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Quartic threefolds containing two skew double lines

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Quartic Threefolds Containing Two Skew Double Lines.

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1. – Introduction.

The problem of rationality for algebraic threefolds is still an open problem in Algebraic Geometry. However the conic bundle theory, developed by Beauville (see [B_1], [B_2] and also [C-M]), gives us a very useful tool to solve this problem in many cases.

Some recent results of Sarkisov and Iskovskih (see [I_1], [I_2] and [Sa]) have improved this technique by giving some answers even when the intermediate Jacobian of the threefold is the Jacobian of a curve. These facts have allowed us to solve the problem of rationality for the Fano threefold of $\mathbf{P}^5$ containing n planes (see [A-B_1] and [A-B_2]).

In this paper we study the rationality of the generic quartic threefold of $\mathbf{P}^4$ containing two skew double lines and containing n planes with all possible configurations. In [C-M] Conte and Murre have proved that a generic quartic threefold of $\mathbf{P}^4$ containing only one double line is not rational, while it is well known that such threefold with two incident double lines is rational. Our work is a natural prosecution of [C-M] and it was suggested by remark (6, 3) of [A-B_2], in which we showed that a generic quartic threefold of $\mathbf{P}^4$ containing two skew double lines, and no planes, is not rational.

Our proofs are based on this idea: there exists a birational morphism (due to Fano, [F]) between $\mathbf{P}^4$ and the quadric hypersurface of $\mathbf{P}^5$,

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identified with the Grassmannian $G(1, 3)$ of lines of $\mathbf{P}^3$. By this morphism some quartic hypersurfaces with two skew double lines correspond to cubic complexes containing two planes, meeting two by two at one point only; these singular varieties have a well known conic bundle structure (see [C], [A-B$_1$] and [A-B$_2$]); the existence of some plane in the quartics changes this structure; by studying these new structures we get our results; they are described in § 4.

We use these conventions: by the word « n-fold » we mean a projective algebraic variety (singular or not) defined on $\mathbf{C}$; by the word « generic » we mean that what we are saying is true in a suitable open Zarisky set.

2. – Fano birational morphism.

We choose $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$ as coordinates in $\mathbf{P}^5$, we fix a smooth quadric hypersurface Q and we choose three planes contained in Q, meeting two by two at one point only; we can always suppose that Q has this equation:

$$Q) \quad x_0x_5 - x_1x_4 + x_2x_3 = 0$$

and that the three planes, P_0, P_1, P_2, have equations:

$$P_0) \quad x_0 = x_2 = x_4 = 0$$

$$P_1) \quad x_3 = x_4 = x_5 = 0$$

$$P_2) \quad x_1 = x_2 = x_5 = 0 \,.$$

Now in $\mathbf{P}^4$ we choose $(z_1 : z_2 : z_3 : z_4 : z_5)$ as coordinates, (this unusual choice will be very useful in the sequel), and we choose three skew lines, not two of them lying in the same hyperplane; we can always suppose that the three lines have equations:

$$L_1) \quad z_3 = z_4 = z_5 = 0$$

$$L_2) \quad z_1 - z_3 = z_2 = z_5 = 0$$

$$L_3) \quad z_1 = z_2 = z_4 = 0 \,.$$

We consider the rational map Φ: $\mathbf{P}^4 \to \mathbf{P}^5$ given by:

$$x_0 = z_4(z_3 - z_1) \qquad x_1 = - z_1 z_5$$

$$x_2 = - z_4 z_5 \qquad x_3 = z_2 z_3$$

$$x_4 = z_2 z_4 \qquad x_5 = z_2 z_5 .$$

Φ is a well known birational morphism between $\mathbf{P}^4$ and Q (see [F]), its inverse is:

$$z_1 = x_1 x_4 \qquad z_2 = - x_4 x_5$$

$$z_3 = x_2 x_3 \qquad z_4 = x_2 x_4$$

$$z_5 = x_2 x_5 .$$

In fact Φ is a quadratic transformation; its base locus in $\mathbf{P}^4$ is given by: L_1, L_2, L_3 and by the only line L_4 which is incident to them, the equations of L_4 are: $z_2 = z_4 = z_5 = 0$.

