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Fully Rigid Systems of Modules.

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Let R be a commutative ring, A an R-algebra (both with 1).

DEFINITION. A *fully rigid system for A on* a set I is a family $G^X(X \subseteq I)$ of faithful right A-modules indexed by the subsets X of I such that, for $X, Y \subseteq I$,

$$G^X \leqslant G^Y \qquad (X \subseteq Y),$$

$$\operatorname{Hom}_R(G^X, G^Y) = \begin{cases} A & (X \subseteq Y), \\ 0 & (X \nsubseteq Y), \end{cases}$$

where of course A acts on G^X by scalar multiplication: in particular we have $\operatorname{End}_R(G^X) = A \ (X \subseteq I)$.

Clearly, if I is infinite then a fully rigid system $G^X(X \subseteq I)$ will contain a rigid system of size $2^{|I|}$: choose $2^{|I|}$ pairwise incomparable subsets of I.

REMARK 1. If $\varphi \colon G^X \to G^Y$ is a nonzero homomorphism between members of a fully rigid system for A, then $X \subseteq Y$ and there exists an element $a \in A$ such that $\varphi \colon g \mapsto ga \ (g \in G^X)$. Since G^X is a sub-A-module of G^Y it is immediate that φ maps G^X into itself, and by faithfulness a is completely determined by the restriction $\varphi \restriction G^\emptyset$.

THEOREM. *Suppose that A admits a fully rigid system H^X $(X \subseteq I)$ on a set I with $|I| \geqslant 5$. Then, for every infinite cardinal $\lambda \geqslant |H^I|$, A admits a fully rigid system G^X $(X \subseteq \lambda)$ on λ with $|G^X| = \lambda$ $(X \subseteq \lambda)$.*

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We start by reducing the theorem to the following proposition, which is based on a construction to be found in the second section of Shelah's remarkable paper [7].

(Note that all our tensor products are over $\mathbb{Z}$).

PROPOSITION. *Let F be a free abelian group of arbitrary infinite rank λ. Then there exist direct summands U_i^X $(i = 0, 1, 2, 3, 4;\ X \subseteq \lambda)$ of F such that $U_i^X \leqslant U_i^Y$ whenever $X \subseteq Y$, and with the property that for any commutative ring R and any R-module H*

$$0 \neq \theta \in \mathrm{End}_R(F \otimes H) \text{ and } (U_i^X \otimes H)\theta \leqslant U_i^Y \otimes H \ (i = 0, \ldots, 4) \Rightarrow$$
$$\Rightarrow X \subseteq Y \text{ and } \theta = \mathrm{id}_F \otimes \varphi \text{ for some } \varphi \in \mathrm{End}_R(H).$$

PROP $\Rightarrow$ THM. We may assume without loss that $I = \{0, \ldots, 4\}$. Choose F free abelian of rank $\lambda \geqslant |H^I|$, construct the A-module $F \otimes H^I$, and for each $X \subseteq \lambda$ take the sub-A-module

$$G^X = F \otimes H^\emptyset + \sum_{i \in I} U_i^X \otimes H^{\{i\}}.$$

This contains $F \otimes H^\emptyset$, a direct sum of λ copies of the faithful $H^\emptyset$. Therefore G^X is a faithful A-module with $|G^X| = \lambda$, and certainly $X \subseteq Y \Rightarrow U_i^X \leqslant U_i^Y \ (i \in I) \Rightarrow G^X \leqslant G^Y$.

Now suppose that $0 \neq \theta \in \mathrm{Hom}_R(G^X, G^Y)$. Since $F \otimes H^I$ is a direct sum of copies of H^I, the image of any R-homomorphism $H^\emptyset \to F \otimes H^I$ must lie in $F \otimes H^\emptyset$, and it follows that the composite $F \otimes H^\emptyset \hookrightarrow G^X \xrightarrow{\theta} G^Y \hookrightarrow F \otimes H^I$ must map $F \otimes H^\emptyset$ into itself. In other words,

$$\theta \restriction F \otimes H^\emptyset \in \mathrm{End}_R(F \otimes H^\emptyset).$$

