RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

PIERANGELO MARCATI

Convergence of approximate solutions to scalar conservation laws by degenerate diffusion

Rendiconti del Seminario Matematico della Università di Padova, tome 81 (1989), p. 65-78

<http://www.numdam.org/item?id=RSMUP_1989__81__65_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 81 (1989)

Convergence of Approximate Solutions to Scalar Conservation Laws by Degenerate Diffusion.

PIERANGELO MARCATI (*)

0. Introduction.

This paper is concerned with the existence of weak solutions to the scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0 & x \in \mathbf{R}\,,\ t > 0\,, \\ u(x, 0) = u_0(x)\,, \end{cases} \tag{0.1}$$

in the framework provided by Compensated Compactness theory recently developed by Tartar [8], [9] and Di Perna [2]. We want to show that the unique (see [17] or [18]) weak « entropic » solution to (0.1) can be obtained by replacing the usual viscous approximation by means of the porous media operator. Namely, we are going to study the limits as $\varepsilon\downarrow 0$ of the convective-parabolic equation

$$\begin{cases} u_t + f(u)_x = \varepsilon\big(|u|^{m-1}u\big)_{xx}\,, \\ u(x, 0) = u_0(x)\,. \end{cases} \tag{0.2}$$

(*) Indirizzo dell'A.: Dipartimento di Matematica Pura e Applicata, Università dell'Aquila, Via Roma 33, 67100 L'Aquila, Italy.

Partially supported by C.N.R.-G.N.A.F.A.

This paper is the definitive version of a preprint, with the same title dated on November 1985. Part of these results have been communicated at the « International Conference on Hyperbolic Equations and Related Topics », Padova, December 1985.

This type of equation has been investigated by several authors, however, for our purposes we will, mainly, refer to the foundamental papers of Vol'pert and Hudjaev [11] and Osher and Ralston [7].

In particular in [7] the existence theory of traveling waves solutions to (0.2) has been carried out revealing the advantage of using the degenerate diffusive approximation (0.2) with respect to the usual one. Indeed, the latter method provides an approximating solution which, in some situations, coincides with the exact solution of (0.1) outside a compact set (in the space variable, for fixed time), while the perturbation effects of the usual viscosity, always yield to undesired modifications of the far fields. This motivated further investigations, towards possible numerical applications, particularly modifications of the classical Lax-Friedrichs monotone scheme to allow degenerate diffusion operators. The convergence of such a scheme is investigated in a separate paper [19], together some complimentary analytical results which include the case, not considered here, of the fast diffusion operators.

The slow diffusion case considered here has a relevant property which fails to be true in the case considered in [19], namely the finiteness of the solutions support propagation speed, proved in section 2 of this paper.

Finally it is interesting to notice how this property (although proved in an elementary way) has been deduced without any positivity assumption on the initial datum. Let us recall that a bounded measurable function u is called a *weak solution* to (0.1) if and only if, for all test functions $\varphi \in C_0'(\mathbb{R}\times\mathbb{R})$, one has

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\, dx\, dt + \int_{-\infty}^\infty u_0(x)\,\varphi x(x, 0)\, dx = 0\,. \tag{0.3}$$

For physical as well as for mathematical reasons (we want the uniqueness of the weak solution), u is required to satisfy the *Lax Entropy inequality* (see [3]), for a given entropy pair (η, q).

Namely

$$\partial_t \eta(u) + \partial_x q(u) \leqslant 0\,, \quad \text{in } D'(\mathbb{R}\times\mathbb{R}_+)\,. \tag{0.4}$$

for a convex *entropy* $\eta\colon \mathbb{R}\to\mathbb{R}$ and the related *entropy flux* $q\colon \mathbb{R}\to\mathbb{R}$ (endowed by the relation $q'(u) = \eta'(u)f'(u)$). If (0.4) is fulfilled for all convex η, therefore the Oleinik's condition (E) holds (see [3]) and hence one has the uniqueness of the weak solution in $L^\infty \cap BV$. In particular we have the same solution obtained by means of the Lax [3] representation formula or by vanishing viscosity method. Moreover, if we solve the Hamilton-Jacobi equation $v_t + f(v_x) = 0$, the unique viscosity solution (see P. L. Lions [4]) provides, by setting $v_x = u$, the entropy solution to (0.1).

