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Quartic Threefolds Containing Two Skew Double Lines.

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1. – Introduction.

The problem of rationality for algebraic threefolds is still an open problem in Algebraic Geometry. However the conic bundle theory, developed by Beauville (see [B1], [B2] and also [C-M]), gives us a very useful tool to solve this problem in many cases.

Some recent results of Sarkisov and Iskovskih (see [I1], [I2] and [Sa]) have improved this technique by giving some answers even when the intermediate Jacobian of the threefold is the Jacobian of a curve. These facts have allowed us to solve the problem of rationality for the Fano threefold of $\mathbb{P}^5$ containing $n$ planes (see [A-B1] and [A-B2]).

In this paper we study the rationality of the generic quartic threefold of $\mathbb{P}^4$ containing two skew double lines and containing $n$ planes with all possible configurations. In [C-M] Conte and Murre have proved that a generic quartic threefold of $\mathbb{P}^4$ containing only one double line is not rational, while it is well known that such threefold with two incident double lines is rational. Our work is a natural prosecution of [C-M] and it was suggested by remark (6, 3) of [A-B2], in which we showed that a generic quartic threefold of $\mathbb{P}^4$ containing two skew double lines, and no planes, is not rational.

Our proofs are based on this idea: there exists a birational morphism (due to Fano, [F]) between $\mathbb{P}^4$ and the quadric hypersurface of $\mathbb{P}^5$.

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identified with the Grassmannian $G(1,3)$ of lines of $\mathbb{P}^3$. By this morphism some quartic hypersurfaces with two skew double lines correspond to cubic complexes containing two planes, meeting two by two at one point only; these singular varieties have a well known conic bundle structure (see [C], [A-B$_1$] and [A-B$_2$]); the existence of some plane in the quartics changes this structure; by studying these new structures we get our results; they are described in § 4.

We use these conventions: by the word « $n$-fold » we mean a projective algebraic variety (singular or not) defined on $\mathbb{C}$; by the word « generic » we mean that what we are saying is true in a suitable open Zarisky set.

2. – Fano birational morphism.

We choose $(x_0:x_1:x_2:x_3:x_4:x_5)$ as coordinates in $\mathbb{P}^5$, we fix a smooth quadric hypersurface $Q$ and we choose three planes contained in $Q$, meeting two by two at one point only; we can always suppose that $Q$ has this equation:

$$Q) \quad x_0x_5 - x_1x_4 + x_2x_3 = 0$$

and that the three planes, $P_0, P_1, P_2$, have equations:

$$P_0) \quad x_0 = x_2 = x_4 = 0$$
$$P_1) \quad x_1 = x_4 = x_5 = 0$$
$$P_2) \quad x_1 = x_2 = x_5 = 0 .$$

Now in $\mathbb{P}^4$ we choose $(z_1:z_2:z_3:z_4:z_5)$ as coordinates, (this unusual choice will be very useful in the sequel), and we choose three skew lines, not two of them lying in the same hyperplane; we can always suppose that the three lines have equations:

$$L_1) \quad z_3 = z_4 = z_5 = 0$$
$$L_2) \quad z_1 - z_2 = z_3 = z_5 = 0$$
$$L_3) \quad z_1 = z_2 = z_4 = 0 .$$
We consider the rational map $\Phi: \mathbb{P}^4 \to \mathbb{P}^5$ given by:

\[
\begin{align*}
    x_0 &= z_4(z_3 - z_1) & x_1 &= -z_1 z_6 \\
    x_2 &= -z_4 z_5 & x_3 &= z_2 z_3 \\
    x_4 &= z_2 z_4 & x_5 &= z_2 z_5.
\end{align*}
\]

$\Phi$ is a well known birational morphism between $\mathbb{P}^4$ and $Q$ (see [F]), its inverse is:

\[
\begin{align*}
    z_1 &= x_1 x_4 & z_2 &= -x_4 x_5 \\
    z_3 &= x_2 x_3 & z_4 &= x_2 x_4 \\
    z_5 &= x_2 x_5.
\end{align*}
\]

In fact $\Phi$ is a quadratic transformation; its base locus in $\mathbb{P}^4$ is given by $L_1, L_2, L_3$ and by the only line $L_4$ which is incident to them, the equations of $L_4$ are: $z_2 = z_4 = z_5 = 0$.