Again, if we choose a direct complement F_i^Y for U_i^Y in F, then for $i \neq j$ in I we have $U_j^Y \otimes H^{\{j\}} \leqslant F \otimes H^{I \setminus \{i\}}$ and $U_i^Y \otimes H^{\{i\}} \leqslant U_i^Y \otimes H^I$, whence $G^Y \leqslant (U_i^Y \otimes H^I) \oplus (F_i^Y \otimes H^{I \setminus \{i\}})$. Since $\{i\} \nsubseteq I \setminus \{i\}$, the composite

$$U_i^X \otimes H^{\{i\}} \hookrightarrow G^X \xrightarrow{\theta} G^Y \hookrightarrow (U_i^Y \otimes H^I) \oplus (F_i^Y \otimes H^{I \setminus \{i\}}) \xrightarrow{\text{proj}} F_i^Y \otimes H^{I \setminus \{i\}}$$

must vanish. Therefore θ maps $U_i^X \otimes H^{\{i\}}$ into $U_i^Y \otimes H^I$, and necessarily into $U_i^Y \otimes H^{\{i\}}$. But then $\theta \restriction F \otimes H^\emptyset$ maps $U_i^X \otimes H^\emptyset$ into $U_i^Y \otimes H^\emptyset$ $(i \in I)$, whence $\theta \restriction F \otimes H^\emptyset = \mathrm{id}_F \otimes \varphi$ for some $\varphi \in \mathrm{End}_R(H^\emptyset) = A$. This means that θ agrees on $F \otimes H^\emptyset$ with scalar multiplication by some $a \in A$, where $a = 0$ unless $X \subseteq Y$. Remark 1 now implies that, regarded as a homomorphism $G^X \to F \otimes H^I$, θ agrees with scalar multiplication by a. Since $\theta \neq 0$, we have $a \neq 0$; therefore $X \subseteq Y$. /

REMARK 2. If $H^I \in \mathcal{C}$, where $\mathcal{C}$ is a class of modules closed under arbitrary direct sums and submodules, then also each $G^X \in \mathcal{C}$ $(X \subseteq \lambda)$. In particular, if $R = \mathbb{Z}$ (or any PID) and H^I is $\aleph_\alpha$-free, then all the G^X are $\aleph_\alpha$-free. Similarly for slenderness, cotorsion-freeness, and so on.

REMARK 3. For a topological R-algebra (with R discrete) call an A-module H *topologically faithful* if the mapping $a \mapsto$ scalar multiplication by a is a topological embedding $A \to \mathrm{End}_R(H)$, where the endomorphism algebra is taken in its finite topology. If H is topologically faithful, so is any direct sum of copies of H, in particular $F \otimes H$. And since

$$\{\mathrm{Ann}_A(h) : h \in F \otimes H^\emptyset\} \subseteq \{\mathrm{Ann}_A(h) : h \in G^X\} \subseteq \{\mathrm{Ann}_A(h) : h \in F \otimes H^I\},$$

it follows that *if $H^\emptyset$ and H^I are topologically faithful, then so are all the G^X $(X \subseteq \lambda)$*. In this case the algebra identifications $\mathrm{End}_R(G^X) = A$ are topological.

The following technical lemma simplifies the proof of the proposition.

LEMMA α. *Let F be a free abelian group with direct summands $U_i \leqslant U_i^* \leqslant F$ $(i \in I)$ such that for any commutative ring R with 1*

$$(Hyp) \quad \theta \in \mathrm{End}_R(F \otimes R) \text{ and } (U_i \otimes R)\theta \leqslant U_i^* \otimes R \ (i \in I) \Rightarrow$$
$$\Rightarrow \theta \text{ is scalar multiplication by some } a \in R.$$

Then for any ring R and any R-modules N, M,

$$(Con) \quad \theta \in \mathrm{Hom}_R(F \otimes N, F \otimes M) \text{ and } (U_i \otimes N)\theta \leqslant U_i^* \otimes M \ (i \in I) \Rightarrow$$
$$\Rightarrow \theta = \mathrm{id}_F \otimes \varphi \text{ for some } \varphi \in \mathrm{Hom}_R(N, M).$$

PROOF. An R-homomorphism is no more than a $\mathbb{Z}$-homomorphism which commutes with scalar multiplications from R. Therefore it is enough to establish (Con) with $R = \mathbb{Z}$. And we need only prove (Con) with $N = \mathbb{Z}_{\mathbb{Z}}$. For if (Con) holds in this special case, consider any

