

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova,*  
tome 81 (1989), p. 85-93

[http://www.numdam.org/item?id=RSMUP\\_1989\\_\\_81\\_\\_85\\_0](http://www.numdam.org/item?id=RSMUP_1989__81__85_0)

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## Uniqueness of the Cauchy Problem for a Second Order Operator.

DANIELE DEL SANTO (\*)

### 1. Introduction and statement of the result.

Uniqueness and non-uniqueness of the Cauchy problem have been the subject of a large number of studies during the last thirty years (see [8]). In the present communication a uniqueness theorem for a second order operator in two variables with roots with variable multiplicity, is proved.

Let  $U$  be an open set of  $\mathbf{R}^2 = \mathbf{R}_x \times \mathbf{R}_t$ ; suppose  $0 \in U$ . Consider the differential operator:

$$(1.1) \quad P(x, t, D_x, D_t) = \\ = D_t^2 + a(x, t) D_x^2 + b(x, t) D_x + c(x, t) D_t + d(x, t)$$

where  $a \in C^\infty(U)$  and  $b, c, d \in L_{loc}^\infty(U)$ . Here, as usual,  $D_x$  stands for  $(1/i)(\partial/\partial x)$ .

**THEOREM 1.** *Let  $k$  be an integer positive number. Suppose that*

$$(1.2) \quad a(x, t) = t^k \alpha(x, t)$$

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This communication is part of the *Tesi di Laurea* presented by the A. in Padova University on the 13 February 1987. The A. would like to thank Prof. F. Colombini for his support and encouragement.

and, writing  $\alpha(x, t) = \alpha_1(x, t) + i\alpha_2(x, t)$ ,  $\alpha_j \in \mathbf{R}$  for  $j = 1, 2$ , suppose:

$$(1.3) \quad (\alpha_2(x, t) \equiv 0 \quad \text{and} \quad \alpha_1(0, 0) \neq 0) \quad \text{or} \quad \alpha_2(0, 0) \neq 0.$$

Suppose also that there exists  $C > 0$  so that

$$(1.4) \quad |b(x, t)| \leq Ct^{k/2-1}.$$

Let  $u \in C^\infty(U)$ ,  $\text{supp}(u) \subseteq \{(x, t) \in U : t \geq 0\}$  and  $Pu \equiv 0$  in  $U$ . Then  $u \equiv 0$  in a neighbourhood of 0, i.e. there is the uniqueness of the Cauchy problem for  $P$  with respect to the surface  $\{t = 0\}$  in 0.

REMARK 1. If  $k$  is even, theorem 1 can be deduced from a more general result of Nakane ([3], theorem 1) concerning operators in several variables and of order  $m$ .

REMARK 2. Conditions (1.2) and (1.3) are, in a way, analogous to the conditions on roots in Calderon's uniqueness theorem. Infact if  $k = 0$ , (1.2) and (1.3) are exactly the requests of Calderon's theorem.

Condition (1.4) on the term of the first order is known as Levi type condition.

REMARK 3. It is of particular interest to make a comparison between theorem 1 and some known result in the case that  $\alpha_2 \equiv 0$ , i.e. in the case of operators with real principal part.

If  $\alpha_1(0, 0) < 0$ , the operator (1.1) is weakly hyperbolic: condition (1.4) is then sufficient to imply also the well-posedness of the  $C^\infty$ -Cauchy problem ([6]). On the contrary, Nakane ([4]) proves that, when:

$$\tilde{P} = D_t^2 - t^k D_x^2 + t^m D_x$$

if  $m < k/2 - 1$  then there exist  $d, u \in C^\infty$  so that

$$0 \in \text{supp}(u) \subseteq \{(x, t) : t \geq 0\} \quad \text{and} \quad \tilde{P}u + du \equiv 0$$

i.e. non-uniqueness holds for  $\tilde{P}$  with respect to  $\{t = 0\}$  (for non-solvability of the Cauchy problem for the operator  $\tilde{P}$  see also [2]).