The base locus of $\Phi^{-1}$ in $\mathbb{P}^5$ is given by $P_0, P_1, P_2$ and by the plane $\Pi$ passing through the points $P_0 \cap P_1, P_0 \cap P_2, P_1 \cap P_2$; the equations of $\Pi$ are: $x_2 = x_4 = x_5 = 0$.

All cubic hypersurfaces $X$ in $\mathbb{P}^5$ containing $P_1$ and $P_2$ have this equation:

\[
\begin{align*}
    e x_0^2 x_5 + x_1^2 F + x_2^2 G + x_0 x_1 H + x_0 x_2 L + x_1 x_2 M + x_0 x_5 N + \\
    + x_1 P + x_2 Q + x_5 R = 0
\end{align*}
\]

where $e \in \mathbb{C}; F = F(x_5; x_4; x_3) = f_1 x_5 + f_2 x_4 + f_3 x_3$ is a degree one homogeneous polynomial; $G, H, L, M, N$ are analogous to $F$; $P = P(x_5; x_4; x_3) = p_{11} x_5^2 + p_{12} x_5 x_4 + p_{13} x_4^2 + x_3(p_4 + p_5 x_4 + p_6 x_3)$ is a degree two homogeneous polynomial; $Q$ and $R$ are analogous to $P$.

$\Phi(X)$ is the following quartic hypersurface $Y$ of $\mathbb{P}^4$:

\[
\begin{align*}
    e(x_1 - z_3) z_4^2 + z_1^2 z_4 F + z_4^2 z_5 G + z_1 (x_1 - z_3) z_4 H + (x_1 - z_3) z_4^2 L + \\
    + z_1 z_4 z_5 M - z_2 (x_1 - z_3) z_4 N - z_1 z_2 P - z_2 z_4 Q + z_2^2 R = 0
\end{align*}
\]

where $F = F(x_5; x_4; x_3)$ etc.
It is easy to see that $Y$ contains $L_1, L_2, L_3, L_4$ and that $L_1, L_3$ are double lines for $Y$, without $n$-ple points ($n > 3$). We can prove:

**Proposition (2.1).** $Y$ is smooth out of $L_1, L_3$ and it is the more general quartic hypersurface of $\mathbb{P}^4$ containing two skew double lines (and no other singularities) and another simple line, no two of them lying in the same hyperplane.

**Proof.** In $\mathbb{P}^4$ we choose $(x:y:z:w:u)$ as coordinates; we can always suppose that the three skew lines, no two of them lying in the same hyperplane, have equations:

$$x = y = u = 0, \quad z = w = u = 0, \quad x = z = y - w = 0.$$  

All quartic hypersurfaces containing $x = y = u = 0$ and $z = w = u = 0$ as double lines have equation:

$$z^2 A + zwB + w^2 C + zwD + wuE + u^2 F = 0$$

where $A = a_{11}x^2 + a_{12}xy + a_{22}y^2 + a_{13}xz + a_{33}yz + a_{23}zu + a_{32}uw$ and $B, C, D, E, F$ are analogous to $A$.

This hypersurface contains the third line if and only if

$$e_{22} = f_{22} + e_{33} = c_{33} + e_{23} + f_{22} = e_{23} + e_{22} = f_{33} = 0.$$

It is easy to see that it is smooth out of the two double lines.

Now if we put: $z_2 = x, z_4 = u, z_3 = y, z_2 = z, z_4 = w$, we see that the equation (2.2), with the conditions (2.3), becomes the equation of $Y$ after a suitable linear, invertible, transformation on its coefficients; so we get our thesis. □

**Remark (2.4).** Obviously the existence of $L_4$ in $Y$ is a direct consequence of the existence of $L_2$ and the double lines $L_1, L_3$.

If we intersect $Y$ with the plane containing $L_1$ and $L_4$ we get an other line $L_6$ whose equations are: $r_{11}z_2 - p_{11}z_4 = z_6 = z_5 = 0$.

If we intersect $Y$ with the plane containing $L_3$ and $L_1$ we get an other line $L_6$ whose equations are: $z_3 = z_4 = f_{11}z_3 + f_{13}z_3 = 0$.