From the physical point of view, this result describes the behavior of the solution of the convective porous media equation in terms of the solution to the related conservation law, when the permeability of the medium tends to zero.

The main result of this paper is the following

THEOREM. *Assume that $f: \mathbb{R} \to \mathbb{R}$ is a C^1 function and $f(0) = 0$. Let us denote by u^ε the solution to the convective porous media equation*

$$\begin{cases} \partial_t u + \partial_x f(u) = \varepsilon(|u|^{m-1}u)_{xx} , \\ u(x, 0) \qquad\quad = u_0(x) , \qquad t \geqslant 0 , \ x \in \mathbb{R} , \ \varepsilon > 0 , \ m > 1 . \end{cases} \tag{0.5}$$

*Then, extracting if necessary a subsequence, we can find $u \in L^\infty$ such that $u^\varepsilon \to u$, *-weakly in L^∞, and u is a weak solution to the scalar conseravtion Law*

$$\partial_t u + \partial_x f(u) = 0 . \tag{0.6}$$

Moreover if there is no interval on which f is affine, then, taking eventually a subsequence,

$$u^\varepsilon \to u \tag{0.7}$$

strongly in L^p, for all $p < +\infty$. In this case, for all entropy pair (η, q), with convex η, u satisfies the Lax entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) \leqslant 0 \tag{0.8}$$

in the sense of distributions.

The proof will be done in section 4.

1. Compensated compactness.

We shall recall, briefly, some basic ideas recently developed by Tartar [8], [9] and Murat [5], that can be considered the main tool of this paper.

By first, we recall the relations occurring between the weak star convergence in L^∞ and the theory of L. C. Young probability measures. Let $\Omega \subset \mathbf{R}^N$ and (u^n) a sequence in $L^\infty(\Omega, \mathbf{R}^m)$ such that

$$(1.1) \qquad u^n \overset{*}{\rightharpoonup} u \quad w^* \text{ in } L^\infty$$

and there exists a relatively compact open set D, such that

$$u^n(\Omega) \subset \bar{D}\,.$$

Then a family of probability measures $\{\nu_x\}_{x\in\Omega}$ can be found, such that, for every continuous function $F\colon \mathbf{R}^m \to \mathbf{R}$, one has (extracting, if necessary, a subsequence)

$$(1.2) \qquad F(u^n) \overset{*}{\rightharpoonup} \bar{F}\,, \quad \operatorname{supp} \nu_x \subset \bar{D}$$

where, we set

$$(1.3) \qquad \bar{F}(x) = \int_{\mathfrak{R}^m} F(\lambda)\, d\nu_x(\lambda) = \langle \nu_x, F(\lambda)\rangle\,.$$

If $\nu_x = \delta_{u(x)}$, a.e. in $x \in \Omega$, therefore $u^n \to u$, strongly in L^p, for every $p < +\infty$ and viceversa.

Assume that, for all convex η, one has

$$(1.4) \qquad \partial_t \eta(u^n) + \partial_x q(u^n) \in \text{compact of } W^{-1,2}_{\text{loc}}(\Omega)\,,$$

hence, in particular,

$$\partial_t u^n + \partial_x f(u^n) \in \text{compact of } W^{-1,2}_{\text{loc}}(\Omega)\,,$$

by virtue of a result due to Van Hove [10] and Tartar [8], we are able to say

$$(1.5) \qquad w^*\text{-lim}\,\{u^n q(u^n) - \eta(u^n) f(u^n)\} = uz - vw$$

where

$$(1.6) \qquad v = w^*\text{-lim}\, f(u^n)\,, \quad w = w^*\text{-lim}\, \eta(u^n)\,, \quad z = w^*\text{-lim}\, q(u^n)\,.$$