If  $\alpha_1(0, 0) > 0$  the operator (1.1) is degenerate elliptic with real principal part. Watanabe ([7]) shows that, if  $k$  is even, theorem 1

is true without conditions on lower order terms, i.e. without any Levi type condition. If  $k$  is odd the only known result is Nirenberg's one ([5]) but in this case theorem 1 is more general, in the way that, Nirenberg's result is a compact-uniqueness result and the condition on  $b(x, t)$  is

$$(1.4)' \quad |b(x, t)| \leq Ct^{k/2}$$

which is more restrictive than (1.4).

## 2. Proof of the Theorem.

The technique of the proof is inspired to [9], and it is based on a Carleman estimate in which function  $t^{-N}$  is the weight function.

It is possible to apply the Carleman estimate directly to the operator (1.1), but it would give only a particular uniqueness result: in that case we obtain that there exists a fixed neighbourhood,  $U$ , of the origin so that there are no solutions,  $u$ , of:

$$(2.1) \quad Pu = 0$$

when  $0 \in \text{supp}(u)$  and  $\text{supp}(u)$  is a compact in  $\{t \geq 0\} \cap U$ .

To reach our result we need to make this change of variables (singular change of variables: see [1])

$$(2.2) \quad \begin{aligned} x &= y, \\ t &= (r - y^2)s, \end{aligned}$$

where  $r > 0$  and it will be fixed later.

With this transformation it is possible to obtain from a solution of (2.1) with support in  $t \geq 0$  a new solution,  $w$ , of:

$$Qw = 0$$

with  $\text{supp}(w) \subseteq \{(y, s) : s \geq 0, |y|^2 \leq r\}$ ; here  $Q = (r - y^2)^2 \tilde{P}$  and  $\tilde{P}$  is the operator  $P$  in the new variables.

Theorem 1 follows, with standard technique, from next lemma.

**LEMMA 1.** *There exist  $S_0 > 0$ ,  $N_0 > 0$ ,  $C > 0$  so that if  $0 < S < S_0$ ,  $N > N_0$ , and  $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, S])$ , then*

$$(2.3) \quad N^2 \|s^{-N-2} u\|^2 \leq C \|s^{-N} Q u\|^2$$

where  $\|\cdot\| = \|\cdot\|_{L^2_{\mathbb{R}_y \times \mathbb{R}_x}$ .

If we write  $x$  and  $t$  instead of  $y$  and  $s$ ,  $Q$  has the following form:

$$(2.4) \quad Q = (r - x^2)^2 \tilde{P} = [1 + 4x^2 t^{k+2} (r - x^2)^k \alpha(x, t)] D_t^2 + \\ + 4xt^{k+1} \alpha(x, t) (r - x^2)^{k+1} D_x D_t + \alpha(x, t) (r - x^2)^{k+2} t^k D_x^2 + \\ + (r - x^2)^2 b(x, t) D_x + c^*(x, t) D_t + d^*(x, t)$$

where  $c^*, d^* \in L_{\text{loc}}^\infty(U)$ .

Calling  $q$  the principal part of  $Q$ , we have:

$$(2.5) \quad q = A \partial_1 \partial_2 - B D_x$$

where

$$(2.6) \quad A(x, t) = 1 + 4x^2 t^{k+2} (r - x^2)^k \alpha(x, t), \\ B(x, t) = A(x, t) [D_t (-t^{k/2} (r - x^2)^{k/2+1} \sigma_2) + \\ + t^{k/2} (r - x^2)^{k/2+1} \sigma_1 D_x (-t^{k/2} (r - x^2)^{k/2+1} \sigma_2)], \\ \partial_j = D_t - t^{k/2} (r - x^2)^{k/2+1} \sigma_j(x, t) D_x, \quad j = 1, 2,$$

and

$$(2.7) \quad \sigma_j(x, t) = \frac{-2xt^{k/2+1} \alpha(x, t) (r - x^2)^{k/2} + (-1)^j \sqrt{-\alpha(x, t)}}{1 + 4x^2 t^{k+2} (r - x^2)^k \alpha(x, t)}.$$

