RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

V. M. TORTORELLI

A characterization of internal sets

Rendiconti del Seminario Matematico della Università di Padova, tome 81 (1989), p. 193-200

<http://www.numdam.org/item?id=RSMUP_1989__81__193_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 81 (1989)

A Characterization of Internal Sets.

V. M. TORTORELLI (*)

SUMMARY - In this paper, which sets out from an elementary Model Theory viewpoint, a characterization of the class of internal sets is given for first order structures elementarily equivalent to Standard Analysis.

1. Introduction.

In early definitions of nonstandard structures, which employed Types Theory language, internal sets coincide with the nonstandard elements in a L-structure (see [L], [R]); it is, however, a meaningful concept, as an L-structure is generally not full. A set-theoretical characterization of nonstandard structure has been given in [RZ] and [Z]. These authors, in accordance with the type-theoretical exposition, define the standard superstructure $\mathcal{A}$, the nonstandard superstructure $\mathcal{M}$:

$$\mathcal{M} = \bigcup_{n\in\mathbf{N}} \mathcal{M}_n\,, \qquad \mathcal{M}_{n+1} = \wp\left(\bigcup_{k=0}^{n} \mathcal{M}_k\right),$$

and the superstructure monomorphism $*$ operating between these. They next define the collection made up by internal sets:

$$\{x \in \mathcal{M}\colon \exists n(n \in \mathbf{N} \wedge x \in {}^*\mathbf{R}_n)\}\,.$$

From an elementary Model Theory point of view, if we examine

(*) Indirizzo dell'A.: S.N.S., Piazza dei Cavalieri 7, 56100 Pisa.

the structure constituted by internal sets, it is not always elementarily equivalent to the theory of the standard structure, being so only for those formulas where all quantifiers are restricted to constant symbols: $\forall x \in c$, $\exists x \in c$. This phenomenon reflects the fact that the Types Theory formulas are « restricted » by definition. If, on the other hand, we want to introduce, by means of set-theoretical language, nonstandard models as traditional first-order structures, then the first requirement would be that of elementary equivalence with the standard structure. We shall call these models nonstandard *strong models*. The second requirement would be to obtain « natural » models, i.e. transitive models, where the interpretation of the symbol $\in$ is that of the « natural membership ». Now if we admit that this « universal relation » is capable of representing, *via* isomorphisms, all extensional binary relations (Free Construction Principles, see [B] and [FH]), we axiomatically obtain natural extension which are elementarily equivalent to Standard Analysis (see [T]). In such a case the idea of an internal set would seem to be unnecessary at the level of the elements belonging to the extension.

In this paper the Regularity Axiom is given preference over the Free Construction Principles. Thus, if we also assume the elementary equivalence condition, the nonstandard models are necessarily not well founded (corollary 2); hence they can not be isomorphic to a natural structure. In order to obtain Mostowsky's collapse we must find substructures of these models which are well founded. Moving in this direction we arrive at the central conclusion of this paper, which is stated in theorem 3, and concerns a characterization of the class of internal sets, requiring not only the condition of being well founded, but also the condition of containing all standard elements and the transitivity.

2. Notations and definitions.

We consider a universe $\mathcal{U}$ verifying the usual axioms of Zermelo-Fraenkel Set Theory with atoms (*Urelementen*). We suppose $\mathbb{R}$, the set of the real numbers, to be included in the class of the atoms in $\mathcal{U}$, and we denote the set of natural numbers by $\mathbb{N}$. We suppose $\mathbb{N} \subset \mathbb{R}$.

Given that:

$$\mathbb{R}_0 = \mathbb{R}\,, \quad \mathbb{R}_{n+1} = \wp(\mathbb{R}_0 \cup \mathbb{R}_n)\,, \quad A = \bigcup_{n\in\mathbb{N}} \mathbb{R}_n\,;$$

we term the theory of $\mathcal{A} = (A; \in|_{A^2}; \mathrm{id}|_A)$ *Standard Analysis.* This is a first-order structure on the first-order language $\mathfrak{L}_A$, having as its symbols for constants all the elements of A, and as the only binary predicative symbol $\in|_{A^2}$. We denote an arbitrary $\mathfrak{L}_A$-structure by $\mathcal{M} = (M; \in_{\mathcal{M}}; \{{}^*a: a \in A\})$, where $*$ is the function of the interpretation of the symbols for the constants.

An $\mathfrak{L}_A$-structure is *natural* if $\in_{\mathcal{M}} = \in|_{M^2}$, and M is transitive except for the atoms, *i.e.* $x \in y \in M \to x \in M$ if y is not an atom.