The base locus of Φ^{-1} in $\mathbf{P}^5$ is given by P_0, P_1, P_2 and by the plane Π passing through the points $P_0 \cap P_1$, $P_0 \cap P_2$, $P_1 \cap P_2$; the equations of Π are: $x_2 = x_4 = x_5 = 0$.

All cubic hypersurfaces X in $\mathbf{P}^5$ containing P_1 and P_2 have this equation:

$$e x_0^2 x_5 + x_1^2 F + x_2^2 G + x_0 x_1 H + x_0 x_2 L + x_1 x_2 M + x_0 x_5 N + \\ + x_1 P + x_2 Q + x_5 R = 0$$

where $e \in \mathbf{C}$; $F = F(x_3 : x_4 : x_5) = f_1 x_3 + f_2 x_4 + f_3 x_5$ is a degree one homogeneous polynomial; G, H, L, M, N are analogous to F; $P = P(x_3 : x_4 : x_5) = p_{11} x_3^2 + p_{12} x_3 x_4 + p_{22} x_4^2 + x_5(p_1 x_3 + p_2 x_4 + p_3 x_5)$ is a degree two homogeneous polynomial; Q and R are analogous to P.

$\Phi(X)$ is the following quartic hypersurface Y of $\mathbf{P}^4$:

$$e(z_1 - z_3) z_4^2 + z_1^2 z_5 F + z_4^2 z_5 G + z_1(z_1 - z_3) z_4 H + (z_1 - z_3) z_4^2 L + \\ + z_1 z_4 z_5 M - z_2(z_1 - z_3) z_4 N - z_1 z_2 P - z_2 z_4 Q + z_2^2 R = 0$$

where $F = F(z_3 : z_4 : z_5)$ etc.

It is easy to see that Y contains L_1, L_2, L_3, L_4 and that L_1, L_3 are double lines for Y, without n-ple points ($n \geqslant 3$). We can prove:

PROPOSITION (2.1). *Y is smooth out of L_1, L_3 and it is the more general quartic hypersurface of $\mathbf{P}^4$ containing two skew double lines (and no other singularities) and another simple line, no two of them lying in the same hyperplane..*

PROOF. In $\mathbf{P}^4$ we choose $(x:y:z:w:u)$ as coordinates; we can always suppose that the three skew lines, no two of them lying in the same hyperplane, have equations:

$$x = y = u = 0\,, \qquad z = w = u = 0\,, \qquad x = z = y - w = 0\,.$$

All quartic hypersurfaces containing $x = y = u = 0$ and $z = w = u = 0$ as double lines have equation:

$$z^2\mathcal{A} + zw\mathcal{B} + w^2\mathcal{C} + zu\mathcal{D} + wu\mathcal{E} + u^2\mathcal{F} = 0 \tag{2.2}$$

where $\mathcal{A} = a_{11}x^2 + a_{12}xy + a_{22}y^2 + a_{13}xu + a_{23}yu + a_{33}u^2$ and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ are analogous to $\mathcal{A}$.

This hypersurface contains the third line if and only if

$$c_{22} = f_{23} + e_{33} = c_{33} + e_{23} + f_{22} = c_{23} + e_{22} = f_{33} = 0\,. \tag{2.3}$$

It is easy to see that it is smooth out of the two double lines.

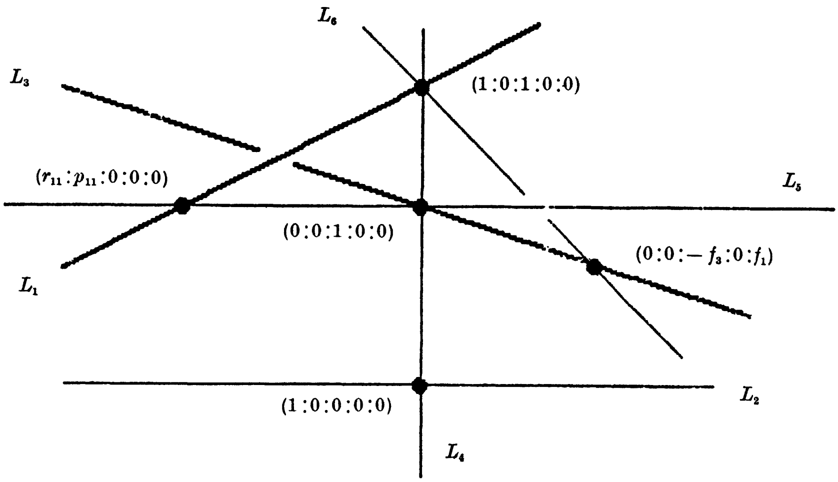
Now if we put: $z_5 = x$, $z_4 = u$, $z_3 = y$, $z_2 = z$, $z_1 = w$, we see that the equation (2.2), with the conditions (2.3), becomes the equation of Y after a suitable linear, invertible, transformation on its coefficients; so we get our thesis. $\square$