The following picture shows the configuration of these six lines.
and their incidence points in $Y$:

In the sequel we will need to know the action of $\Phi$ on some plane in $Y$, so we prove the following:

**Proposition (2.5).** Let $p$ be a plane in $Y$.

Suppose that $p$ does not belong to the hyperplane $z_4 = 0$. If $p$ cuts $L_1$ and $L_3$ but not $L_2$, then $\Phi(p)$ is a quadric (irreducible or not), in $V = Q \cap X$; if $p$ cuts $L_1$, $L_2$ and $L_3$ then $\Phi(p)$ is a plane in $V$ meeting $P_0$, $P_1$, $P_3$ at one point only.

Suppose that $p$ belongs to the hyperplane $z_4 = 0$. If $p$ does not contain $L_1$ or $L_3$ then $V$ contains $P_0$ and therefore $Y$ splits into a cubic hypersurface and a hyperplane.

**Proof.** In the first case it suffices to consider the equations of a plane $p$ with the above conditions and to write down the equations of $\Phi(p)$ in $P^5$ by using the previously fixed coordinate system.

In the second case a direct calculation shows that the existence of a plane $p$ in $Y$, with the above conditions, implies that $V$ contains $P_0$: in this case $\Phi^{-1}(V)$ is a cubic hypersurface, hence $Y$ is reducible. $\square$

Now let $p$ be a plane in $Y$; if $p$ contains $L_1$ and it is incident with $L_3$ but it is not $z_4 = z_5 = 0$ (i.e. the plane containing $L_1$ and $L_4$)
we call it a «λ-plane». If p contains $L_3$ and it is incident with $L_4$, but it is not $z_2 = z_3 = 0$ (i.e. the plane containing $L_3$ and $L_4$) we call it a «μ-plane». Obviously all these planes belong to the hyperplane $z_4 = 0$. We have this:

**Proposition (2.6).** Let $(a, b)$ be the numbers of λ-planes and respectively μ-planes contained in $Y$, by keeping it irreducible. If $Y$ does not contain $z_4 = z_3 = 0$ or $z_2 = z_4 = 0$ we have only these couples: $(a, b) = (0, 0); (1, 0); (0, 1); (1, 1)$. If $Y$ contains $z_4 = z_5 = 0$ we have $(a, b) = (0, 0); (1, 0); (0, 1); (1, 1); (2, 0); (1, 1)$. If $Y$ contains both of them we have $(a, b) = (0, 0); (1, 0); (0, 1); (1, 1)$.

**Proof.** Obviously when $V$ contains $P_1$ and $P_3$ only, among the three planes which are the base locus of $\Phi$ in $P^5$, we can state that $Y$ is irreducible if and only if $V$ is irreducible; then our strategy is the following: to consider the generic $Y$ containing $a$ λ-planes and $b$ μ-planes, to consider the corresponding $V$ and to check if it, i.e. $X$ because $Q$ is fixed, is irreducible.

A λ-plane has equations: $z_4 = z_5 = 0 \lambda \rho_2 = 0$ $\lambda \in C$; $Y$ contains it if and only if: $\lambda f_1 + f_2 = \lambda^2 p_{11} + \lambda p_1 + p_3 = \lambda^2 r_{11} + \lambda r_1 + r_3 = 0$; while $Y$ contains $z_4 = z_5 = 0$ if and only if: $p_{11} = r_{11} = 0$. $\Phi$ sends the λ-plane into the line $x_5 = \lambda x_3$ on the plane $P_0$, while $\Phi$ blow down the plane $z_4 = z_5 = 0$ in the point $(0:0:0:1:0:0)$ of $P^5$.

A μ-plane has equations: $z_4 = z_5 = 0 \mu \rho_2 = 0$ $\mu \in C$; $Y$ contains it if and only if: $-\mu p_{11} + r_1 = \mu^2 f_1 - \mu p_1 + r_1 = \mu^2 f_3 - \mu p_3 + r_3 = 0$; while $Y$ contains $z_2 = z_3 = 0$ if and only if: $f_1 = f_3 = 0$. $\Phi$ sends the μ-plane into the line $x_1 = -\mu x_3$ on the plane $P_0$, while $\Phi$ blow down the plane $z_2 = z_3 = 0$ in the point $(0:1:0:0:0:0)$ of $P^5$.