An $\mathfrak{L}_A$-structure is a *non-standard strong model of Analysis* if:

i) there exists an $m \in M$ such that for every $n \in \mathbb{N}$ we get

$$\mathcal{M} \models y \in \mathbb{N} \wedge y > n[y = m] \, ;$$

ii) there is an embedding of $\mathcal{A}$ in $\mathcal{M}$;

iii) every formula is satisfied in $\mathcal{A}$ if and only if it is satisfied in $\mathcal{M}$ (elementary equivalence $\mathcal{M} \equiv \mathcal{A}$).

The existence of such a structure is guaranteed by Completeness Theorems, and by Łos' Theorem.

The usual definition of *the class of the internal sets of* a $\mathfrak{L}_A$-structure $\mathcal{M}$ is:

$$\{m \in M: \exists n \in \mathbb{N}\ (m \in_{\mathcal{M}} {}^*\mathbb{R}_n)\} = \bigcup_{n \in \mathbb{N}} \{m \in M: m \in_{\mathcal{M}} {}^*\mathbb{R}_n\} \, ,$$

that we simply denote by A' throughout the paper. As usual we say that an element of M is *standard* if it is in the range of $*$.

If C is an arbitrary set, and R is a binary relation on C, $R \subset C \times C$, we say that C is *well-founded by R*, if and only if every non void $C \subset B$ has an R-minimal element. We simply say that a structure $\mathcal{M}$ is *well-founded* if it is well-founded by $\in_{\mathcal{M}}$.

3. A characterization of internal sets.

We first see that is not possible to obtain natural nonstandard strong models.

PROPOSITION 1. *Let* $\mathcal{N} = (N; \in|_{N^2}; \{{}^*a: a \in A\})$ *be a natural* $\mathcal{L}_A$*-structure with an infinite natural number, i.e.:* $\exists m \in {}^*\mathbf{N}$ $(\forall n \in \mathbf{N}$ $(m \mathbin{{}^*{>}} {}^*n))$. *Then* $\mathcal{N}$ *cannot be elementarily equivalent to Standard Analysis.*

PROOF. We have:

$$\mathcal{A} \models \forall x\Big(x \subset \mathbf{N} \wedge \text{« } x \text{ finite »} \wedge \exists f\Big(\operatorname{dom} f = \\ = x \wedge \forall z \in x \ \ \forall y \in x\big(y < z \to f(y) \in f(z)\big)\Big)\Big)\,.$$

If *ab absurdo* $\mathcal{N} \equiv \mathcal{A}$, then $\mathcal{N}$ verifies the same formula. Let us now take an $m \in {}^*\mathbf{N}$ such that $m \mathbin{{}^*{>}} {}^*n$ for every $n \in \mathbf{N}$, and also suppose $x = \{0, \ldots, m\}$.

It follows that $x \in N$, and $\mathcal{N} \models$ « x finite ». Then we would get $f(m) \ni f(m \mathbin{{}^*{-}} {}^*1) \in f(m \mathbin{{}^*{-}} {}^*2) \ldots$, and this is absurd as $\mathcal{N}$ is well-founded. QED

COROLLARY 2. *If* $\mathcal{M}$ *is a nonstandard strong model, it cannot be well-founded.*

PROOF. If $\mathcal{M}$ is well founded then we can obtain a natural structure which is isomorphic to it by means of a Mostowsky's collapse, and this is absurd. QED

Our central conclusion is the following characterization theorem.

THEOREM 3. *If* $\mathcal{M} \equiv \mathcal{A}$ *then* A' *is the only subclass of* M *such that:*

i) *it is well-founded by* $\in_{\mathcal{M}}$;

ii) *it is* $\in_{\mathcal{M}}$*-transitive, i.e.* $x \in_{\mathcal{M}} y \wedge y \in A' \to x \in A'$;

iii) *it contains all the standard elements.*

PROOF. We first prove that A' verifies the three conditions above stated.

i) The thesis follows on an absurd hypothesis, and by applying the elementary equivalence to the formulas φ_n defined as follows:

$$\forall x \in \mathbf{R}_{n+1} \forall y (y \in x \to y \in \mathbf{R}_n \vee y \in \mathbf{R}_0)\,.$$

ii) If $y \in A'$, then there exists an $n \in \mathbf{N}$ such that $y \in_{\mathcal{M}} {}^*\mathbf{R}_n$. We obtain by means of the elementary equivalence on φ_n:

$$x \in_{\mathcal{M}} y \to x \in_{\mathcal{M}} {}^*\mathbf{R}_{n-1} \vee x \in_{\mathcal{M}} {}^*\mathbf{R}_0 .$$

In particular we obtain $x \in A'$.

iii) Straightforward.