Note that choosing  $r$  small enough, if  $\text{Im } \sigma_j(0, 0) \neq 0$ , there exists  $T_0 > 0$  so that, if  $0 \leq t \leq T_0$  and  $x \in [-\sqrt{r}, \sqrt{r}]$  then

$$(28) \quad |\text{Im } (\sigma_j(x, t))| \geq \varepsilon > 0.$$

With the notations stated before, it is possible to prove the following:

LEMMA 2. *There exist  $T_0 > 0$ ,  $N_0 > 0$ ,  $C > 0$ ,  $C' > 0$  so that if  $0 < T < T_0$ ,  $N > N_0$  and  $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, T])$ , then:*

$$(2.9) \quad N \|t^{-N-1} u\|^2 \leq C \|t^{-N} \partial_j u\|^2 \quad j = 1, 2.$$

$$(2.10) \quad N^2 \|t^{-N-2} u\|^2 \leq C' \|t^{-N} \partial_1 \partial_2 u\|^2.$$

PROOF. Let  $v = t^{-N}u$ , and  $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, T])$ .

Let:

$$(2.11) \quad \begin{aligned} (r - x^2)^{k/2+1} \sigma_{j1}(x, t) &= \operatorname{Re}((r - x^2)^{k/2+1} \sigma_j(x, t)) = A_j(x, t), \\ (r - x^2)^{k/2+1} \sigma_{j2}(x, t) &= \operatorname{Im}((r - x^2)^{k/2+1} \sigma_j(x, t)) = B_j(x, t), \end{aligned}$$

note that  $(r - x^2)^{k/2+1} \sigma_j$  are  $C^1$  in  $x$  and  $t$ .

Writing:

$$(2.12) \quad \begin{aligned} X_j &= D_t - t^{k/2} A_j(x, t) D_x, \\ Y_j &= \frac{N}{t} + t^{k/2} B_j(x, t) D_x, \end{aligned}$$

we obtain

$$\|t^{-N} \partial_j u\|^2 = \|X_j v - i Y_j v\|^2 = \|X_j v\|^2 + \|Y_j v\|^2 + 2 \operatorname{Re} \langle X_j v, -i Y_j v \rangle.$$

Let's calculate the last term and we find:

$$(2.13) \quad 2 \operatorname{Re} \left\langle D_t v, -i \frac{N}{t} v \right\rangle = N \left\| \frac{v}{t} \right\|^2,$$

$$(2.14) \quad \left| 2 \operatorname{Re} \left\langle -t^{k/2} A_j(x, t) D_x v, -i \frac{N}{t} v \right\rangle \right| \leq N C_1 \|t^{k/4-1/2} v\|^2,$$

$$(2.15) \quad 2 \operatorname{Re} \langle -t^{k/2} A_j(x, t) D_x v, -i t^{k/2} B_j(x, t) D_x v \rangle = 0.$$

If  $B_j \equiv 0$  (i.e.  $P$  is weakly hyperbolic) it is easy to reach (2.9) from (2.13), (2.14) and (2.15). Suppose that  $B_j(0, 0) \neq 0$ ; knowing that, in this case, (2.8) is true and using a punctual inequality, we obtain that there exists  $C_2 > 0$  so that:

$$(2.16) \quad \|t^{k/2} A_j(x, t) D_x v\| \leq C_2 \|t^{k/2} B_j(x, t) D_x v\|$$

$$(2.17) \quad \|t^{k/2} D_t(B_j(x, t)) D_x v\| \leq C_2 \|t^{k/2} B_j(x, t) D_x v\|.$$