As we have seen, all these planes, belonging to the hyperplane $z_4 = 0$, are sent in $P_0$ by $\Phi$. The section of $X$ with $P_0$ is the following plane cubic $E$:

$$\begin{align*}
x_1^2(f_1x_3 + f_2x_5) + x_1(p_{11}x_2^2 + p_1x_2x_5 + p_3x_3^2) + \\
+ x_5(r_{11}x_2^2 + r_1x_3x_5 + r_3x_5^2) = 0.
\end{align*}$$

For generic $Y$, passing through $(0:0:0:1:0:0)$ and $(0:1:0:0:0:0)$, is smooth; if $Y$ contains some λ-plane, some μ-plane or the two particular planes $z_4 = z_5 = 0$ or $z_2 = z_3 = 0$, then $E$ splits in a obvious way. The values $(a, b)$ quoted in (2.6) are the only possibilities to avoid
that $X$ contains $P_0$ entirely: it would imply $Y$ reducible. In all these cases it is easy to see that $X$ is in fact irreducible by looking at the possible hyperplanes contained in $X$ which would cut one of the lines into which $E$ splits on $P_0$.

If $Y$ contains $z_4 = z_5 = 0$ only or $z_2 = z_4 = 0$ only, $E$ does not split and hence $X$ is irreducible.

We will give an example of this reasoning: let us suppose that $Y$ contains a $\lambda$-plane, then $E$ splits into the line $x_3 = \lambda x_5$ and into the smooth conic $(x_3 + \lambda x_5)(p_{11}x_1 + r_{11}x_5) + f_1x_1^2 + p_1x_3x_5 + p_3x_5^3 = 0$. If $X$ is reducible it splits into a hyperplane of $P^5$ and something other; this hyperplane has to cut the line $x_3 = \lambda x_5$ on $P_0$, hence its equation is: $x_3 = \lambda x_5 + ax_0 + bx_2 + cx_4$; but there exists no choice of the three numbers $a, b, c$ such that the generic $X$ contains this hyperplane, in spite of conditions imposed on $Y$ by containing the $\lambda$-plane, (i.e.: $\lambda f_1 + f_3 = \lambda^2 p_{11} + \lambda p_1 + p_3 = \lambda^2 r_{11} + \lambda r_1 + r_3 = 0$), even when $Y$ contains $z_4 = z_5 = 0$ or $z_2 = z_4 = 0$ or both.

The other cases are solved in the same way. □

**Remark (2.7).** By a simple check of the partial derivatives of the equations of $V$ we see that, in spite of the existence in $Y$ of the planes quoted in (2.6), $V$ has ordinary double points only, (see also [A-B₁] and [A-B₂]).

### 3. - The conic bundle structures.

We need some definitions and basic facts about conic bundle theory.

**Definition (3.1).** Let $W$ be a threefold, let $S$ be a smooth surface. If there exists a surjective morphism $\tau: W \to S$ such that for every point $t \in S$ the fibre $\tau^{-1}(t)$ is isomorphic to a conic in $P^2$, possibly degenerated, then $W$ is called a conic bundle over $S$; we will use the symbol: $(W, \tau, S)$.

**Definition (3.2).** Let $(W, \tau, S)$ and $(W', \tau', S')$ be two conic bundles; if there exists a commutative diagram as follows:

$$
\begin{array}{ccc}
W & \longrightarrow & W' \\
\downarrow & & \downarrow \\
S & \longrightarrow & S'
\end{array}
$$
in which the horizontal arrows are birational morphisms, then we say that \((W, \tau, S)\) and \((W', \tau', S')\) are birationally equivalent.

**Remark (3.3).** Let \((W, \tau, S)\) be a singular conic bundle; suppose that \(W\) has only a finite number of ordinary double points such that none of them is the intersection point of the two lines into which a degenerate fibre splits. Then, if we solve the singularities of \(W\) by blowings up, we get a smooth conic bundle over \(S\) which is birationally equivalent to \((W, \tau, S)\).

**Definition (3.4).** Let \((W, \tau, S)\) be a conic bundle; the set of the points \(t \in S\) such that the fibre \(\tau^{-1}(t)\) is a degenerate conic is called the discriminant locus of the conic bundle. It can be shown (see [Sa], p. 358) that it is always a divisor of \(S\); from now on we will refer to it as the discriminant divisor \(D_w\) of \((W, \tau, S)\).

