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Groups with finite automorphism classes of subgroups

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1. Introduction.

The *automorphism class* of a subgroup H of a group G is the orbit $\{H^\alpha : \alpha \in \operatorname{Aut} G\}$ of H under the action of $\operatorname{Aut} G$. One of the main aims of [7] was to classify those groups for which the automorphism classes are boundedly finite, the answer being as follows.

THEOREM A. *The following properties of a group G are equivalent.*

(i) *The automorphism classes of subgroups of G are boundedly finite.*

(ii) *The automorphism classes of abelian subgroups of G are boundedly finite.*

(iii) *Either* (*a*) $\operatorname{Aut} G$ *is finite,*

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or (*b*) *there is a direct decomposition* $G = G_1 \times G_2$, *where* G_1 *is a locally cyclic torsion group*, G_2 *is a finite central extension of a direct product of finitely many groups of type* Z_{p^∞} *for different primes* p, *and* G_1 *and* G_2 *do not contain elements of the same prime order.*

The problem of determining the groups in which the automorphism classes are merely finite was left open in [7]. We shall prove here that the two classes are the same:

THEOREM *B. The following properties of a group G are equivalent.*

(i) *The automorphism classes of subgroups of G are boundedly finite.*

(ii) *The automorphism classes of abelian subgroups of G are finite.*

(iii) *Either* (*a*) *G is not periodic and* Aut G *is finite,*

or (*b*) *G is periodic and of the structure described in* (iii) (*b*) *of Theorem A.*

To prove this, we need only show that a group with finite automorphism classes of abelian subgroups is of the type mentioned in (iii) of Theorem B, and then apply Theorem A.

The analogous results for conjugacy of subgroups and of abelian subgroups are due to B. H. Neumann [6] and Eremin [2]; groups with finite classes of abelian subgroups are just the centre-by-finite groups. We shall make much use of this result in our proof, which is, surprisingly, a good deal harder than that of the Neumann-Eremin theorem.

By an *abus de langage* we call the set $\{H^\alpha : \alpha \in \operatorname{End} G\}$ of endomorphic images of a subgroup H of a group G the endomorphism class of H; note that endomorphism classes need not be disjoint. The following theorem is an analogue of Theorems A and B for endomorphism classes:

THEOREM C. *The following properties of a group G are equivalent.*

(i) *The endomorphism classes of subgroups of G are boundedly finite.*

(ii) *The endomorphism classes of abelian subgroups of G are finite.*

(iii) *G is a finite central extension of a direct product of finitely many groups of type* Z_{p^∞} *for different primes* p.

Finally, we have the following curious result:

THEOREM D. *Let G be a group in which the endomorphism classes of elements are boundedly finite. Then G is finite.*

This generalizes a theorem of Baer [1] stating that a group is finite if its endomorphism set is finite. See [7] for groups in which the automorphism classes of elements are boundedly finite. The boundedness condition in Theorem D is essential, as a glance at the direct product of infinitely many finite groups of coprime orders shows.

All our methods are elementary and notation standard. We use without further comment the fact that groups of the type under discussion are centre-by-finite and thus have finite commutator subgroups. Further, if α is any endomorphism of a group G into the centre of G such that $\alpha^2 = 0$, then the map $1 + \alpha$ is an automorphism of G. [4] is a good reference for facts on abelian groups that we use.

We thank Joachim Neubüser for pointing out a serious error in a first version of Theorem C.

2. Proof of Theorem B: the periodic case.

In this section, G is a periodic group in which the orbits of abelian subgroups are finite, and we show that G has the structure defined in (iii) (b) of Theorem A. The proof is accomplished in several stages.

1) *G does not have a divisible p-subgroup of rank 2, for any prime p.*

Suppose that G contains a subgroup $U \times V$, where $U \cong V \cong Z_{p^\infty}$. Since G is central-by-finite, U and V are central; and since G' is finite, $UVG'/VG' \cong Z_{p^\infty}$. Thus $G/VG' = UVG'/VG' \times R/VG'$ for some $R \leqslant G$, and maps φ such that $UVG'/VG' \twoheadrightarrow V$, $R/VG' \to 1$ give rise to endomorphisms $\alpha\colon G \to Z(G)$ such that $\alpha^2 = 0$. But

$$U^{1+\alpha} = \{u(uVG')^\varphi : u \in U\};$$

since there are $2_{\aleph_0}$ different choices for the complement of U in $U \times V$, this means that the Aut G-orbit of U is infinite, a contradiction.

