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Pseudo-Closure-Operators.

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In the algebra of R -modules closure operators are used frequently; their common foundation is based as follows: let M be an R -module, $L(M)$ the complete lattice of the sub-modules of M , $\varphi: L(M) \rightarrow L(M)$ a mapping, written as

$$\varphi(N) = N^c \quad (N \in L(M)),$$

such that

- (i) $N_1 \subseteq N_2 \Rightarrow N_1^c \subseteq N_2^c$;
- (ii) $N \subseteq N^c$;
- (iii) $(N^c)^c = N^c$,

then φ is called a *closure operator* on the lattice $L(M)$. The submodule $N \in L(M)$ is *φ -closed* if $N = N^c$.

A majority of the closure operators on $L(M)$ can be defined by means of a Gabriel filter \mathcal{F} on the ring R ; the filter \mathcal{F} defines a hereditary torsion functor $\tau = \tau_{\mathcal{F}}$, such that

$$\tau_{\mathcal{F}}(M) = \{m \in M : \text{Ann}_R(m) \in \mathcal{F}\}$$

is the $\tau_{\mathcal{F}}$ -torsion submodule of M . Any Gabriel filter \mathcal{F} on the ring R induces on the lattice $L(M)$ a *closure operator*, defined by

$$N \mapsto N^c = \{m \in M : (N; m) \in \mathcal{F}\}, \quad N \in L(M).$$

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Then N^c is a submodule of M , the $\tau_{\mathcal{F}}$ -closure (or the $\tau_{\mathcal{F}}$ -saturation) of $N \in L(M)$. The submodule $N \in L(M)$ is $\tau_{\mathcal{F}}$ -closed, if $N = N^c$, i.e. if and only if $\tau_{\mathcal{F}}(M/N) = 0$.

The family of the $\tau_{\mathcal{F}}$ -closed submodules N of M is denoted by

$$\text{Sat}_{\mathcal{F}}(M) = \{N \in L(M) : N = N^c\} \quad (1).$$

We find in the literature (see: Y. Miyashita [1]) a *pseudo-closure operator*, defined on the set $\Sigma(M)$ of all subsets of an R -module M , properly containing the element $\{0\} \in M$. For the subset $S \in \Sigma(M)$ we define the *pseudo-closure* S^c of the subset $S \in \Sigma(M)$ by

$$(1) \quad S \mapsto S^c = \{0\} \cup \{m \in M : Rm \cap S \neq \emptyset\},$$

and we set $0^c = 0$. Then we have for subsets $S, T \in \Sigma(M)$:

$$(i) \quad S \subseteq T \Rightarrow S^c \subseteq T^c;$$

$$(ii) \quad S \subseteq S^c;$$

$$(iii) \quad (S^c)^c = S^c.$$

If S is a subset of M , then S^c is called the *pseudo-closure* of S , and S will be called *pseudo-closed*, if $S = S^c$.

If N is a submodule of the R -module M , then the pseudo-closure N^c of the submodule N is—in general—not a submodule of M .

EXAMPLE. — Let $R = Z$, $M = Z \oplus Z/2Z$, $N = Z(2; \bar{0})$; then $N_1 = Z(1; \bar{0})$ and $N_2 = Z(1; \bar{1})$ are maximal essential extensions of N in M . We have $N_1 \subseteq N^c$, $N_2 \subseteq N^c$, but N^c is *not* a submodule of N ; for $(1; \bar{0}) \in N_1 \subseteq N^c$, $(1; \bar{1}) \in N_2 \subseteq N^c$, but $(0; \bar{1}) = (1; \bar{1}) - (1; \bar{0}) \notin N^c$, since $Z(0; \bar{1}) \cap Z(2; \bar{0}) = (0; \bar{0})$.

THEOREM 1. *If $N \in L(M)$ then we have:*

(i) *the pseudo-closure N^c of N in M contains every essential extension of N in M ;*

(ii) *if in particular N^c is a submodule of M , then N^c is the unique maximal essential extension of N in M ;*

(1) See e.g. B. Stenström [2], p. 207.

(iii) if N^c is the unique maximal essential extension of N in M for every submodule N of M , then M has the «intersection property» for essentially closed submodules of M (i.e. the intersection of any collection of essentially closed submodules of M is essentially closed in M).

PROOF. (i) is a consequence of the definition (1).

(ii) If N^c is a submodule of M , then N^c cannot have a proper essential extension in M : i.e. then N^c is essentially closed in M . That implies that N^c is a complement of some submodule $N' \subseteq M$, and we may choose N' in such a way, that N' contains an element $0 \neq m_0 \in M \setminus N^c$. Then $Rm_0 \cap N^c = 0$ implies that $Rm_0 \cap N = 0$, i.e. $m_0 \notin N^c$, and thus N^c is the unique maximal essential extension of N in M .

(iii) If N^c is a submodule of M for all submodules $N \subseteq M$, then any submodule N of M has a unique maximal essential extension $\bar{N} = N^c$ in M , and that implies that M has the «intersection property» for essentially closed submodules of M .

THEOREM 2. *If the R -module M satisfies the «intersection property» for essentially closed submodules, then for every submodule $N \subseteq M$ the pseudo-closure N^c is the unique maximal essential extension of N in M .*

Indeed, in this situation N^c is the (unique) intersection of all essentially closed submodules of M containing N .

