_i , \quad \text{with } c_i' \in R . \]

From (1), (6), and the linear independence of \( x_1, \ldots, x_k \) we obtain

(7) \[ r(a_i' - \bar{p}c_i') x_i = 0 \quad \text{for } i = 1, \ldots, k ; \]

since \( a_i' \) is a unit for all \( i \) and \( r \), we have that \( a_i' - \bar{p}c_i' \) is a unit, too, hence (7) implies \( r \in \text{Ann } x_i \) for all \( r \in A_{k+1} \). But this means that \( rx_{k+1} \in B \) implies \( rx_{k+1} = 0 \), from which \( \langle x_{k+1} \rangle \cap B = 0 \), and \( B \) is not essential, against the definition of basic submodule. We conclude that, necessarily, \( X = B \), as desired. It remains to prove that, if \( A = \text{Ann } X \), then \( X \cong (R/A)^n \). By contradiction, let us suppose that \( X = \bigoplus_{i=1}^{n} \langle x_i \rangle \), where, for a suitable \( j < n \), \( \text{Ann } x_j > A \). Let us assume, without loss of generality, that \( \text{Ann } x_1 = A \). Let \( \eta : X \to X/PX \) be the homomorphism which extends by linearity the assignments

\[ \eta : x_i \mapsto 0 \quad \text{if } i \neq j ; \quad \eta : x_j \mapsto x_1 + PX . \]
If \( \theta \in \text{End} X \) is such that \( \eta = \pi \circ \theta \), then we have
\[
\theta(x_i) = x_1 + p \sum_{i=1}^{n} a_i x_i, \quad \text{with} \quad p \in P, \quad a_i \in \mathcal{R}.
\]

Choose now \( r \in \text{Ann} x_i \setminus A \); from (8) we obtain
\[
0 = \theta(rx_i) = r(1 + pa_x)x_1 + rp \sum_{i=2}^{n} a_i x_i,
\]
from which \( r(1 + pa_x)x_i = 0 \), which is impossible, because \( r \notin A = \text{Ann} x_i \). This concludes the proof. ///

As an easy consequence of the preceding result we obtain the following

**Theorem 3.** Let \( \mathcal{R} \) be a valuation ring. A finitely generated \( \mathcal{R} \)-module \( X \) is a \( * \)-module if and only if for suitable \( n \geq 0 \) and \( A \) ideal of \( \mathcal{R} \).

**Proof.** For any ring \( \mathcal{R} \), modules of the form \( (\mathcal{R}/A)^n \) are \( * \)-modules as a consequence of Theorem 1, observing that \( \text{Gen} ((\mathcal{R}/A)^n) = \mathcal{R}/A - \text{Mod} \), and \( \text{Hom}_{\mathcal{R}} ((\mathcal{R}/A)^n, -) \approx \text{Hom}_{\mathcal{R}/A} ((\mathcal{R}/A)^n, -) \), if \( n \geq 1 \).

Conversely, let us note that \( PX \in \text{Gen} (X) \), as it is easy to verify. Therefore, if \( X \) is a finitely generated \( * \)-module, then, by Theorem 1, \( X \) must satisfy the condition in the hypothesis of Theorem 2, hence \( X \) has the desired form. ///

The problem of finding \( * \)-modules which are not finitely generated remains open. We actually think that a \( * \)-module over a valuation ring must be finitely generated; this opinion is mainly based on the following remark, derived from discussions with L. Salce.

**Remark 4.** The simplest non finitely generated \( \mathcal{R} \)-modules are the uniserial ones, i.e. those \( \mathcal{R} \)-modules whose lattice of submodules is linearly ordered. Fuchs and Salce proved that, if \( U \) is a divisible uniserial module over a valuation domain \( \mathcal{R} \), whose elements have nonzero principal annihilators, then there is an equivalence of categories
\[
\text{Gen} (U) \xrightarrow{\mathcal{F}} \mathcal{C}
\]
where $C$ is the class of complete torsion-free reduced $R$-modules, $F \approx \text{Hom}_R(U,-)$ and $G \approx U \otimes_R -$ (see [6]; this equivalence was inspired by Matlis equivalence in [7]; see also [5], p. 99). Moreover, $U$ is small if and only if it is not countably generated. Nevertheless, for any choice of $R$ we notice that $U$ is not a $\ast$-module. This is clear if $U$ is countably generated (see Theorem 1). If $U$ is not countably generated, then also $Q$, the field of fractions of $R$, is not countably generated as an $R$-module; in this case we get that $C$ is not closed for submodules, hence $C$ cannot be cogenerated by any module. It is worth giving a check of this last fact: let us suppose, by contradiction, that $C$ is closed for submodules, for a convenient $R$, with $Q$ not countably generated as an $R$-module; with these assumptions, $R$ must be complete, and each free $R$-module $F$ is complete, too, in view of Cor. 2.2 of [6]. Let us consider a short exact sequence

$$0 \to K \to F \to Q \to 0$$

with $F$ free; then $F \in C$ implies $K \in C$, hence $K$ is closed in $F$ and $Q \approx F/K$ must be Hausdorff in the natural topology, i.e. $\{0\} = \bigcap_{r \in R^*} rQ = Q$, a contradiction.

REFERENCES


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