Therefore, we have (dropping the indices x in ν_x)

$$\langle \nu, \lambda q(\lambda) - \eta(\lambda) f(\lambda)\rangle = \langle \nu, \lambda\rangle\langle \nu, q(\lambda)\rangle - \langle \nu, f(\lambda)\rangle\langle \nu, \eta(\lambda)\rangle . \tag{1.7}$$

Using this relation Tartar [8] achieved the following result.

PROPOSITION. (1.1) *Let Ω be a bounded open set in $\mathbf{R}\times\mathbf{R}_+$ and let f be a real valued C^1 function defined in* $\mathbf{R}$. *Suppose that $\{u^\varepsilon\}$ is a sequence of functions such that $u^\varepsilon \overset{*}{\rightharpoonup} u$, w^* in $L^\infty(\Omega)$ and for all convex $\eta: \mathbf{R} \to \mathbf{R}$ in the class C^2*

$$\partial_t(u^\varepsilon) + \partial_x q(u^\varepsilon) \in \text{compact set of } W^{-1,2}(\Omega) , \tag{1.8}$$

where q is defined by the relation $q'(u) = \eta'(u) f'(u)$. Then

$$f(u^\varepsilon) \overset{*}{\rightharpoonup} f(u) , \quad w^* - L^\infty(\Omega) . \tag{1.9}$$

In addition if there is no interval in which f is affine, then, extracting if necessary a subsequence

$$u^\varepsilon \to u \quad \textit{in } L^p(\Omega) \ (\textit{strongly}) . \tag{1.10}$$

The proof of this result may be found in Tartar [8] or in the book of Dacorogna [1].

We wish to remark that in the above result, the limit in (1.9) has been achieved without making any restriction on the function f but it suffices to say that u is a weak solution to (0.1). If, in addition, we are interested to verify the entropy inequality, we are forced to impose some extra conditions on f. These conditions, however, are sufficiently general to cover the genuine nonlinear case and the case $f(u) = u^p$, $p > 1$.

In the paper of Tartar [8] the above result has been applied to prove the convergence of the viscous approximate solutions to the admissible solution to (0.1).

The result of Di Perna [2] for systems is more difficult, but it can be expressed in a similar way. Moreover in [2] the method of compensated compactness has been used to get the convergence of several numerical schemes.

2. Propagation results.

This section is devoted to prove a result of finite propagation speed for the parabolic degenerate convective equation (0.2). In this result seems to be crucial the assumption that f is a C^1 function. Indeed this result is known to be false when $f(u) = u^p$ and $0 < p < 1$, how proved Diaz and Kersner [12]. Let $T > 0$, we shall consider $t \in [0, T]$. In the sequel we shall drop the index ε in u^ε. Denote by

$$v = \frac{m}{m-1} |u|^{m-1} \tag{2.1}$$

and by

$$E[z] = z_t + f'(u) z_x - \varepsilon z_x^2 - \varepsilon(m-1) z z_{xx} . \tag{2.2}$$

One has $E[v] = 0$ in the sense of distribution.

If we set

$$\hat{v}(x, t) = \varphi(t) \left(1 - \frac{x^2}{\varphi(t)}\right)^+ , \qquad x \in \mathbb{R} ,\ t \in [0, T] \tag{2.3}$$

we can determine a positive $\psi(t)$ such that

$$v(x, t) \leqslant \hat{v}(x, t) \tag{2.4}$$

Indeed, for all $x \in \{x^2 \leqslant \psi(t)\}$, one has

$$\begin{aligned} E[\hat{v}] = \psi' - 2xf'(u) - 4\varepsilon x^2 + 2\varepsilon(m-1)\psi \left(1 - \frac{x^2}{\psi}\right)^+ \geqslant \\ \geqslant \psi' - 2\sqrt{\psi(t)}\, K - 4\varepsilon\psi(t) \end{aligned} \tag{2.5}$$