2) *The reduced part of every Sylow p-subgroup of the centre of G is finite.*

If not, then G/G' has the same property, and we can write $G/G' = \langle aG' \rangle \times X/G'$, where $\langle aG' \rangle$ is a non-trivial cyclic p-group: since $|G:X| < \infty$, $X \cap Z(G)$ must have an infinite elementary abelian subgroup Y, because a basic subgroup of an abelian p-group is finite if and only if it is a direct factor and thus the reduced part is finite. Any homomorphism $\varphi\colon \langle aG' \rangle \to Y$, together with the trivial map $X/G' \to 1$, gives rise to an endomorphism α of G such that $\alpha^2 = 0$, and $\langle a \rangle^{1+\alpha} = \langle a \cdot (aG')^{\varphi} \rangle$; since Y is infinite, $\langle a \rangle$ has infinite Aut G-orbit.

A consequence of 1 and 2 is:

3) *Every infinite Sylow p-subgroup of $Z(G)$ is of the form $Z_{p^\infty} \times F_p$, where F_p is finite.*

For every $p > |G'|$, the Sylow p-subgroups of G are central. We use this fact to prove:

4) *The subgroup $S = \langle F_p : p > |G'| \rangle$ is a direct factor of G.*

Every subgroup $F_p G'/G'$ is a direct factor of G/G', since it is finite and the reduced part of a Sylow p-subgroup, so that SG'/G' is a direct factor of G/G', say $G/G' = SG'/G' \times L/G'$. But then $G = SG'L = SL$ and $SG' \cap L = G'$, so that $S \cap G'L = 1$ and S is a direct factor.

The next step is proved in a very similar way to 4 and we omit the proof.

5) *For every $p > |G'|$, a divisible p-subgroup of G is a direct factor of G.*

For our final step, note that S is the direct product of the F_p.

6) *Almost all the F_p are cyclic.*

It is enough to show that almost all the F_p with $p > |G'|$ are cyclic. Suppose that infinitely many of them are non-cyclic, so that F_p has a direct factor of the form $\langle a_p, b_p : a_p^{n_p} = b_p^{m_p} = [a_p, b_p] = 1 \rangle$, $n_p \geqslant m_p \geqslant 1$, for p lying in an infinite set π. Consider an automorphism φ of G that extends the automorphisms $a_p \to a_p b_p^{\lambda_p}$, $b_p \to b_p$, $p \in \pi$, where $(\lambda_p, p) = 1$. The image of the abelian subgroup $A = \langle a_p : p \in \pi \rangle$ under φ is $\langle a_p b_p^{\lambda_p} : p \in \pi \rangle$, and it is clear that suitable choices of the λ_p give rise to infinitely many Aut G-images of A.

A quick glance now shows that G has the structure required, and the proof of the periodic case of Theorem B is complete.

3. Proof of Theorem B: the nonperiodic case.

Throughout this section, G is a non-periodic group in which the orbits of abelian subgroups are finite and T its periodic part (which is a subgroup since G is an FC-group). Again we proceed in several stages to the proof that $\operatorname{Aut} G$ is finite.

1) *T has no subgroups isomorphic to a Z_{p^∞}.*

Let $A = \langle a_1, a_2, \ldots\rangle$ be such a subgroup, with $a_1^p = 1$, $a_{i+1}^p = a_i$ for $i \geq 1$; as before, A is central.

If $(G/T)^p \neq G/T$, take $x \in G$ such that $xT \notin (G/T)^p$. For $k \geq 1$, set $X_k/T = (G/T)^{p^k}$; then xX_k has maximal order p^k in G/X_k, so that $G/X_k = \langle xX_k\rangle \times Y_k/X_k$ for some subgroup Y_k of G. If σ_k is the endomorphism of G defined by $x^{\sigma_k} = a_k$, $Y_k^{\sigma_k} = 1$, then $1 + \sigma_k \in \operatorname{Aut} G$ and $x^{1+\sigma_k} = xa_k$; but this gives the contradiction that $\langle x\rangle$ has infinitely many images under $\operatorname{Aut} G$.