**Definition (3.5).** A smooth conic bundle \((W, \tau, S)\) is called standard if for every curve \(C\) of \(S\), the surface \(\tau^{-1}(C)\) is irreducible.

**Proposition (3.6)** (see [Sa], p. 366-367, see also [A-B2] prop. (2.6)). Let \((W, \tau, S)\) be a smooth conic bundle, such that \(D_w\) is the disjoint union of smooth curves \(D_i\), \(i = 1, 2 \ldots n\); if \(\tau^{-1}(D_i)\), for instance, is reducible then necessarily \(D_1 \cap (D_w - D_1)\) is empty and we can blow down one of the two components of \(\tau^{-1}(D_1)\) to obtain a new smooth conic bundle, birationally equivalent to \((W, \tau, S)\), whose \(D\) is \(D_2 \cup D_2 \cup \ldots D_n\). We can repeat this process until to obtain a smooth standard conic bundle birationally equivalent to \((W, \tau, S)\).

**Theorem (3.7)** (see [Ia], p. 742). Let \((W, \tau, S)\) be a smooth, standard, conic bundle, let \(S\) be a rational surface, let \(D_w\) be a curve. Then \(W\) is rational if there exists a pencil of rational curves \(C_t\) on \(S\), \((t \in \mathbb{P}^1)\), without fixed components, such that \(C_t \cdot D_w < 3 \ \forall t\).

Now we consider the conic bundle structures of \(X\) and \(Y\).

It is well known that every quartic hypersurface in \(\mathbb{P}^4\) with a double line has a conic bundle structure (see [C-M]): we fix the plane \(\pi\) whose equations are: \(z_1 = z_2 = 0\); it is skew with \(L_1\). If we project \(Y\) from \(L_1\) to \(\pi\) we have that the fibre over a point of \(\pi\) is a quartic plane curve which splits into \(L_1\), counted twice, and into another conic; if we blow up \(Y\) along \(L_1\) we get a smooth conic bundle according to definition (3.1).
Now we want to determine $D_r$. The generic point of the plane containing a point $(0:0:z_3:z_4:z_5)$ of $\pi$ and $L_1$, has coordinates $(h:k:tx_3:tx_4:tx_5)$; the intersection between $Y$ and this plane is the following plane quartic (where $F = F(z_3:z_4)$ etc.):

$$t^2[(ez_4^2 + z_3F + z_4H)h^2 - (z_4N + P)hk + Rhk^2 -$$

$$- (2ez_3z_4 + z_3z_4H + z_4^2L + z_4z_5M)ht + (z_3z_4N - z_4Q)kt +$$

$$+ (ez_3^2 + z_3^2G - z_3z_4L)] = 0;$$

$t^2 = 0$ gives $L_0$ counted twice, the remaining curve is a conic; it is degenerated if and only if:

$$(3.8) \quad z_4^2(4R(4z_4^2 + z_3F + z_4H)(e^2 + z_3G - z_3L) -$$

$$- (z_4N + P)(z_3N - Q)(-2ez_3z_4 + z_3H + z_4L + z_5M) -$$

$$- R(-2ez_3z_4 - z_3H + z_4L + z_5M)^2 - (z_3N - Q)^2(e^2 + z_3F + z_4H) -$$

$$- (z_4N + P)^2(e^2 + z_3G - z_3L) = 0.$$

Therefore $D_r$ splits into the line $z_4 = 0$ counted twice (whose existence is an obvious consequence of the double lines $L_1$ and $L_3$ in $Y$) and into a sestic $\Gamma$; we remark that the existence of a double line in $D_r$ makes very difficult to apply all known theorems about the rationality of the conic bundles.

Now let us consider $V = X \cap Q$, as $\Phi(X) = Y$ we have that $V$ is birational to $Y$. $V$ has a conic bundle structure too; it is well known (see [C], [A-B1]): we fix the plane $\pi'$, whose equations are $x_0 = x_1 = x_2 = 0$; we project $V$ from $P_1$ to $\pi'$; by blowing up $V$ along $P_1$ and at the ordinary double points which $V$ has on $P_2$ (see [A-B1]) we get a smooth conic bundle.