We divide the proof of the unicity into two steps.

Step 1. If $B \subset M$ verifies ii) and iii), then $A' \subset B$. Indeed, every element of A' $\mathcal{M}$-belongs to some standard element. But B is $\in_{\mathcal{M}}$-transitive and it contains all the standard elements.

Step 2. If $B \subset M$ verifies i) and ii), then $B \subset A'$. We now state a preliminar lemma to prove this point, its proof is given at the end of the theorem.

LEMMA 4:

$$\forall a \notin A' \ \exists b \in_{\mathcal{M}} a \wedge b \notin A' .$$

Thus if $a \in B \setminus A'$, we have:

$$\exists b_1 \in_{\mathcal{M}} a \wedge b_1 \notin A' ,$$

also obtaining $b \in B$, for B is $\in_{\mathcal{M}}$-transitive. Repeating this procedure we get a sequence $\{b_n\}_{n \in \mathbf{N}} \subset B$ which is increasing with respect to $\in_{\mathcal{M}}$, and this contradicts the hypothesis that B is well-founded by $\in_{\mathcal{M}}$. QED

Let us set out the proof of the lemma, that is based on the interchange between inner and outer « stratification » of the nonstandard strong model.

PROOF OF LEMMA 4. Given by definition that:

$$\forall a \in A \qquad \varrho(a) = \min\{n : a \in \mathbf{R}_n\}, \ \ \forall b \in A' \ \ \varrho'(b) = \min\{n : b \in_{\mathcal{M}} {}^*\mathbf{R}_n\} ,$$

using φ_n we get $\varrho(a) = \max\{\varrho(x) : x \in a\} + 1$.

Let $y \prec \varrho_x$ be the following formula:

$$[\text{« } y \text{ function »} \wedge \mathbf{R}_0 \cup \{x\} \subset \operatorname{dom} y \wedge \text{« dom } y \text{ transitive »} \wedge \operatorname{ran} y \subset \mathbf{N} \wedge \\ \wedge \forall z \in \mathbf{R}_0(y(z) = 0) \wedge \forall z \in \operatorname{dom} y \setminus \mathbf{R}_0(y(z) = \max\{y(t) : t \in z\} + 1)] .$$

And let ψ_1, ψ_2, ψ_3 be respectively:

$$\forall x \, \exists y (y \prec \varrho_x) ;$$

$$\forall x \, \forall y \, \forall z \big(z \prec \varrho_x \wedge y \prec \varrho_x \to \forall t (t \in \operatorname{dom} z \cap \operatorname{dom} y \to z(t) = y(t))\big) ;$$

$$\forall x \, \forall y \, \forall t \, \forall s \big(t \prec \varrho_x \wedge s \prec \varrho_y \to \forall z (z \in \operatorname{dom} t \cap \operatorname{dom} s \to t(z) = s(z))\big) .$$

And finally, for every natural number n, we define ψ_{3+n} as:

$$\forall x \, \forall t \big(t \prec \varrho_x \to \forall y (t(y) \leqslant n \to y \in \mathbf{R}_n)\big) .$$

We have that $\mathcal{A} \models \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_{3+n}$, hence, by elementary equivalence, this formula is also verified by $\mathcal{M}$.

By using ψ_2, when $b \in A'$, we are able to choose in $\mathcal{M}$ ϱ_b from among the various $\mathcal{M}$-extensions or $\mathcal{M}$-restrictions of $^*(\varrho|_{\mathbf{R}_{\varrho'(\bar{b})}})$.

Now if $a \notin A'$, for some ϱ_a we have

$$\exists \bar{b} \in_{\mathcal{M}} a (\varrho_a(\bar{b})^* + {}^*1 = \varrho_a(a)) .$$

If we suppose by absurd hypothesis: $\forall c \in_{\mathcal{M}} a (c \in A')$, then in particular we get:

$$\exists \bar{n} \in \mathbf{N}(\bar{b} \in_{\mathcal{M}} {}^*\mathbf{R}_{\bar{n}}) .$$

Using ψ_3 we obtain:

$$\varrho_a(\bar{b}) = \varrho_{\bar{b}}(\bar{b}) = {}^*(\varrho|_{\mathbf{R}_{\varrho'(\bar{b})}})(\bar{b}) .$$

But the range of $\varrho|_{\mathbf{R}_{\varrho'(\bar{q})}}$ is a subset of $\{0, \ldots, \varrho'(\bar{b})\}$. Thus $^*(\varrho|_{\mathbf{R}_{\varrho'(\bar{b})}})$ has its $\mathcal{M}$-range $\mathcal{M}$-contained in $^*\{0, \ldots, \varrho'(\bar{b})\}$.