$\mathbb{Z}$-homomorphism $\theta: F \otimes N \to F \otimes M$ mapping $U_i \otimes N$ into $U_i^* \otimes M$ for all i. Then for each $n \in N$, the map $f \otimes 1 \to (f \otimes n)\theta$ is a homomorphism $F \otimes \mathbb{Z} \to F \otimes M$ mapping $U_i \otimes \mathbb{Z}$ into $U_i^* \otimes M$ for all i, so must be of the form $f \otimes 1 \to f \otimes (n\varphi)$ for some $n\varphi \in M$. Here $n\varphi$ is clearly unique, and it follows that $\varphi: N \to M$ is a homomorphism such that $\theta = \mathrm{id}_F \otimes \varphi$.

Given a $\mathbb{Z}$-module M, take R to be what Nagata [6] felicitously calls the *idealisation* of M, viz. R is the additive group $\mathbb{Z} \oplus M$ with the multiplication $(r, x)(s, y) = (rs, ry + sx)$. Then R is a commutative ring with « one » (1,0), and with the usual identifications M is an ideal of R whose square vanishes. Any $\mathbb{Z}$-homomorphism $\theta: F \otimes \mathbb{Z} \to F \otimes M$ mapping $U_i \otimes \mathbb{Z}$ into $U_i^* \otimes M$ $(i \in I)$ extends to a $\mathbb{Z}$-endomorphism $\bar{\theta}$ of $F \otimes R = (F \otimes \mathbb{Z}) \oplus (F \otimes M)$ vanishing on $F \otimes M$. This $\bar{\theta}$ is obviously an R-endomorphism of $F \otimes R$ mapping $U_i \otimes R$ into $U_i^* \otimes R$ $(i \in I)$, so by (Hyp) it is scalar multiplication by some element $(r, m) \in R$, in other words

$$(0, (f \otimes 1)\theta) = (f \otimes 1, 0)\bar{\theta} = (f \otimes 1, 0)(r, m) = (f \otimes r, f \otimes m) .$$

This implies that ($r = 0$ and) $\theta = \mathrm{id}_F \otimes \varphi$ where $\varphi: \mathbb{Z} \to M$ is the homomorphism mapping $1 \mapsto m$. /

DEFINITION. Let F be a free $\mathbb{Z}$-module of infinite rank. A *starch* (1) for F shall be a system of direct summands of F,

$$U;\ U_i,\ U_i^* \quad (i = 0, \ldots, 4) ,$$

satisfying the condition (Hyp) of Lemma α and such that

$$\mathrm{rk}(U) = \mathrm{rk}(F) ,$$

$$U_i^* = U_i \oplus \delta_{0i} U \quad (i = 0, \ldots, 4) .$$

The last of these conditions requires that $U_0 \cap U = 0$ and that $U_0 \oplus U$ be a direct summand of F.

We now reduce the proposition to

(1) I am indebted to Claudia Metelli for proposing this crisply apt solution to a problem of nomenclature.

PROPOSITION β. *Every free $\mathbb{Z}$-module of infinite rank admits a starch.*

PROP. $\beta \Rightarrow$ PROP. Let U; U_i, $U_i^* = U_i \oplus \delta_{0i} U$ $(i = 0, \dots, 4)$ be a starch on a free $\mathbb{Z}$-module F of infinite rank λ. Choose a free basis v_α $(\alpha \in \lambda)$ of U, and for each subset $X \subseteq \lambda$ take

$$U^X = \bigoplus_{\alpha \in X} v_\alpha \mathbb{Z} ,$$

$$U_i^X = U_i \oplus \delta_{0i} U^X \qquad (i = 0, \dots, 4) .$$

Certainly these are direct summands of F, and $X \subseteq Y \Rightarrow U^X \leqslant U^Y \Rightarrow \Rightarrow U_i^X \leqslant U_i^Y$ $(i = 0, \dots, 4)$.