THEOREM 3. *Let R be a left Ore domain, M a torsionfree R -module, and $N \neq 0$ a submodule of M ; then:*

- (i) N^c is the unique maximal essential extension of N in M ;
- (ii) M has the «intersection property» for essentially closed submodules.

PROOF. If $0 \neq m \in N^c$, then for some $0 \neq r \in R$ we have $0 \neq rm \in N$. If $0 \neq r' \in R$, $Rr \cap Rr' \neq 0$ implies that $0 \neq r''r = r'''r'$ for some $0 \neq r'' \in R$, $0 \neq r''' \in R$. Then $0 \neq r''rm = r'''r'm \in N$, and, using the torsion-freeness of M , we have $0 \neq rm \in N^c$.

If m_1, m_2 are in N^c , then $0 \neq r_1m_1 \in N$, $0 \neq r_2m_2 \in N$ for some

$0 \neq r_1, r_2 \in R$. Then $Rr_1 \cap Rr_2 \neq 0$ implies that

$$0 \neq \varrho_1 r_1 = \varrho_2 r_2$$

for some $\varrho_1, \varrho_2 \in R$. Therefore

$$0 \neq \varrho_1 r_1 (m_1 + m_2) = \varrho_1 r_1 m_1 + \varrho_2 r_2 m_2 \in N;$$

i.e. $m_1 + m_2 \in N^c$. Thus N^c contains with any $m \neq 0$ also rm , and with $m_1, m_2 \in N^c$ also $m_1 + m_2 \in N^c$. Thus N^c is a submodule of M , and (by theorem 1 (ii)), this implies (i) and (ii).

COROLLARY 4. *If R is a commutative integral domain, $N \neq 0$ a submodule of the torsionfree R -module M , then the pseudo-closure N^c of N is a submodule of M , and M has the «intersection property» for essentially closed submodules.*

From the theorems 1 and 2 it follows that the «intersection property» for essentially closed submodules of M is a necessary and sufficient condition therefore that the pseudo-closure N^c of any submodule N of M is a submodule of M .

We will give some other examples of sufficient conditions for R (resp. M) in order that M has the «intersection property» for essentially closed submodules of M . Therefore we define:

DEFINITION (β). Let N be a submodule of M ; then the pair $(N; M)$ satisfies the condition (β), if 1_M is the only R -automorphism of M inducing 1_N .

The condition (β) of the pair $(N; M)$ is equivalent with each of the following conditions:

(β') every R -endomorphism f of N has at most one extension f on M ;

(β'') $\text{Hom}_R(M/N; M) = 0$.

We give some examples:

1) If N is an essential submodule of the non-singular R -module M , then M/N is singular, i.e. $\text{Hom}(M/N; M) = 0$, and the pair $(N; M)$ satisfies (β).

2) If M is a rational extension of the submodule N , then $\text{Hom}_R(M/N; M) = 0$, i.e. the pair $(N; M)$ satisfies (β) .

The following result proves that the property (β) —in a special sense—is a sufficient condition therefore that the pseudo-closure N^c of a submodule N of M is a submodule of M .

THEOREM 5. *Let $N \neq 0$ be a submodule of M , \hat{M} an injective hull of M ; if we assume that the pair $(N; \hat{M})$ satisfies the condition (β) , then:*

- (i) N has a unique maximal essential extension in M ;
- (ii) M has the « intersection property » for essentially closed submodules of M if the condition (β) holds for any pair $(N; \hat{M})$;
- (iii) the pseudo-closure N^c of a submodule N is a submodule of M .

PROOF. Let N_1, N_2 be two maximal essential extensions of N in M , and \hat{N}_1 , resp. \hat{N}_2 the corresponding injective hulls of N in \hat{M} . Then $N_i = M \cap \hat{N}_i$ ($i = 1, 2$). Furthermore there exists an isomorphism $\varphi: \hat{N}_1 \cong \hat{N}_2$, $\varphi(n) = n$ ($\forall n \in N$). Let N^* be an injective hull of a complement of N in M . Then $\hat{M} = \hat{N}_1 \oplus N^* \cong \hat{N}_2 \oplus N^*$. Define: $\alpha \in \text{End}_R(\hat{M})$ by $\alpha(\hat{N}_1) = \varphi(\hat{N}_1) = \hat{N}_2$, $\alpha(n^*) = n^*$ ($\forall n^* \in N^*$). Then α is an R -automorphism of \hat{M} , inducing 1_N . Since the pair $(N; \hat{M})$ satisfies the condition (β) , we have $\alpha = 1_{\hat{M}}$, i.e. $\hat{N}_1 = \hat{N}_2$, and therefore $N_1 = \hat{N}_1 \cap M = \hat{N}_2 \cap M = N_2$. Hence any submodule N of M has a unique maximal essential extension in M . Then the proof of (ii) and of (iii) follows from theorem 1.

LITERATURE

- [1] Y. MIYASHITA, *On quasi-injective modules*, Journ. of the Fac. of Sc. Hokkaido University, Ser. 1, Math., **18** (1965), pp. 158-187.
- [2] B. STENSTRÖM, *Rings of Quotients*, Springer, 1975.