REMARK (2.4). Obviously the existence of L_4 in Y is a direct consequence of the existence of L_2 and the double lines L_1, L_3.

If we intersect Y with the plane containing L_1 and L_4 we get an other line L_5 whose equations are: $r_{11}z_2 - p_{11}z_1 = z_4 = z_5 = 0$.

If we intersect Y with the plane containing L_3 and L_4 we get an other line L_6 whose equations are: $z_2 = z_4 = f_1z_3 + f_3z_5 = 0$.

The following picture shows the configuration of these six lines and their incidence points in Y:



In the sequel we will need to know the action of Φ on some plane in Y, so we prove the following:

PROPOSITION (2.5). *Let p be a plane in Y.*

Suppose that p does not belong to the hyperplane $z_4 = 0$. If p cuts L_1 and L_3 but not L_2, then $\Phi(p)$ is a quadric *(irreducible or not), in $V = Q \cap X$; if p cuts L_1, L_2 and L_3 then $\Phi(p)$ is a* plane *in V meeting P_0, P_1, P_2 at one point only.*

Suppose that p belongs to the hyperplane $z_4 = 0$. If p does not contain L_1 or L_3 then V contains P_0 and therefore Y splits into a cubic hypersurface and a hyperplane.

PROOF. In the first case it suffices to consider the equations of a plane p with the above conditions and to write down the equations of $\Phi(p)$ in $\mathbf{P}^5$ by using the previously fixed coordinate system.

In the second case a direct calculation shows that the existence of a plane p in Y, with the above conditions, implies that V contains P_0: in this case $\Phi^{-1}(V)$ is a cubic hypersurface, hence Y is reducible. □

Now let p be a plane in Y; if p contains L_1 and it is incident with L_3 but it is not $z_4 = z_5 = 0$ (i.e. the plane containing L_1 and L_4)

we call it a « λ-plane ». If p contains L_3 and it is incident with L_1 but it is not $z_2 = z_4 = 0$ (i.e. the plane containing L_3 and L_4) we call it a « μ-plane ». Obviously all these planes belong to the hyperplane $z_4 = 0$. We have this:

PROPOSITION (2.6). *Let (a, b) be the numbers of λ-planes and respectively μ-planes contained in Y, by keeping it irreducible. If Y does not contain $z_4 = z_5 = 0$ or $z_2 = z_4 = 0$ we have only these couples: $(a, b) = (0, 0); (1, 0); (0, 1); (1, 1)$. If Y contains $z_4 = z_5 = 0$ we have $(a, b) = (0, 0); (1, 0); (0, 1); (1, 1); (0, 2)$. If Y contains $z_2 = z_4 = 0$ we have $(a, b) = (0, 0); (1, 0); (0, 1); (2, 0); (1, 1)$. If Y contains both of them we have $(a, b) = (0, 0); (1, 0); (0, 1); (1, 1)$.*

PROOF. Obviously when V contains P_1 and P_3 only, among the three planes which are the base locus of Φ in $\mathbf{P}^5$, we can state that Y is irreducible if and only if V is irreducible; then our strategy is the following: to consider the generic Y containing a λ-planes and b μ-planes, to consider the corresponding V and to check if it, i.e. X because Q is fixed, is irreducible.

A λ-plane has equations: $z_4 = z_3 - \lambda z_5 = 0$ $\lambda \in \mathbb{C}$; Y contains it if and only if: $\lambda f_1 + f_3 = \lambda^2 p_{11} + \lambda p_1 + p_3 = \lambda^2 r_{11} + \lambda r_1 + r_3 = 0$; while Y contains $z_4 = z_5 = 0$ if and only if: $p_{11} = r_{11} = 0$. Φ sends the λ-plane into the line $x_3 = \lambda x_5$ on the plane P_0, while Φ blow down the plane $z_4 = z_5 = 0$ in the point $(0:0:0:1:0:0)$ of $\mathbf{P}^5$.