Given a commutative ring R, consider any R-homomorphism θ of $F \otimes R$ mapping $U_i^X \otimes R$ into $U_i^Y \otimes R$ for each I. Then θ maps $U_i \otimes R$ into $U_i^* \otimes R$ for each i, and the definition of a starch implies that θ is scalar multiplication by some $a \in R$. Therefore θ leaves the sub-R-module $U \otimes R$ invariant, and it follows that it carries $(U \otimes R) \cap \cap (U_0^X \otimes R) = U^X \otimes R$ into $(U \otimes R) \cap (U_0^Y \otimes R) = U^Y \otimes R$. In particular for each $\alpha \in X$ we have

$$v_\alpha \otimes a = (v_\alpha \otimes 1)\theta \in U^Y \otimes R = \bigoplus_{\beta \in Y} (v_\beta \otimes 1) R .$$

Since $v_\alpha \otimes 1$ $(\alpha \in \lambda)$ is a free basis of the free R-module $U \otimes R$ this gives the conclusion that either $X \subseteq Y$ or $a = 0$; and if $a = 0$, then of course $\theta = 0$. By a trivial extension of Lemma α we obtain the proposition. /

The proof of Proposition β makes heavy use of the following lemma, the essential content of which was first brought to my attention over twenty years ago by Sheila Brenner.

LEMMA γ. *In a free $\mathbb{Z}$-module F let V be a direct summand of rank 2 with a given ordered basis v_0, v_1. Let M' be a direct summand of an R-module M, and $\tau\colon M' \to M$ an R-homomorphism. Write*

$$V[\tau] = M'(v_0 \otimes \mathrm{id} + v_1 \otimes \tau) = \{v_0 \otimes x + v_1 \otimes x\tau : x \in M'\} .$$

Then

(*a*) *$V[\tau]$ is a direct summand of $F \otimes M$ contained in $V \otimes M$;*

(*b*) *if* $\varphi \in \mathrm{End}_R(M)$ *is such that* $\mathrm{id}_F \otimes \varphi$ *leaves invariant* $V[\tau]$, *then* φ *leaves invariant* M' *and commutes with* τ *on* M'.

PROOF. (*a*) If M'' is a direct complement of M' in M, then $(v_0\mathbb{Z}\otimes M'')\oplus(v_1\mathbb{Z}\otimes M)$ is a direct complement of $V[\tau]$ in $V\otimes M$, which is itself a direct summand of $F\otimes M$.

(*b*) If $\mathrm{id}_F\otimes\varphi$ leaves invariant $V[\tau]$, then for every $x\in M'$ there exists $x'\in M'$ such that $v_0\otimes x\varphi + v_1\otimes x\tau\varphi = v_0\otimes x' + v_1\otimes x'\tau$, whence, equating coefficients, $x\varphi = x'\in M'$ and $x\tau\varphi = x'\tau = x\varphi\tau$. /

Note that in the situation of (*b*) above, if φ is known to leave a certain submodule N of M' invariant, then it must also leave $N\tau$ invariant: for then $N\tau\varphi = N\varphi\tau \leqslant N\tau$.

NOTATION. Given a set I we write F_I for a free $\mathbb{Z}$-module with a free basis f_α $(\alpha\in I)$ indexed by I; if J is a subset of I, then F_J will denote the obvious direct summand of F_I. For a commutative ring R, $F_I^R := F_I\otimes R$ is the free R-module with basis $f_\alpha^R = f_\alpha\otimes 1$ $(\alpha\in I)$, but we shall consistently abuse notation and identify $f_\alpha = f_\alpha\otimes 1$ so that f_α $(\alpha\in I)$ is also a basis of F_I^R. If we are given a partial function on I, i.e. a function $p\colon D\to I$ where D is a subset of I, we write $p^{\mathbb{Z}}\colon F_D\to F_I$ and $p^R\colon F_D^R\to F_I^R$ for the homomorphisms given on the « common basis » by $f_\alpha\mapsto f_{\alpha p}$ $(\alpha\in D)$; clearly then

$$p^R = p^{\mathbb{Z}}\otimes \mathrm{id}_R\,.$$

Now $p^{\mathbb{Z}}\colon F_D\to F_I$ is a homomorphism of the direct summand F_D of F_I into F_I, so with F, V as in Lemma γ, the « graph » $V[p^{\mathbb{Z}}]$ is a direct summand of $F\otimes F_I$ contained in $V\otimes F_I$. An obvious check shows that if we tensor with a commutative ring R, then

$$V[p^{\mathbb{Z}}]\otimes R = V[p^R] \quad \text{in } F\otimes F_I\otimes R = F\otimes F_I^R\,.$$