where $K = K(T) > 0$ is determined in the following way. Owing to the results of [7] and [11] the functions

$$t \in [0, T] \mapsto \int_{-\infty}^{\infty} |u(x, t)| dx , \qquad t \in [0, T] \mapsto \int_{-\infty}^{\infty} |u_x(x, t)| dx$$

are bounded, therefore we can find $M = M(\varepsilon, T) > 0$ such that

$$|u(x, t)| \leqslant M , \quad \text{a.e. in } (x, t) \in \mathbf{R} \times \mathbf{R}_+ .$$

Hence it is natural to assume

$$K = \sup \{|f'(u)|: |u| \leqslant M\} \tag{2.6}$$

moreover let us denote by

$$\psi_0 = \sup \{|x|: x \in \operatorname{supp} (u_0)\}$$

therefore if we choose $\psi(t)$ as the solution of the following Cauchy Problem

$$\begin{cases} \psi'(t) - 2K(\psi(t))^{\frac{1}{2}} - 4\varepsilon\psi(t) = 0 , \\ \psi(0) = \psi_0 , \end{cases} \tag{2.7}$$

we obtain $E[\hat{v}] \geqslant 0$. This implies that (2.4) holds. In this way the following result has been proved.

PROPOSITION. *Let $m > 1$, u_0 a continuous function with compact support, therefore there exists a function $\zeta: \mathbf{R} \to \mathbf{R}_+$ such that for all $t \geqslant 0$*

$$\operatorname{supp} u(\cdot, t) \subseteq [-\zeta(t), \zeta(t)] . \tag{2.8}$$

REMARK. We observe that the above result does not provide any existence theorem concerning the nature of the « interfaces ». In the case of positive solutions, we refer for a more accurate analysis to Diaz and Kersner [12] [13], Gilding [14], Gilding and Peletier [15], Di Benedetto [16].

3. « A priori » estimates.

This section deals with some « a priori » bounds on the amplitude of the solutions to the convective diffusive equation

$$\begin{cases} \partial_t u + \partial_x f(u) = \varepsilon \partial_{xx}(|u|^{m-1}u) , \\ u(x, 0) \qquad = u_0(x) , \qquad x \in \mathbf{R} , \ t \geqslant 0 , \ \varepsilon > 0 , \ m > 1 . \end{cases} \tag{3.1}$$

Even in this section we shall write u in place of u^ε to denote the solutions to (3.1).

Before to go further, we recall some regularity results proved by Vol'pert and Hudjaev [11] and by Osher and Ralston [7]. They proved that $[T(t)u_0](x) = u(x, t)$ is a nonlinear contraction semigroup in L^1 and u_x, u_t are in $L^1(\omega_\lambda dx)$ where $\omega_\lambda(x) = \exp[-\lambda(1+x^2)]$. Moreover one has $|u|^{m/2}\partial_x u$ is in $L^2_{\text{loc}}(\omega_\lambda dx\, dt)$.

The next proposition regards the existence of an L^∞ bound for u, independent from ε.

PROPOSITION. (3.1) *Assume that u_0 is continuous and has a compact support, therefore one has*

$$\sup_{x,t} |u^\varepsilon(x, t)| \leqslant \sup_x |u_0(x)| . \tag{3.2}$$

If in addition $u_0 \in L^2$, then

$$\sup_t \|u^\varepsilon(x, t)\|_{L^2} \leqslant \|u_0\|_{L^2} \tag{3.3}$$

and for all $T > 0$

$$\varepsilon m \iint_{S_T} |u|^{m-1} u_x^2\, dx\, dt \leqslant 2\|u_0\|^2_{L^2} \tag{3.4}$$