A μ-plane has equations: $z_4 = z_1 - \mu z_2 = 0$ $\mu \in \mathbb{C}$; Y contains it if and only if: $-\mu p_{11} + r_{11} = \mu^2 f_1 - \mu p_1 + r_1 = \mu^2 f_3 - \mu p_3 + r_3 = 0$; while Y contains $z_2 = z_4 = 0$ if and only if: $f_1 = f_3 = 0$. Φ sends the μ-plane into the line $x_1 = -\mu x_5$ on the plane P_0, while Φ blow down the plane $z_2 = z_4 = 0$ in the point $(0:1:0:0:0:0)$ of $\mathbf{P}^5$.

As we have seen, all these planes, belonging to the hyperplane $z_4 = 0$, are sent in P_0 by Φ. The section of X with P_0 is the following plane cubic E:

$$x_1^2(f_1 x_3 + f_3 x_5) + x_1(p_{11} x_3^2 + p_1 x_3 x_5 + p_3 x_5^2) + \\ + x_5(r_{11} x_3^2 + r_1 x_3 x_5 + r_3 x_5^2) = 0 .$$

For generic Y E, passing through $(0:0:0:1:0:0)$ and $(0:1:0:0:0:0)$, is smooth; if Y contains some λ-plane, some μ-plane or the two particular planes $z_4 = z_5 = 0$ or $z_2 = z_4 = 0$, then E splits in a obvious way. The values (a, b) quoted in (2.6) are the only possibilities to avoid that X contains P_0 entirely: it would imply Y reducible. In all these cases it is easy to see that X is in fact irreducible by looking at the possible hyperplanes contained in X which would cut one of the lines into which E splits on P_0.

If Y contains $z_4 = z_5 = 0$ only or $z_2 = z_4 = 0$ only, E does not split and hence X is irreducible.

We will give an example of this reasoning: let us suppose that Y contains a λ-plane, then E splits into the line $x_3 = \lambda x_5$ and into the smooth conic $(x_3 + \lambda x_5)(p_{11}x_1 + r_{11}x_5) + f_1 x_1^2 + p_1 x_3 x_5 + p_3 x_5^2 = 0$. If X is reducible it splits into a hyperplane of $\mathbf{P}^5$ and something other; this hyperplane has to cut the line $x_3 = \lambda x_5$ on P_0, hence its equation is: $x_3 = \lambda x_5 + a x_0 + b x_2 + c x_4$; but there exists no choice of the three numbers a, b, c such that the generic X contains this hyperplane, in spite of conditions imposed on Y by containing the λ-plane, (i.e.: $\lambda f_1 + f_3 = \lambda^2 p_{11} + \lambda p_1 + p_3 = \lambda^2 r_{11} + \lambda r_1 + r_3 = 0$), even when Y contains $z_4 = z_5 = 0$ or $z_2 = z_4 = 0$ or both.

The other cases are solved in the same way. $\square$

REMARK (2.7). By a simple check of the partial derivatives of the equations of V we see that, in spite of the existence in Y of the planes quoted in (2.6), V has ordinary double points only, (see also [A-B$_1$] and [A-B$_2$]).

3. – The conic bundle structures.

We need some definitions and basic facts about conic bundle theory.

DEFINITION (3.1). Let W be a threefold, let S be a smooth surface. If there exists a sur^{5}ective morphism $\tau\colon W \to S$ such that for every point $t \in S$ the fibre $\tau^{-1}(t)$ is isomorphic to a conic in $\mathbf{P}^2$, possibly degenerated, then W is called a conic bundle over S; we will use the symbol: (W, τ, S).

DEFINITION (3.2). Let (W, τ, S) and (W', τ', S') be two conic bundles; if there exists a commutative diagram as follows:

$$\begin{tikzcd} W \arrow[r, leftrightarrow] \arrow[d] & W' \arrow[d] \\ S \arrow[r, leftrightarrow] & S' \end{tikzcd}$$

in which the horizontal arrows are birational morphisms, then we say that (W, τ, S) and (W', τ', S') are birationally equivalent.