LEMMA δ. *F_ω admits a starch.*

PROOF. In fact we prove the equivalent assertion that $F_2\otimes F_\omega$ admits a starch. Let $s\colon\omega\to\omega$ be the successor function $k\mapsto k+1$, and in $F_2 \otimes F_\omega$ take the direct summands

$$V_0 = f_1\mathbf{Z} \otimes f_0\mathbf{Z}\,,$$
$$V = V_1 = f_0\mathbf{Z} \otimes F_\omega\,,$$
$$V_2 = f_1\mathbf{Z} \otimes F_\omega\,,$$
$$V_3 = (f_0 + f_1)\mathbf{Z} \otimes F_\omega\,,$$
$$V_4 = F_2[s^{\mathbf{Z}}]\,.$$

Clearly $V \cong F_\omega \cong F_2 \otimes F_\omega$, and $V_0 \oplus V$ is a direct summand of $F_2 \otimes F_\omega$. Write $V_i^* = V_i \oplus \delta_{0i} V$.

Consider any commutative ring R with 1, and any R-endomorphism θ of $F_2 \otimes F_\omega \otimes R = F_2 \otimes F_\omega^R$ which maps $V_i \otimes R$ into $V_i^* \otimes R$ for all i. Then θ leaves invariant

$$V_1 \otimes R = f_0\mathbf{Z} \otimes F_\omega^R, \quad V_2 \otimes R = f_1\mathbf{Z} \otimes F_\omega^R, \quad V_3 \otimes R = (f_0 + f_1)\mathbf{Z} \otimes F_\omega^R,$$

and a trivial calculation shows that $\theta = \mathrm{id}_{F_2} \otimes \varphi$ for some $\varphi \in \mathrm{End}_R(F^R)$. But $V_0 \leqslant V_2$, so $\theta = \mathrm{id}_{F_2} \otimes \varphi$ maps $V_0 \otimes R = (f_1 \otimes f_0)R$ into $(V_0^* \otimes R) \cap \cap (V_2 \otimes R) = V_0 \otimes R = (f_1 \otimes f_0)R$, i.e. $f_0\varphi = f_0 a$ for some $a \in R$. Finally, since $\mathrm{id}_{F_2} \otimes \varphi$ leaves invariant $V_4 \otimes R = F_2[s^R]$, Lemma γ implies that φ commutes with s^R: thus, if $f_k\varphi = f_k a$ for some $k < \omega$, then also $f_{k+1}\varphi = f_k s^R \varphi = f_k \varphi s^R = (f_k a) s^R = (f_k s^R) a = f_{k+1} a$. Hence by induction φ is scalar multiplication by a. Therefore so is $\theta = \mathrm{id}_{F_2} \otimes \varphi$. /

We come now to the proof of Proposition β - which is where Shelah's ideas come into play. We have just handled the countable case in Lemma δ, so let λ be an uncountable cardinal. Write

$$\varrho = \begin{cases} \omega & (\lambda \text{ regular } > \omega)\,, \\ \mathrm{cf}(\lambda) & (\lambda \text{ singular})\,. \end{cases}$$

We shall prove that if F_ϱ admits a starch, then so does $F_\varrho \otimes F_\lambda$. Now in either case $F_\varrho \otimes F_\lambda \cong F_\lambda$, so it will follow from Lemma δ that F_λ admits a starch whenever λ is regular, and then, as the cofinality of a singular cardinal is regular, that F_λ admits a starch whenever λ is singular; and Proposition β will be proved.