where

$$S_T = \{-\infty < x < +\infty\} \times (0. T) .$$

PROOF. Let $k > 0$ and «multiply» the equation (3.1) by the test function $\varphi(x, t) = (|u(x, t)| - k)^+$. In the interior of $\{u(x, t) \geqslant k\}$, one has

$$\begin{aligned}(u-k)u_t + (u-k)f(u)_x &= \\ &= \tfrac{1}{2}\partial_t(u-k)^2 + \partial_x\big(uf(u) - F(u)\big) - kf(u)_x = \\ &= \varepsilon(u-k)(u^m)_{xx} = \varepsilon[(u-k)(u^m)_x]_x - \varepsilon u_x(u^m)_x .\end{aligned} \tag{3.5}$$

Hence

$$\begin{aligned}\tfrac{1}{2}\partial_t(u-k)^2 + \partial_x[(u-k)f(u)] - f(u) &= \\ &= \varepsilon m[(u-k)u^{m-1}u_x]_x - \varepsilon m u^{m-1}u_x^2 .\end{aligned} \tag{3.6}$$

Repeating the same calculus on the interior of $\{u(x, t) \leqslant - k\}$, one has

$$(3.7) \quad \partial_t \eta_k(u) + \partial_x q_k(u) = \varepsilon \mathrm{m}[((|u| - k)^+ |u|^{m-1} u_x]_x - \varepsilon m |u|^{m-1} u_x^2 .$$

where

$$(3.8) \quad \eta_k(u) = \frac{1}{2}[((|u| - k)^+]^2 , \qquad q_k(u) = (|u| - k)_+ f(u) - \left(\int_k^u f(s)\, ds\right)^+ .$$

Therefore integrating on $\{|u(x, t)| > k\}$ and using the results of section 2, it follows

$$(3.9) \qquad \frac{d}{dt}\int_{-\infty}^{\infty} \eta_k(u(x, t))\, dx = - \varepsilon m \int_{\{|u| > k\}} |u(x, t)|^{m-1} u_x^2(x, t)\, dx .$$

An integration in t, yealds

$$(3.10) \qquad \int_{-\infty}^{\infty} \eta_k(u(t, x))\, dx =$$

$$= \int_{-\infty}^{\infty} \eta_k(u_0(x))\, dx - \varepsilon m \int_0^t \int_{\{|u| > k\}} |u(s, x)|^{m-1} u_x^2(s, x)\, dx\, ds .$$

If we choose

$$k = \sup_x |u_0(x)|$$

the following inequality holds

$$(3.11) \qquad \eta_k(u(x, t)) \leqslant \eta_k(u_0(x)) \equiv 0 , \quad \text{a.e. in } (x, t) ,$$

namely

$$(3.12) \qquad |u(x, t)| \leqslant \sup_x |u_0(x)| , \quad \text{a.e. in } (x, t) ,$$

The estimates (3.3) and (3.4) are easily achieved by choosing $k = 0$, namely

$$(3.13)\qquad \int_{-\infty}^{\infty}\eta_0(u(x,t))\,dx = \int_{-\infty}^{\infty}\eta_0(u_0(x))\,dx - \iint_{S_t}(\varepsilon m)|u|^{m-1}u_x^2\,dx\,ds$$

where $\eta_0(u) = |u|^2$

An additional information can be deduced for the genuine nonlinear case.

PROPOSITION. (3.2) *If the function f verifies the genuine nonlinearity condition $f''(u) \neq 0$, for every $u \in \mathbb{R}$, there exists $K > 0$ such that*

$$(3.14)\qquad v_x \leqslant \frac{K}{t} \qquad \text{a.e. in } (x,t)$$

where $v = (m/(m-1))\,u^{m-1}$, provided that $u_0 \geqslant 0$.