REMARK (3.3). Let (W, τ, S) be a singular conic bundle; suppose that W has only a finite number of ordinary double points such that none of them is the intersection point of the two lines into which a degenerate fibre splits. Then, if we solve the singularities of W by blowings up, we get a smooth conic bundle over S which is birationally equivalent to (W, τ, S).

DEFINITION (3.4). Let (W, τ, S) be a conic bundle; the set of the points $t \in S$ such that the fibre $\tau^{-1}(t)$ is a degenerate conic is called the *discriminant locus* of the conic bundle. It can be shown (see [Sa], p. 358) that it is always a divisor of S; from now on we will refer to it as the discriminant divisor $\boldsymbol{D}_W$ of (W, τ, S).

DEFINITION (3.5). A smooth conic bundle (W, τ, S) is called standard if for every curve C of S, the surface $\tau^{-1}(C)$ is irreducible.

PROPOSITION (3.6) (see [Sa], p. 366-367, see also [A-B$_2$] prop. (2.6)). Let (W, τ, S) be a smooth conic bundle, such that $\boldsymbol{D}_W$ is the disjoint union of smooth curves D_i, $i = 1, 2 \dots n$; if $\tau^{-1}(D_1)$, for instance, is reducible then necessarily $D_1 \cap (\boldsymbol{D}_W - D_1)$ is empty and we can blow down one of the two components of $\tau^{-1}(D_1)$ to obtain a new smooth conic bundle, birationally equivalent to (W, τ, S), whose $\boldsymbol{D}$ is $D_2 \cup D_3 \cup \dots D_n$. We can repeat this process until to obtain a smooth *standard* conic bundle birationally equivalent to (W, τ, S).

THEOREM (3.7) (see [I$_2$], p. 742). Let (W, τ, S) be a smooth, standard, conic bundle, let S be a rational surface, let $\boldsymbol{D}_W$ be a curve. Then W is rational if there exists a pencil of rational curves C_t on S, $(t \in \mathbf{P}^1)$, without fixed components, such that $C_t \cdot \boldsymbol{D}_W \leqslant 3$ $\forall t$.

Now we consider the conic bundle structures of X and Y.

It is well known that every quartic hypersurface in $\mathbf{P}^4$ with a double line has a conic bundle structure (see [C-M]): we fix the plane π whose equations are: $z_1 = z_2 = 0$; it is skew with L_1. If we project Y from L_1 to π we have that the fibre over a point of π is a quartic plane curve which splits into L_1, counted twice, and into another conic; if we blow up Y along L_1 we get a smooth conic bundle according to definition (3.1).

Now we want to determine $\boldsymbol{D}_Y$. The generic point of the plane containing a point $(0:0:z_3:z_4:z_5)$ of π and L_1, has coordinates $(h:k:tz_3:tz_4:tz_5)$; the intersection between Y and this plane is the following plane quartic (where $F = F(z_3:z_4:z_5)$ etc.):

$$t^2[(ez_4^2 + z_5F + z_4H)h^2 - (z_4N + P)hk + Rk^2 -$$
$$- (2ez_3z_4^2 + z_3z_4H + z_4^2L + z_4z_5M)ht + (z_3z_4N - z_4Q)kt +$$
$$+ (ez_3^2z_4^2 + z_4^2z_5G - z_3z_4^2L)] = 0;$$

$t^2 = 0$ gives L_1 counted twice, the remaining curve is a conic; it is degenerated if and only if:

$$\text{(3.8)} \quad z_4^2[4R(ez_4^2 + z_5F + z_4H)(ez_3^2 + z_5G - z_3L) -$$
$$- (z_4N + P)(z_3N - Q)(- 2ez_3z_4 - z_3H + z_4L + z_5M) -$$
$$- R(- 2ez_3z_4 - z_3H + z_4L + z_5M)^2 - (z_3N - Q)^2(ez_4^2 + z_5F + z_4H) -$$
$$- (z_4N + P)^2(ez_3^2 + z_5G - z_3L)] = 0\,.$$

Therefore $\boldsymbol{D}_Y$ splits into the line $z_4 = 0$ counted twice (whose existence is an obvious consequence of the double lines L_1 and L_3 in Y) and into a sestic Γ; we remark that the existence of a double line in $\boldsymbol{D}_Y$ makes very difficult to apply all known theorems about the rationality of the conic bundles.