Assume then that F_ϱ admits a starch V; V_i, $V_i^* = V_i \oplus \delta_{0i} V$ $(i = 0, \ldots, 4)$. We shall choose a set Π of ϱ partial functions $p: D(p) \to \lambda$, where each domain $D(p) \subseteq \lambda$. Correspondingly we take a direct decomposition of V,

$$V = v_0 \mathbf{Z} \oplus \bigoplus_{p \in \Pi} V^p ,$$

where $v_0 \mathbf{Z}$ is of rank 1 and each V^p is of rank 2 with a fixed ordered basis v_0^p, v_1^p. In $F_\varrho \otimes F_\lambda$ take the direct summands

$$U = v_0 \mathbf{Z} \otimes F_\lambda ,$$

$$U_0 = (V_0 \otimes F_\lambda) \oplus \bigoplus_{p \in \Pi} V^p [p^{\mathbf{Z}}] ,$$

$$U_i = V_i \otimes F_\lambda \quad (i = 1, \ldots, 4) .$$

Obviously $U \simeq F_\lambda$. And since $V_0^* = V_0 \oplus v_0 \mathbf{Z} \oplus \bigoplus_{p \in \Pi} V^p$ is a direct summand of F_ϱ, it is clear from Lemma γ that the direct sum defining U_0 makes sense; and $U_0 \oplus U$ is a direct summand of $F_\varrho \otimes F_\lambda$. Moreover writing as usual $U_i^* = U_i \oplus \delta_{0i} U$, we have

$$V_i \otimes F_\lambda \leqslant U_i \leqslant U_i^* \leqslant V_i^* \otimes F_\lambda \qquad (i = 0, \ldots, 4) .$$

For the rest of the proof θ is an arbitrary R-endomorphism of $F_\varrho \otimes F_\lambda \otimes R = F_\varrho \otimes F_\lambda^R$ mapping $U_i \otimes R$ into $U_i^* \otimes R$ $(i = 0, \ldots, 4)$. The last display shows that θ maps $V_i \otimes F_\lambda^R$ into $V_i^* \otimes F_\lambda^R$ $(i = 0, \ldots, 4)$. Therefore by Lemma α

$$\theta = \mathrm{id}_{F_\varrho} \otimes \varphi \qquad \text{for some } \varphi \in \mathrm{End}_R(F_\lambda^R) .$$

Since $\mathrm{id}_{F_\varrho} \otimes \varphi$ certainly leaves each $V^p \otimes F_\lambda^R$ invariant, it must map the submodule

$$(V^p \otimes F_\lambda^R) \cap (U_0 \otimes R) = V^p [p^R] = (V^p \otimes F_\lambda^R) \cap (U_0^* \otimes R)$$

into itself. By Lemma γ this means that, for each $p \in \Pi$, *φ leaves $F_{D(p)}^R$ invariant and commutes thereon with p^R.* The proof now reduces to showing that it is possible to choose a set Π of ϱ partial functions

$p: D(p) \to \lambda$ on λ in such a way that the corresponding invariance-cum-commutation conditions force φ to be a scalar multiplication: and it is essentially this that Shelah has shown us how to do.

Let Λ be the set of all finite subsets of λ. Then $|\Lambda| = \lambda > \aleph_0 = |\omega| = |\Lambda \cap \lambda|$, so we may relabel the f_α $(\alpha \in \lambda \setminus (\Lambda \cap \lambda))$ as f_σ $(\sigma \in \Lambda \setminus (\Lambda \cap \lambda))$ to obtain a relabelling f_σ $(\sigma \in \Lambda)$ of the original basis f_α $(\alpha \in \lambda)$ of F_λ, whence an identification $F_\lambda = F_\Lambda$. For any nonempty $\sigma \in \Lambda$ write $\sigma_{\min}$ for the least element of σ, and $\sigma h = \sigma \setminus \{\sigma_{\min}\}$. Then, writing

$$\Lambda(\alpha) = \{\sigma \in \Lambda \colon \sigma \neq \emptyset \text{ and } \sigma_{\min} \geqslant \alpha\} \qquad (\alpha < \lambda)\,,$$

so that in particular $\Lambda(0)$ is the set of all nonempty finite subsets of λ, we note that the images of $\Lambda(\alpha)$ under the partial maps $h: \Lambda(0) \to \Lambda$, $\min: \Lambda(0) \to \lambda$ are

$$\Lambda(\alpha)h = \{\emptyset\} \cap \Lambda(\alpha+1)\,,$$

$$\Lambda(\alpha)_{\min} = \lambda \setminus \alpha = \{\xi \in \lambda \colon \xi \geqslant \alpha\}\,.$$