PROOF. Since $u_0 \geqslant 0$ one has $u(x,t) \geqslant 0$. Define $g(v)$ by the relation

$$g'(v) = f'\left(\left(\frac{m-1}{m}v\right)^{1/(m-1)}\right), \qquad g'(0) = 0\,.$$

If we set

$$L[w] = w_t + g'(v)\,w_x + g''(v)\,w^2 - \varepsilon(m-1)\,ww_x - \varepsilon(m-1)\,vw_{xx}$$

one has $L[v_x] = 0$ in the sense of distribution. Then it follows

$$(3.15)\qquad L\left[\frac{K}{t}\right] = -\frac{K}{t^2} + g''(v)\frac{K^2}{t^2}$$

without loss of generality, we may assume $f''(u) \geqslant \alpha > 0$, hence by a standard computation, one has

$$g''(v) \geqslant \left(\left(\frac{m-1}{m}\right)^{1/(m-1)}\left(\frac{\alpha}{m-1}\right)\right)\|v_0\|_{L^\infty}^{-m/(m-1)} = \alpha_m\,.$$

Therefore if we choose $K = 2/\alpha m$, by (3.15) it follows the inequality (3.14).

4. Entropy estimates and main theorem.

We want to prove now, that for all convex entropy η, with entropy flux q, one has

$$\partial_t \eta(u) + \partial_x q(u) \in \text{compact set of } W^{-1,2}(\Omega)\,, \tag{4.1}$$

for every $\Omega \subset\subset \mathbf{R}\times\mathbf{R}_+$.

By first we « multiply » our approximating equation (0.2) by $\eta'(u)$, then

$$\partial_t \eta(u) + \partial_x q(u) = \varepsilon\big(\eta'(u)\, m|u|^{m-1} u_x\big)_x - \varepsilon\eta''(u)\, m|u|^{m-1} u_x^2\,. \tag{4.2}$$

Let us define $\psi'(u) = m\eta'(u)|u|^{m-1}$, $\psi(0) = 0$, therefore

$$\partial_t \eta(u) + \partial_x q(u) = \varepsilon\psi(u)_{xx} - \varepsilon m\eta''(u)|u|^{m-1} u_x^2\,. \tag{4.3}$$

In order to prove (4.1), we shall use a lemma due to Murat [5] (see also Tartar [8]).

LEMMA. (4.1) *Let Ω be an open bounded set of $\mathbf{R}^N$ and let $\{\varphi_n\}$ a sequence of distribution satisfying*

(*a*) *$\{\varphi_n\}$ is a bounded subset of $W^{-1,r}(\Omega)$, for some $r > 2$.*

(*b*) *$\varphi_n = \psi_n + \chi_n$, where $\{\psi_n\}$ is relatively compact in $W^{-1,2}(\Omega)$ and $\{\chi_n\}$ is bounded in the space of Radon measures $M(\Omega)$.*

Then $\{\varphi_n\}$ is relatively compact in $W^{-1,2}(\Omega)$.

Let $\varepsilon_n \downarrow 0$, we shall denote by

$$\psi_n = \varepsilon_n \psi(u)_{xx}\,, \qquad \chi_n = -\,\varepsilon_n m\eta''(u)|u|^{m-1} u_x^2\,, \tag{4.5}$$

where, for brevity we wrote u in place of u^{ε_n}.

One has

$$\|\psi_n(u)\|_{W^{-1,2}} = \sup\left\{\left|\iint_\Omega \varepsilon_n \psi(u)_{xx}\varphi\, dx\, dt\right| : \varphi \in W_0^{1,2},\ \|\varphi\| = 1\right\},$$

hence we find

$$(4.6)\qquad \|\psi_n(u)\|_{W^{-1,2}}\leqslant \varepsilon_n m\left|\iint_\Omega \eta'(u)|u|^{m-1}u_x\varphi_x\,dx\,dt\right|\leqslant$$

$$\leqslant (\varepsilon_n)^{\frac12} m\|\eta'(u)\|_{L^\infty(\Omega)}\|\,|u|^{(m-1)/2}\|_{L^\infty(\Omega)}\left(\iint_\Omega \varepsilon_n|u|^{m-1}u_x^2\,dx\,dt\right)^{\frac12}\|\varphi_x\|_{L^2(\Omega)}\,.$$