Consider then:

$$\begin{aligned}
 (2.18) \quad &= 2 \operatorname{Re} \langle D_t v, -it^{k/2} B(x, t) D_x v \rangle = \\
 &= \langle D_t v, -it^{k/2} B(x, t) D_x v \rangle + \langle -it^{k/2} B(x, t) D_x v, D_t v \rangle = \\
 &= i \langle t^{k/2} D_x (B(x, t)) D_t v, v \rangle - \frac{k}{2} \langle t^{k/2-1} B(x, t) D_x v, v \rangle + \\
 &\quad - i \langle t^{k/2} D_t (B(x, t)) D_x v, v \rangle = a + b + c.
 \end{aligned}$$

Using (2.16) we have:

$$\begin{aligned}
 |a| &\leq C_3 T^{k/2} \|D_t v\| \|v\| \leq C_3 T^{k/2} \|D_t v - t^{k/2} A(x, t) D_x v\| \|v\| + \\
 &\quad + C_3 T^{k/2} \|t^{k/2} A(x, t) D_x v\| \|v\| \leq \\
 &\leq C_3 T^{k/2} \|Xv\| \|v\| + C_2 C_3 T^{k/2} \|t^{k/2} B(x, t) D_x v\| \|v\| \leq \\
 &\leq C_3 T^{k/2} \|Xv\| \|v\| + C_2 C_3 T^{k/2} \left( \|Yv\| \|v\| + N \left\| \frac{v}{t} \right\| \|v\| \right)
 \end{aligned}$$

and:

$$\begin{aligned}
 b &= -\frac{k}{2} \langle t^{k/2-1} B(x, t) D_x v, v \rangle = \\
 &= -\frac{k}{2} \left\langle t^{k/2} B(x, t) D_x v + N \frac{v}{t}, \frac{v}{t} \right\rangle + \frac{k}{2} \left\langle N \frac{v}{t}, \frac{v}{t} \right\rangle = -\frac{k}{2} \left\langle Yv, \frac{v}{t} \right\rangle + \frac{k}{2} N \left\| \frac{v}{t} \right\|^2.
 \end{aligned}$$

Using (2.17) we have:

$$|c| \leq C_2 \|t^{k/2} B(x, t) D_x v\| \|v\| \leq C_2 \|Yv\| \|v\| + C_2 N \left\| \frac{v}{t} \right\| \|v\|.$$

Finally:

$$\left| a + b + c - \frac{k}{2} N \left\| \frac{v}{t} \right\|^2 \right| \leq (2.19)$$

where:

$$\begin{aligned}
 (2.19) \quad &= 2C_3 T^{k/2} (\|Xv\|^2 + \|v\|^2) + 2C_2 C_3 T^{k/2} (\|Yv\|^2 + \|v\|^2) + \\
 &\quad + 2NC_2 C_3 T^{k/2} \left( \left\| \frac{v}{t} \right\|^2 + \|v\|^2 \right) + \frac{k}{2} \left( \varepsilon \|Yv\|^2 + C_\varepsilon \left\| \frac{v}{t} \right\|^2 \right) + \\
 &\quad + C_2 (\varepsilon \|Yv\|^2 + C_\varepsilon \|v\|^2) + C_2 N \left( \varepsilon \left\| \frac{v}{t} \right\|^2 + C_\varepsilon \|v\|^2 \right)
 \end{aligned}$$

$\varepsilon$  and  $C_\varepsilon$  are constants.

We obtain:

$$(2.18) \geq \frac{k}{2} N \left\| \frac{v}{t} \right\|^2 - (2.19)$$

and from (2.13), (2.14), (2.15) and (2.18) we have:

$$\|t^{-N} \partial_t u\|^2 \geq \|Xv\|^2 + \|Yv\|^2 + N \left(1 + \frac{k}{2}\right) \left\| \frac{v}{t} \right\|^2 - [(2.19) + NC_1 \|t^{k/4 - \frac{1}{2}} v\|^2].$$

It is easy to see that choosing  $T_0$  and  $N_0^{-1}$  small enough we have:

$$\|Xv\|^2 + \|Yv\|^2 + \left(\frac{1}{2} + \frac{k}{2}\right) N \left\| \frac{v}{t} \right\|^2 - [(2.19) + NC_1 \|t^{k/4 + \frac{1}{2}} v\|^2] \geq 0$$

by which (2.9) is proved.