Hence

$$\Lambda(\alpha)h \cap \Lambda(0) = \Lambda(\alpha+1) \qquad (\alpha < \lambda)\,;$$

and certainly

$$\bigcap_{\alpha<\beta} \Lambda(\alpha) = \Lambda(\beta) \qquad \text{for a limit ordinal } \beta < \lambda\,.$$

Assume then that Π contains the partial functions h and min. Then, reverting to the considerations of the paragraph before last, we find that φ leaves $F^R_{\Lambda(0)}$ invariant and commutes thereon with h^R and $\min^R$. Thus if, for some $\alpha < \lambda$, φ leaves $F^R_{\Lambda(\alpha)}$ invariant, then it leaves invariant also $F^R_{\Lambda(\alpha)}h^R \cap F^R_{\Lambda(0)} = F^R_{\Lambda(\alpha)h \cap \Lambda(0)} = F^R_{\Lambda(\alpha+1)}$. Taking intersections at limit ordinals we deduce by induction that φ leaves invariant each $F^R_{\Lambda(\alpha)}$ $(\alpha < \lambda)$. Hence φ leaves invariant also the images $F^R_{\Lambda(\alpha)}\min^R = F^R_{\lambda\setminus\alpha} (\alpha < \lambda)$. This means that for each $\alpha < \lambda$ there is an expression of the form

$$f_\alpha \varphi = \sum_{i=0}^{m} f_{\alpha(i)}\, a_i$$

where the $a_i \in R$, and

$$\alpha = \alpha(0) < \alpha(1) < \ldots < \alpha(m) = \alpha^{\#} \text{ (say)} < \lambda$$

(and where everything in sight of course depends on α).

To exploit this, let

$$\Delta = \{\delta \in \lambda : cf(\delta) = \omega\} ,$$

and for each $\delta \in \Delta$ choose a strictly increasing sequence of ordinals δs_n $(n < \omega)$ such that

$$\delta s_0 = 0 \quad \text{and} \quad \sup_{n<\omega} \delta s_n = \delta .$$

We now distinguish the two cases.

(a) Assume first that λ is regular, so that $\varrho = \omega$. It is well known that Δ is stationary in λ, and by a theorem of Solovay (quoted as Theorem 85 on p. 433 of Jech [5]), any stationary subset of an uncountable regular cardinal λ may be partitioned into λ stationary subsets. Therefore there exists a function $z : \Delta \to \lambda$ such that

$$\gamma \in \lambda \Rightarrow \gamma z^{-1} \text{ is stationary in } \lambda .$$

Take Π to consist of

$$h : \Lambda(0) \to \Lambda , \quad \min : \Lambda(0) \to \lambda , \quad s_n : \Delta \to \lambda \ (n < \omega) , \quad z : \Delta \to \lambda .$$

With the notation introduced at the top of the page, for each $\beta < \lambda$ write

$$\beta^* = \sup \{\alpha^{\#} : \alpha < \beta\} .$$

Since λ is regular, we have $\beta \leqslant \beta^* < \lambda$; and the mapping $\beta \mapsto \beta^*$ is obviously continuous. So $\{\beta < \lambda : \beta^* = \beta\}$ is closed unbounded in λ, therefore for each $\gamma < \lambda$, this « club » meets the stationary set γz^{-1} ($\subseteq \Delta$). In other words, for each $\gamma \in \lambda$ there is an ordinal $\delta \in \Delta$ such that

$$\delta^* = \delta \quad \text{and} \quad \delta z = \gamma .$$

The first equation here means that $\alpha < \delta \Rightarrow \alpha\varphi < \delta$ or, equivalently, that *F_δ^R is invariant under φ.*

Write

$$f_\delta \varphi = \sum_{i=0}^{m} f_{\delta_i} a_i ,$$

where the $a_i \in R$, and $\delta = \delta_0 < \delta_1 < \dots < \delta_m$ $(< \lambda)$ are in Δ (because F_Δ^R is invariant under φ). Since φ commutes with each s_n^R on F_Δ^R, we have $f_\delta s_n^R \varphi = f_\delta \varphi s_n^R$, in other words