Thus we may assume that G/T is p-divisible. Let X/G' be a complement of the quasicyclic subgroup AG'/G' in G/G'; then $G = AX$, $X \cap A = \langle a_m\rangle$ is finite, $X/(T \cap X)$ is p-divisible and there is a subgroup $Y \leqslant X$ with $T \cap X \leqslant Y$ such that $X/Y = \langle c_1 Y, c_2 Y, \ldots\rangle \cong A$, the isomorphism being given by $a_i = (c_i Y)^\varphi$ for all i. For a p-adic integer α, let γ_α be the automorphism of G that is 1 on A and such that $x^{\gamma_\alpha} = x(xY)^{\varphi\alpha}$ for $x \in X$. Then $X^{\gamma_\alpha} = \langle a_1^\alpha c_1, a_2^\alpha c_2, \ldots\rangle Y$ and $X^{\gamma_\alpha} \cap A = X \cap A$. We claim that $X^{\gamma_\alpha} \neq X^{\gamma_\beta}$ if $\alpha \neq \beta$. Indeed, if $X^{\gamma_\alpha} = X^{\gamma_\beta}$ we have $a_i^\beta c_i \in X^{\gamma_\alpha}$, so that $\alpha_i^{\alpha-\beta} \in X^{\gamma_\alpha} \cap A = \langle a_m\rangle$, and so $p^m(\alpha - \beta) = 0$. But then $\alpha = \beta$, so our claim is justified and $\{X^\gamma : \gamma \in \operatorname{Aut} G\}$ is infinite; this is not yet a contradiction, since X is non-abelian if G is. However, a very easy argument now shows that the abelian subgroup X^n, $n = |G : Z(G)|$, is such that $(X^n)^\alpha \neq (X^n)^\beta$ if $\alpha \neq \beta$, and this is the required contradiction.

2) *Every Sylow p-subgroup of T is finite.*

Clearly, since there are no divisible p-subgroups by 1, it is enough to show that there are no infinite elementary abelian p-subgroups. By way of contradiction, assume that E is one. Since G' is finite, the p-component T_p/G' of T/G' is not divisible, so that it has a cyclic direct factor $\langle aG'\rangle$ of order $p^k > 1$, and of course we may choose a to be of order p^h, $h \geqslant k$. Write $G/G' = \langle aG'\rangle \times B/G'$. Then the intersection $E_0 = E \cap Z(G) \cap B$ is still infinite, since $B \cap Z(G)$ is of finite index; for each $z \in E_0$, the identity map of B extends to an automorphism φ_z of G sending a to az, and again we have the contradiction that the Aut G-orbit of $\langle a \rangle$ is infinite.

3) *T is finite.*

Let ω be the set of primes p for which $Z(G)$ has a p-element. Our task is to show that ω is finite, so we assume that it is infinite, and without loss of generality that the index n of $Z(G)$ in G is coprime to p, if $p \in \omega$.

CASE 1. *There exist an element g of infinite order and an infinite subset $\omega_0 \subseteq \omega$ such that, for every $p \in \omega_0$, g fails to be infinitely p-divisible.*

Enumerate the elements of ω_0 in some way, $\omega_0 = \{p_1, p_2, \dots\}$, and for each $p_i \in \omega_0$ define k_i to be the largest integer such that $x_i^{p_i^{k_i}} = g$ for some $x \in G$. Without loss of generality we may assume that $g \in Z(G)$. Set $y_0 = g$, and suppose that $y_1^{p_1^{k_1}} = y_0$. The p_2-characteristic of y_1 is again k_2, and we choose y_2 such that $y_2^{p_2^{k_2}} = y_1$. Proceed in this way; once $y_0, y_1, \dots, y_r$ have been chosen, the p_{r+1}-characteristic of y_r is k_{r+1}, and we choose y_{r+1} such that $y_{r+1}^s = y_r$, where s is short for $p_{r+1}^{k_{r+1}}$. By construction, the subgroup $Y = \langle y_0, y_1, \dots, y_r, \dots \rangle$ is locally cyclic and torsion-free, and moreover $y_i \notin G^{p_i}$ for $i > 0$.

Now let t_i be an element of order p_i in $Z(G)$ such that $\langle t_i \rangle G^{p_i} \neq \neq YG^{p_i} = \langle y_i \rangle G^{p_i}$, if one exists, and let φ_i be the usual automorphism induced from a homomorphism $G/G^{p_i} \to \langle t_i \rangle$ with kernel a complement of YG^{p_i}/G^{p_i} containing $\langle t_i \rangle G^{p_i}$: If no such t_i exists, then $G = \langle u_i \rangle \times \times K_i$, where $\langle u_i \rangle$ is the p_i-part $S_{p_i}(G)$ of G, which is of course the Sylow p_i-subgroup of $Z(G)$ since $p_i \nmid n$. In that case, let φ_i be the identity on K_i and a non-trivial power on $\langle u_i \rangle$ (which must exist except in the case $\langle u_i \rangle \simeq Z_2$, and we can ignore this). In both cases, $Y \cdot Y^{\varphi_i} \cap S_{p_i}(G) \neq 1$ and $Y^{\varphi_i} \leqslant Y \times S_{p_i}(G)$, and this shows that our subgroup Y has infinite Aut G-orbit.