Now let us consider $V = X \cap Q$, as $\Phi(X) = Y$ we have that V is birational to Y. V has a conic bundle structure too; it is well known (see [C], [A-B$_1$]): we fix the plane π', whose equations are $x_0 = x_1 = x_2 = 0$; we project V from P_1 to π'; by blowing up V along P_1 and at the ordinary double points which V has on P_2 (see [A-B$_1$]) we get a smooth conic bundle.

Let us determine $\boldsymbol{D}_V$: the generic point of the plane containing a point $(0:0:0:x_3:x_4:x_5)$ of π' and P_1 has coordinates: $(\alpha:\beta:\gamma:\delta x_3:\delta x_4:\delta x_5)$; this point belongs to V if and only if:

$$e\alpha^2\delta x_5 + \beta^2\delta F + \gamma^2\delta G + \alpha\beta\delta H + \alpha\gamma\delta L + \beta\gamma\delta M + \alpha\delta^2x_5N +$$
$$+ \beta\delta^2P + \gamma\delta^2Q + \delta^3x_5R = 0$$

and

$$\alpha\delta x_5 - \beta\delta x_4 + \gamma\delta x_3 = 0\,.$$

$\delta = 0$ gives the plane P_1; if we delete δ we obtain a conic, it is easy to see ([A-B$_1$]) that the conic is degenerate if and only if:

$$\begin{aligned}(3.9)\quad & x_5[4R(ex_4^2 + x_5F + x_4H)(ex_3^2 + x_5G - x_3L) - \\ & - (x_4N + P)(x_3N - Q)(-2ex_3x_4 - x_3H + x_4L + x_5M) - \\ & - R(-2ex_3x_4 - x_3H + x_4L + x_5M)^2 - (x_3N - Q)^2(ex_4^2 + x_5F + x_4H) - \\ & - (x_4N + P)^2(ex_3^2 + x_5G - x_3L)] = 0\end{aligned}$$

where $F = F(x_3 : x_4 : x_5)$ etc.

Therefore D_V splits into the line $x_5 = 0$ and into a smooth plane sestic Γ (see [A-B$_1$] and [A-B$_2$]); it is exactly the same curve into which D_Y splits, in fact if we look at (3.8) and (3.9) and if we put $x_i = z_i$, $i = 3, 4, 5$ we see that the two curves are the same curve.

4. – The main results.

Now we want to prove this:

PROPOSITION (4.1). *The generic quartic hypersurface of* $\mathbf{P}^4$ *containing two skew double lines is not rational.*

As the set of the generic quartic hypersurfaces of $\mathbf{P}^4$, containing two skew double lines and a third simple skew line, (not two of them belonging to the same hyperplane), is a closed Zarisky set of the moduli space of all quartic hypersurfaces of $\mathbf{P}^4$, to prove (4.1) it suffices to prove the following:

PROPOSITION (4.2). *The generic quartic hypersurface of* $\mathbf{P}^4$, *containing two skew double lines and a third simple skew line, not two of them belonging to the same hyperplane, is not rational.*

PROOF. By (2.1) it suffices to show that Y is not rational. By the previous section we have seen that Y is birational to V which is a cubic complex containing two planes only, meeting two by two at one point; therefore it is not rational (see [A-B$_1$] and [A-R]). □

Now we want to study the rationality of the generic quartic hypersurface of $\mathbf{P}^4$ with two skew double lines when it contains some plane;

as we have seen this problem is equivalent to study the rationality of the generic Y containing some plane.

If Y contains a plane which is skew with L_1 (or L_3) it is rational; in fact every line intersecting L_1 and the plane cuts Y in one other point only, so that it is not difficult to see that in this case Y is birational to $\mathbb{P}^2 \times \mathbb{P}^1$. Therefore we can suppose that every plane contained in Y is incident with both double lines, or it is a λ-plane or a μ-plane or it is $z_4 = z_5 = 0$ or $z_2 = z_4 = 0$.