Because of the inequality (3.4), we obtain

$$(4.7)\qquad \varepsilon_n\iint_\Omega |u|^{m-1}u_x^2\,dx\,dt\leqslant\left(\frac{2}{m}\right)\|u_0\|^2_{L^2(\Omega)}\,.$$

The above inequality, together the estimates of proposition (3.1), implies that

$$\|\psi_n(u)\| = O(\varepsilon_n^{\frac12})\qquad \text{as } n\to\infty\,,$$

which is sufficient to ensure the precompactness of $\{\psi_n(u)\}$ in $W^{-1,2}$. Moreover, by (3.4), we know that

$$(4.8)\qquad \begin{cases}\varepsilon_n m|u|^{m-1}u_x^2 & \text{belongs to a bounded set of } L^1(\Omega)\,,\\ \eta''(u) & \text{is in a bounded set of } L^\infty(\Omega)\,,\end{cases}$$

therefore $\{\chi_n(u)\}$ is in a bounded set of $L^1(\Omega)$. Since $L^1(\Omega)\subset(\Omega)$, we M obtain that $\{\chi_n\}$ is a bounded sequence in $M(\Omega)$.

Finally, it is easily checked that $\{\varphi_n(u)\}$ is a bounded sequence in $W^{-1,\infty}(\Omega)$, in view of the boundedness of $\{\eta(u)\}$ and $\{q(u)\}$ in $L^\infty(\Omega)$.

By applying the proposition (1.1) we complete the proof of the main result. Indeed we obtain that the approximating solution converge, strongly in L^p, to the weak solution of (0.1). Let us consider a test function $\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, $\varphi\geqslant 0$, therefore one has (dropping, as above, the index ε in the solution u^ε of (0.2))

$$(4.9)\qquad \iint\{\partial_t\eta(u)+\partial_x q(u)\}\varphi\,dx\,dt = \varepsilon m\iint\eta'(u)(|u|^{m-1}u_x)_x\varphi\,dx\,dt =$$

$$= -\varepsilon m\iint\eta''(u)|u|^{m-1}u_x^2\varphi\,dx\,dt + \varepsilon m\iint(\eta'(u)|u|^{m-1}u_x)_x\varphi\,dx\,dt\leqslant$$

$$\leqslant \varepsilon m\iint\{\eta'(u)|u|^{m-1}u_x\}_x\varphi\,dx\,dt\,.$$

Since, using the estimates of proposition (3.1), one has

$$\left|\varepsilon m\iint\{\eta'(u)|u|^{m-1}u_x\}_x\varphi\,dx\,dt\right|\leqslant$$
$$\leqslant\varepsilon^{\frac{1}{2}}m\|\varepsilon^{\frac{1}{2}}|u|^{(m-1)/2}u_x\|_{L^2}\|\eta'(u)\|_{L^\infty}\||u|^{(m-1)/2}\|_{L^\infty}\|\varphi_x\|_{L^2},$$

then, as $\varepsilon\downarrow 0$, the Lax entropy inequality is fulfilled in $W^{-1,p}_{\mathrm{loc}}$, for every $p<+\infty$.

REMARK 1. The above results may be easily generalized to the following equation

$$u_t+f(u)_x=\varepsilon\varphi(u)_{xx}$$

where φ verifies

$$\frac{\varphi(u)\varphi''(u)}{(\varphi'(u))^2}\geqslant\alpha>0\,.$$

REMARK 2. The paper of Osher and Ralston [7] investigates the existence of a travelling wave solution to (0.2) connecting the two constant states u_- and u_+ of a simple shock for (0.1). As a consequence of our result, these travelling wave solutions converge, in the sense precised in the above results, towards the simple shock wave as ε tends to zero.