Let us determine $D_r$: the generic point of the plane containing a point $(0:0:0:x_3:x_4:x_5)$ of $\pi'$ and $P_1$ has coordinates $(\alpha:\beta:\gamma:\delta x_3:\delta x_4:\delta x_5)$; this point belongs to $V$ if and only if:

$$e\delta^2 + \beta^2\delta F + \gamma^2\delta G + \alpha\beta\delta H + \alpha\gamma\delta L + \beta\gamma\delta M + \alpha\delta^2 x_3N +$$

$$+ \beta\delta^2 P + \gamma\delta^2 Q + \delta^2 x_5R = 0$$

and

$$\alpha\delta x_3 - \beta\delta x_4 + \gamma\delta x_5 = 0.$$
\[ \delta = 0 \] gives the plane \( P_1 \); if we delete \( \delta \) we obtain a conic, it is easy to see ([A-B₁]) that the conic is degenerate if and only if:

\[(3.9) \quad x_3[4R(ex^2_4 + x_5F + x_4H)(ex^2_3 + x_6G - x_3L) - \]
\[ - (x_4N + P)(x_3N - Q)(- 2ex_3x_4 - x_3H + x_4L + x_5M) - \]
\[ - R(- 2ex_3x_4 - x_3H + x_4L + x_5M)^2 - (x_3N - Q)^2(ex^2_4 + x_3F + x_4H) - \]
\[ - (x_4N + P)^2(ex^2_3 + x_6G - x_3L) = 0 \]

where \( F = F(x_3;x_4:x_5) \) etc.

Therefore \( D_r \) splits into the line \( x_3 = 0 \) and into a smooth plane sextic \( I \) (see [A-B₁] and [A-B₂]); it is exactly the same curve into which \( D_r \) splits, in fact if we look at (3.8) and (3.9) and if we put \( x_i = z_i, i = 3, 4, 5 \) we see that the two curves are the same curve.

4. - The main results.

Now we want to prove this:

**Proposition (4.1).** The generic quartic hypersurface of \( \mathbb{P}^4 \) containing two skew double lines is not rational.

As the set of the generic quartic hypersurfaces of \( \mathbb{P}^4 \), containing two skew double lines and a third simple skew line, (not two of them belonging to the same hyperplane), is a closed Zarisky set of the moduli space of all quartic hypersurfaces of \( \mathbb{P}^4 \), to prove (4.1) it suffices to prove the following:

**Proposition (4.2).** The generic quartic hypersurface of \( \mathbb{P}^4 \), containing two skew double lines and a third simple skew line, not two of them belonging to the same hyperplane, is not rational.

**Proof.** By (2.1) it suffices to show that \( Y \) is not rational. By the previous section we have seen that \( Y \) is birational to \( V \) which is a cubic complex containing two planes only, meeting two by two at one point; therefore it is not rational (see [A-B₁] and [A-R]). □

Now we want to study the rationality of the generic quartic hypersurface of \( \mathbb{P}^4 \) with two skew double lines when it contains some plane;
as we have seen this problem is equivalent to study the rationality of
the generic $Y$ containing some plane.

If $Y$ contains a plane which is skew with $L_1$ (or $L_2$) it is rational;
in fact every line intersecting $L_i$ and the plane cuts $Y$ in one other
point only, so that it is not difficult to see that in this case $Y$ is bi-
rational to $\mathbb{P}^2 \times \mathbb{P}^1$. Therefore we can suppose that every plane con-
tained in $Y$ is incident with both double lines, or it is a $\lambda$-plane or a
$\mu$-plane or it is $z_4 = z_5 = 0$ or $z_2 = z_4 = 0$.

We have this:

**Proposition (4.3).** If $Y$ contains some plane incident to both double
lines or containing one of them, then it is rational (or reducible) save
when it contains at most one plane incident with $L_1$ and $L_2$ and all
$\lambda$-planes and $\mu$-planes allowed by (2.6).

Before proving (4.3) we need

**Lemma (4.4).** If $Y$ contains one plane only, intersecting $L_1$ and $L_2$ 
but not intersecting $L_3$, then $Y$ is not rational.