We have this:

PROPOSITION (4.3). *If Y contains some plane incident to both double lines or containing one of them, then it is rational (or reducible) save when it contains at most one plane incident with L_1 and L_3 and all λ-planes and μ-planes allowed by* (2.6).

Before proving (4.3) we need

LEMMA (4.4). *If Y contains one plane only, intersecting L_1 and L_3 but not intersecting L_2, then Y is not rational.*

PROOF. - Let us call p this plane. If p belongs to the hyperplane generated by L_1 and L_3 (i.e. $z_4 = 0$), then $\Phi(p)$ is P_0 and V is a cubic complex containing the three planes which are the base locus of Φ^{-1}, therefore Y is reducible, (see also (2.5)).

In the other cases, by a suitable choice of coordinate system, we can always suppose that p has equations:

1) $z_3 = z_4 - z_1 = 0,$

2) $z_4 - z_1 = z_5 - z_3 = 0,$

3) $z_3 = z_4 - z_1 + z_2 = 0,$

4) $z_5 - z_3 = z_4 - z_1 + z_2 = 0.$

Then $\Phi(p)$ has equations:

1) $x_1 = x_3 = x_0x_5 - x_1x_4 + x_2x_3 = 0,$

2) $x_1 = x_3 + x_4 - x_5 = x_0x_5 - x_1x_4 + x_2x_3 = 0,$

3) $x_1 + x_5 = x_3 = x_0x_5 - x_1x_4 + x_2x_3 = 0,$

4) $x_1 + x_5 = x_3 + x_4 - x_5 = x_0x_5 - x_1x_4 + x_2x_3 = 0.$

In the cases 1) and 3) $\Phi(p)$ splits into a couple of planes and V

is a cubic complex containing four planes. It is easy to see that this is the case $(4,3,1)$ of table R of [A-B$_2$], therefore V is not rational.

In the cases 2) and 4) $\Phi(p)$ is a smooth quadric cutting a line on P_1 and a line on P_2 both passing through $P_1 \cap P_2$. This configuration in V is obtained as follows: by choosing two points A, B in $\mathbf{P}^3$ and two skew lines a, b passing through A and B respectively; by considering the two stars of lines centered in A and in B and the lines intersecting both a and b. If we move a until it cuts b in a third distinct point C we get a cubic complex V containing four planes (the three stars of lines centered in A, B, C and the lines of the plane through A, B, C) with the previously considered configuration. It is easy to see that this degeneration is flat so that V is not rational as in the previous cases. □

PROOF OF (4.3). Let us suppose that Y contains only one plane p intersecting $L_1, L_2 L_3$; by (2.5) $\Phi(p)$ is a plane in V, meeting P_1 and P_2 at one point only, so that Y is birational to a cubic complex containing three planes two by two meeting at one point only (and no other planes), such complex is not rational (see [A-R] and [A-B$_1$]).

Let us suppose that Y contains only one plane intersecting L_1, L_3 but not intersecting L_2: Y is not rational by lemma (4.4).

Now it is easy to see that if we suppose that Y contains two planes intersecting L_1, L_2, L_3, or two planes intersecting L_1, L_3 but not L_2, or one plane of the first type and one plane of the second type, we get that V is a singular conic bundle over $\mathbf{P}^2$ birationally equivalent to a smooth standard conic bundle W over a rational surface S, such that $\boldsymbol{D}_W$ is the pull back of a smooth plane quartic by blowings up; (for the second type we can use a degeneration argument as in the proof of lemma (4.4)).

V is rational by theorem (3.7): it suffices to consider a pencil of lines of $\mathbf{P}^2$ (through a point not belonging to the quartic) and its transformed on S by the blowings up.

Finally we have only to remark that the existence in Y of any plane p quoted in (2.6) does not change the conic bundle structure of V; in fact in all these cases V is irreducible, with ordinary double points only, $\Phi(p)$ is a line or a point (see (2.6)) and when we project V from P_1 to π' we see that $\boldsymbol{D}_V$ is the same divisor (a *smooth* curve plus one or two lines) arising when Y does not contain any plane of this type; this last fact is easy checked by looking directly at (3.8) or (3.9) and by recalling the conditions imposed on Y by the existence of a plane of this type (see (2.6)). □

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