REFERENCES

[1] B. DACOROGNA, *Weak Continuity and Weak Lower Semicontinuity of Non-Linear Functionals*, Lecture Notes in Mathematics, **922**, Springer-Verlag, Berlin-Heidelberg-New York (1982).

[2] R. J. DI PERNA, *Convergence of approximate solutions to conservation laws*, Arch. Rat. Mech. Anal., **82** (1983), pp. 27-70.

[3] P. D. LAX, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM Reg. Conference Series in Applied Mathematics, Philadelphia (1973).

[4] P. L. LIONS, *Generalized Solutions of Hamilton-Jacobi Equations*, Research Notes in Mathematics, **69**, Pitman, London (1982).

[5] F. MURAT, *Compacité par compensation*, Ann. Sc. Norm. Pisa, Sci. Fis. Mat., **5** (1978), pp. 489-507.

[6] O. A. Oleinik, *Discontinuous solutions of nonlinear differential equations*, Uspekhi Mat. Nauk (N.S.), **12**, n. 3(75) (1957), pp. 3-73 (English transl. in Amer. Math. Soc. Transl. Ser. 2, vol. 26, pp. 95-172).
[7] S. Osher - J. Ralston, *L^1 stability of travelling waves with applications to convective porous media flow*, Comm. Pure Appl. Math., **35** (1982), pp. 737-749.
[8] L. Tartar, *Compensated compactness and applications to partial differential equations*, in *Research Notes in Mathematics, Nonlinear Analysis and Mechanics: Herriot-Watt Symposium*, Vol. 4, ed. R. J. Knops, Pitman (1979).
[9] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, in *Nonlinear PDEs*, J. Ball (Ed.), D. Reidel Publishing Company (1983).
[10] L. Van Hove, *Sur l'extension de la condition de Legendre du calcul des variations aux integrales multiples à plusieurs fonction inconnues*, Nederl. Akad. Weten., **50** (1947), pp. 18-23.
[11] A. I. Vol'pert - S. I. Hudjaev, *Cauchy problem for degenerate second order quasilinear parabolic equations*, Math. USSR Sbornik, **7** (1969), pp. 365-387.
[12] J. I. Diaz - R. Kersner, *Nonexistence d'une des frontières libres dans une équation dégenerée en theorie de la filtration*, C. R. Acad. Sc. Paris, **296**, Série I (28 mars 1983), pp. 505-508.
[13] J. I. Diaz - R. Kersner, *On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium*, MRC Tech. Summ. Rep. (University of Wisconsin) (1983).
[14] B. H. Gilding, *A nonlinear degenerate parabolic equation*, Ann. Sc. Norm. Pisa, Sci. Fis. Mat., **4** (1977), pp. 393-432.
[15] B. H. Gilding - L. A. Peletier, *The Cauchy problem for an equation in the theory of infiltration*, Arch. Rat. Mech. Anal., **61** (1976), pp. 127-140.
[16] E. Di Benedetto, *Continuity of weak solutions to a general porous medium equation*, Indiana Univ. Math. J., **32** (1983), pp. 83-118.
[17] O. A. Oleinik, *Uniqueness and stability of the generalized solution of the Cauchy problem for a quasilinear equation*, Usp. Mat. Nauk., **14** (1959), pp. 165-170 (English transl. in Amer. Math. Soc. Transl. Ser. 2, **23** (1964), pp. 285-290).
[18] M. Crandall, *The semigroup approach to first order quasilinear equation in several space variables*, Israel J. Math., **12** (1972), pp. 108-132.
[19] P. Marcati, *Approximate solutions to conservation laws via convective parabolic equations*, to appear Comm. P.D.E. (1988).

Manoscritto pervenuto in redazione il 26 gennaio 1988.