**Proof.** – Let us call $\mathfrak{p}$ this plane. If $\mathfrak{p}$ belongs to the hyperplane
generated by $L_1$ and $L_2$ (i.e. $z_4 = 0$), then $\Phi(\mathfrak{p})$ is $P_0$ and $V$ is a
cubic complex containing the three planes which are the base locus
of $\Phi^{-1}$, therefore $Y$ is reducible, (see also (2.5)).

In the other cases, by a suitable choice of coordinate system, we
can always suppose that $\mathfrak{p}$ has equations:

1) $z_3 = z_4 - z_1 = 0$,
2) $z_4 - z_1 = z_5 - z_3 = 0$,
3) $z_3 = z_4 - z_1 + z_2 = 0$,
4) $z_5 - z_3 = z_4 - z_1 + z_2 = 0$.

Then $\Phi(\mathfrak{p})$ has equations:

1) $x_1 = x_3 = x_0 x_2 - x_1 x_4 + x_2 x_3 = 0$,
2) $x_1 = x_3 + x_4 - x_5 = x_0 x_3 - x_1 x_4 + x_2 x_5 = 0$,
3) $x_1 + x_5 = x_3 = x_0 x_5 - x_1 x_4 + x_2 x_3 = 0$,
4) $x_1 + x_5 = x_3 + x_4 - x_2 = x_0 x_5 - x_1 x_4 + x_2 x_3 = 0$.

In the cases 1) and 3) $\Phi(\mathfrak{p})$ splits into a couple of planes and $V$
is a cubic complex containing four planes. It is easy to see that this is the case (4, 3, 1) of table R of [A-B₂], therefore $V$ is not rational.

In the cases 2) and 4) $\Phi(p)$ is a smooth quadric cutting a line on $P_1$ and a line on $P_2$, both passing through $P_1 \cap P_2$. This configuration in $V$ is obtained as follows: by choosing two points $A, B$ in $\mathbb{P}^n$ and two skew lines $\alpha, \beta$ passing through $A$ and $B$ respectively; by considering the two stars of lines centered in $A$ and in $B$ and the lines intersecting both $\alpha$ and $\beta$. If we move $\alpha$ until it cuts $\beta$ in a third distinct point $C$ we get a cubic complex $V$ containing four planes (the three stars of lines centered in $A, B, C$ and the lines of the plane through $A, B, C$) with the previously considered configuration. It is easy to see that this degeneration is flat so that $V$ is not rational as in the previous cases.

**Proof of (4.3).** Let us suppose that $Y$ contains only one plane $p$ intersecting $L_1, L_2, L_3$; by (2.5) $\Phi(p)$ is a plane in $V$, meeting $P_1$ and $P_2$ at one point only, so that $Y$ is birational to a cubic complex containing three planes two by two meeting at one point only (and no other planes), such complex is not rational (see [A-R] and [A-B₁]).

Let us suppose that $Y$ contains only one plane intersecting $L_1, L_2$ but not intersecting $L_3$: $Y$ is not rational by lemma (4.4).

Now it is easy to see that if we suppose that $Y$ contains two planes intersecting $L_1, L_2, L_3$, or two planes intersecting $L_1, L_2$, or one plane of the first type and one plane of the second type, we get that $V$ is a singular conic bundle over $\mathbb{P}^2$ birationally equivalent to a smooth standard conic bundle $W$ over a rational surface $S$, such that $D_W$ is the pull back of a smooth plane quartic by blowings up; (for the second type we can use a degeneration argument as in the proof of lemma (4.4)).

$V$ is rational by theorem (3.7): it suffices to consider a pencil of lines of $\mathbb{P}^2$ (through a point not belonging to the quartic) and its transformed on $S$ by the blowings up.

Finally we have only to remark that the existence in $Y$ of any plane $p$ quoted in (2.6) does not change the conic bundle structure of $V$; in fact in all these cases $V$ is irreducible, with ordinary double points only, $\Phi(p)$ is a line or a point (see (2.6)) and when we project $V$ from $P_1$ to $\pi'$ we see that $D_\pi$ is the same divisor (a smooth curve plus one or two lines) arising when $Y$ does not contain any plane of this type; this last fact is easy checked by looking directly at (3.8) or (3.9) and by recalling the conditions imposed on $Y$ by the existence of a plane of this type (see (2.6)).

REFERENCES


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