Whence:

$$\varrho_a(\bar{b}) = \varrho_{\bar{b}}(\bar{b}) = {}^*(\varrho|_{\mathbf{R}_{\varrho'(\bar{b})}})(\bar{b}) \;{}^*\!\leqslant {}^*(\varrho'(\bar{b})) \;{}^*\!< {}^*\bar{n} .$$

It follows that using ψ_1:

$$\forall c \in_{\mathcal{M}} a(\varrho_a(c) \,{}^*\!\leqslant \varrho_a(\bar{b}) \,{}^*\!< {}^*\bar{n}) \;,$$

and furthemore using ψ_{3+n}:

$$\forall c \in_{\mathcal{M}} a(c \in_{\mathcal{M}} {}^*\mathbf{R}_{\bar{n}}) \;.$$

So $\mathcal{M} \models y \subset \mathbf{R}_{\bar{n}}[y = a]$, *i.e.* $a \in_{\mathcal{M}} {}^*\mathbf{R}_{n+1}$, and in particular $a \in A'$, which is absurd. QED

4. Final remarks.

As a consequence the substructure of a nonstandard strong model made up of its internal sets is well founded, and we can collapse it into a natural structure. The latter cannot be elementarily equivalent to Standard Analysis. However the following classical result holds (for the proof see [SL], [RZ]):

THEOREM 5. *If $\mathcal{N}$ is the structure of a nonstandard strong model made up of its internal sets, then every formula with all quantifiers restricted to symbols for constants in $\mathfrak{L}_A$, i.e. the elements of A, is satisfied in $\mathcal{N}$ if and only if it is satisfied in $\mathcal{A}$* (weak elementary equivalence condition).

In this way we get the following counterpart of the classical definition of a nonstandard model of the Analysis, with respect to first order structures:

DEFINITION 6. *An $\mathfrak{L}_A$-structure $\mathcal{M}$ is a nonstandard model if:*

i)′ *there exits infinite natural numbers;*

ii)′ *there is an embedding of $\mathcal{A}$ into $\mathcal{M}$;*

iii)′ *$\mathcal{M}$ is elementarily equivalent to $\mathcal{A}$ in weak sense.*

Acknowledgements. This paper is based on a seminar given as part of M. Servi's « Istituto Nazionale di Alta Matematica » course on « Logica Matematica e Teoria delle Categorie », held at the Departement of Mathematics of the University of Pisa in May 1984.

I am most grateful to M. Servi (Università di Parma) for the opportunity of public relating the conclusion reached in this piece of research, and for the stimulating discussions had with him. I am also deeply indebted to M. Forti (Università di Cagliari) and to F. Honsell (Università di Torino) for their encouragement and collaboration.

REFERENCES

[B] M. Boffa, *Sur la Théorie des Ensemble sans Axiom de Fondement*, Bull. Soc. Mat. Belgique, **21** (1) (1969).

[FH] M. Forti - F. Honsell, *Set Theory with Free Construction Principles*, Ann. Scuola Norm. Sup. Pisa, Cl. Sc. (1983).

[L] W. A. J. Luxembourg, *A General Theory of Monads*, in *Applications of Model Theory to Algebra, Analysis and Probability*, W. A. J. Luxemburg Editor, Holt-Rinehart-Winston (1969).

[R] A. Robinson, *Non Standard Analysis*, North-Holland (1966).

[RZ] A. Robinson - E. Zakon, *A Set-Theoretical Characterization of Enlargements*, in *Applications of Model Theory to Algebra, Analysis and Probability*, W. A. J. Luxemburg Editor, Holt-Rinehart-Winston (1969).

[T] V. M. Tortorelli, *Γ-limiti ed analisi non standard*, Tesi di laurea, Pisa (1983).

[Z] E. Zakon, *A New Variant of Non-Standard Analysis*, Lectures Notes in Math., vol. **369**, Springer-Verlag (1970).

Manoscritto pervenuto in redazione il 18 aprile 1988.