$$f_{\delta s_n} \varphi = \sum_{i=0}^{m} f_{\delta_i s_n} a_i .$$

But $\lim\limits_{n<\omega} \delta_i s_n = \delta_i > \delta_{i-1}$ $(i = 1, \dots, m)$. Therefore for any large n we have $\delta_0 s_n < \delta_0 = \delta < \delta_1 s_n < \delta_1 < \dots < \delta_m s_n < \delta_m$: so $\delta_i s_n$ $(i = 1, \dots, m)$ are distinct ordinals $> \delta$ while $\delta s_n < \delta$, so that $f_{\delta s_n} \in F_\delta^R$. This forces $a_1 = \dots a_m = 0$, giving

$$f_\delta \varphi = f_\delta a_0 .$$

Since $\delta s_0 = 0$, for $n = 0$ the penultimate display reduces to $f_0 \varphi = f_0 a_0$; which means that a_0 is independent of δ. Finally, φ commutes with z^R, and since $\delta z = \gamma$ the equation $f_\delta z^R \varphi = f_\delta \varphi z^R$ asserts that

$$f_\gamma \varphi = f_\gamma a_0 \qquad (\gamma \in \lambda) .$$

Thus φ is scalar multiplication by a_0; and Proposition β is proved for every regular cardinal λ.

(b) Assume now that λ is singular, so that $\varrho = cf(\lambda)$ is regular (possibly equal to ω). Choose infinite cardinals $\lambda_\alpha < \lambda$ $(\alpha < \varrho)$ such that $\sup\limits_{\alpha<\varrho} \lambda_\alpha = \lambda$. Replacing each by its cardinal successor, we may assume that each λ_α is regular and $> \omega$. Then, for each $\alpha < \varrho$, $\Delta \cap \lambda_\alpha$ is stationary in λ_α and by Solovay's theorem there exists a function $z_\alpha : \Delta \cap \lambda_\alpha \to \lambda_\alpha$ such that the inverse image of each ordinal in λ_α is stationary in λ_α. Take Π to consist of the partial functions

$$h : \Lambda(0) \to \Lambda , \qquad \min : \Lambda(0) \to \lambda ,$$

$$s_n : \Delta \to \lambda \;\; (n < \omega) , \qquad z_\alpha : \Delta \cap \lambda_\alpha \to \lambda \;\; (\alpha < \varrho) .$$

Note that, for each $\alpha < \varrho$, φ leaves invariant both $F^R_{\Delta \cap \lambda_\alpha}$ and its image under z^R, namely $F^R_{\lambda_\alpha}$; therefore in turn $\xi < \lambda_\alpha \Rightarrow \xi^\# < \lambda_\alpha$ and $\eta < \lambda_\alpha \Rightarrow \eta^* < \lambda_\alpha$. The argument of case (*a*) now goes through with minimal changes.

Post scriptum. This Note was written in response to a query by Laszlo Fuchs at the 1985 Oberwolfach Conference on Abelian Groups, asking whether there were circumstances under which the existence of « small » indecomposables might entail that of « large » indecomposables. Happily, before the advent of Shelah's « Black Box » (see [2] for an exposition), which at the time appeared to render his earlier idea obsolete, I had amused myself by disentangling the various set theoretic, linear algebraic, and group theoretic strands in Shelah's 1974 construction, and by grafting the result on to the ideas of my paper [1], in a rather absurd setting of additive categories. So I was well placed to answer Fuchs's query: see [4] for his application of the present theorem. The comparable Note [3] by Franzen and Göbel stays much closer to its original in Shelah [7].

REFERENCES

[1] A. L. S. Corner, *Endomorphism algebra of large modules with distinguished submodules*, J. Algebra, **11** (1969), pp. 155-185.

[2] A. L. S. Corner - R. Göbel, *Prescribing endomorphism algebras, a unified treatment*, Proc. London Math. Soc., **50** (1985), pp. 447-479.

[3] B. Franzen - R. Göbel, *The Brenner-Buther-Corner theorem and its application to modules*, in *Abelian Group Theory*, Gordon and Brech, New York (1987), pp. 209-227.

[4] L. Fuchs, *Large indecomposable modules in torsion theories*, Aequat. Math., **34** (1987), pp. 106-111.

[5] T. Jech, *Set Theory*, Academic Press, New York (1978).

[6] M. Nagata, *Local Rings*, Interscience, New York (1962).

[7] S. Shelah, *Infinite abelian groups, the Whiteheael problem and some constructions*, Israel J. Math., **18** (1974), pp. 243-256.

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