Using (2.9) twice we get (2.10). Q.E.D.

Consider  $g_1, g_2 \in L^\infty$ . Let us define  $\mathcal{R}$  as the set of operators of the kind:

$$R = \frac{g_1(x, t)}{t} D_t + t^{k/2-1} (r - x^2)^{k/2+1} g_2(x, t) D_x.$$

Note that for each  $R \in \mathcal{R}$  there exist  $h_1, h_2 \in L^\infty$  so that

$$R = h_1(x, t) \frac{\partial_1}{t} + h_2(x, t) \frac{\partial_2}{t}.$$

If  $R \in \mathcal{R}$ , it is easy to see that there exist  $C > 0$ ,  $N_0 > 0$  so that if  $u \in C_0^\infty([-\sqrt{r}, \sqrt{r}] \times [0, T_0])$  and  $N > N_0$  then:

$$(2.20) \quad N \|t^{-N} Ru\|^2 \leq C \|t^{-N} \partial_1 \partial_2 u\|^2$$

with  $T_0$  as in Lemma 2.

Infact from (2.9) we have:

$$(2.21) \quad N \|t^{-N} Ru\|^2 \leq C (\|t^{-N} \partial_1 \partial_2 u\|^2 + \|t^{-N} \partial_2 \partial_1 u\|^2);$$



note now that:

$$(2.22) \quad \partial_2 \partial_1 - \partial_1 \partial_2 = S \in \mathfrak{R};$$

applying (2.21) to  $S$  we deduce:

$$N \|t^{-N} S u\|^2 \leq C(\|t^{-N} \partial_1 \partial_2 u\|^2 + \|t^{-N} \partial_2 \partial_1 u\|^2)$$

so

$$N \|t^{-N} S u\|^2 \leq C(\|t^{-N} \partial_1 \partial_2 u\|^2 + \|t^{-N}(S u + \partial_1 \partial_2 u)\|^2)$$

from which

$$(2.23) \quad N \|t^{-N} S u\|^2 \leq C \|t^{-N} \partial_1 \partial_2 u\|^2.$$

Using (2.22), (2.23) and (2.21) we obtain (2.20).

Let us finally come to the proof of Lemma 1. Using (2.4), (2.5) and (1.4) we can write

$$Q u = A \partial_1 \partial_2 u + R u + t^{-2} \gamma(x, t) u$$

where  $R \in \mathfrak{R}$  and  $\gamma \in L^\infty$ .

Note that  $Q$  has this form only because the Levi type condition is true.

It follows:

$$2 \|t^{-N} Q u\|^2 \geq \|t^{-N} A \partial_1 \partial_2 u\|^2 - 4(\|t^{-N} R u\|^2 + \|t^{-N-2} \gamma u\|^2)$$

but applying (2.20), with  $N$  sufficiently large, we have:

$$\frac{1}{4} \|t^{-N} A \partial_1 \partial_2 u\|^2 > 4 \|t^{-N} R u\|^2$$

and applying (2.10) we obtain:

$$\frac{1}{4} \|t^{-N} A \partial_1 \partial_2 u\|^2 > 4(\|\gamma\|_{L^\infty})^2 \|t^{-N-2} u\|^2.$$

So that again from (2.10):

$$2 \|t^{-N} Q u\|^2 \geq \frac{1}{2} \|t^{-N} A \partial_1 \partial_2 u\|^2 \geq \frac{N^2}{2C'} \|t^{-N-2} u\|^2. \quad \text{Q.E.D.}$$

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Manoscritto pervenuto in redazione il 22 febbraio 1988.