CASE 2. *Every element of infinite order is infinitely p-divisible for almost all p.*

Assume first that G/T has infinite rank. Then there is a countable independent subset of G/T which we may index by elements of ω: $\{x_p T\}_{p \in \omega}$, and we can suppose that each x_p is in $Z(G)$. For each $p \in \omega$, choose $a_p \in S_p(G)$, the Sylow p-subgroup of G, of maximum order, and an integer s_p such that $s_p \not\equiv 1$ and $ps_p \equiv p$, where the congruences are modulo the exponent of $S_p(G)$. Note that $S_p(G)$ is a direct factor of G since $S_p(G) \cap G' = 1$ (p does not divide the index of $Z(G)$ in G and therefore does not divide $|G'|$ [8]) and $S_p(G)G'/G'$ is a direct factor of G/G'. Let C_p be a complement of $S_p(G)$ in G, and define an automorphism φ_p of G that is the identity on C_p and the s_p-th power automorphism on $S_p(G)$. With $X = \langle a_p x_p^p : p \in \omega \rangle$, we have $X^{\varphi_p} \neq X$, $X^{\varphi_p} \leqslant X \times S_p(G)$, and thus X has infinite Aut G-orbit.

Thus, we may assume that G/T has finite rank and is p-divisible for almost all primes in ω. Choose a prime $q \in \omega$, so that $q \nmid |G'| \cdot |G: Z(G)|$, as before, and G/T is q-divisible. Then $G = S_p(G) \times B$, where B is q-divisible, $|B:Z(B)|\,|\,n$, $Z(B)$ is q-divisible and has no q-torsion, by 1. (Recall that $n = |G:Z(G)|$). If r is the order of q mod n, then the mapping $\alpha: B \to B$ given by $b^\alpha = b^{q^r}$ is an automorphism of B (this because $q^r \equiv 1$ mod $|B:Z(B)|$ and α induces an automorphism on $Z(B)$), which clearly extends to an automorphism of G. Every element of infinite order in B has infinitely many images under $\langle\alpha\rangle$.

4) *For every $p \in \omega$, $|G:G^p|$ is finite.*

If $|G:G^p|$ is infinite, then $|G:G^pG'Q_p|$ is infinite, where Q_p is the Sylow p-subgroup of $Z(G)$. Take any subgroup N of index p containing $G^pG'Q_p$, and let φ be a homomorphism of G into $Z(G)$ with kernel N. Write $Z(G) = Q_p \times X$. Then $X \cap X^{1+\varphi} = X \cap N$; and, since there are infinitely many N, there must be infinitely many $X \cap N$ since $|G:X|$ and $|G:N|$ are (boundedly) finite. Thus there are infinitely many $X^{1+\varphi}$.

We come now to the final stage of the proof. Let R be a transversal for $Z(G)$ in G; if K is any subgroup of G containing R, then $C_G(K) = Z(G)$. Set $\Gamma = \operatorname{Aut} G$.

For a finite subset F of G, we define the closure $H(F)$ of F as follows:

$$H(F) = \{x \in G: x^k \in \langle f^\gamma : f \in F \cup R,\ \gamma \in \Gamma \rangle \text{ for some } k \neq 0\}.$$

Clearly $H(F)$ is a characteristic subgroup, $G/H(F)$ is torsion-free, and $Z(H(F)) \leqslant Z(G)$. From here on we assume that Aut G is infinite and find a contradiction.

5) *For every finite subset F of G, $C_\Gamma(H(F))$ has finite index in Γ, and $C_\Gamma(H(F)) \cap C_\Gamma(G/H(F))$ is finite.*

Clearly, $\Omega(F) := \{f^\gamma : f \in F, \gamma \in \Gamma\} \cup \{r^\gamma : r \in R, \gamma \in \Gamma\} \cup T$ is a finite Γ-invariant set. Thus the kernel K of the restriction $\Gamma \to \mathrm{Sym}\,(\Omega(F))$ has finite index; K is the centraliser in Γ of $L := \langle f^\gamma : f \in F \cup R, \gamma \in \Gamma\rangle T$. To establish the first claim we need to show that $|K : C_\Gamma(H(F))|$ is finite. Every element α of K centralises $H(F)/T$ and L, so it induces a homomorphism $H(F)/L \to T \cap Z(L)$, namely $hL \mapsto [h, \alpha]$; it is important here to notice that $Z(L) \leqslant Z(G)$ since $L \supseteq R$. However, $C_\Gamma(H(F))$ is just the kernel of this map, and thus has finite index since its image is in T and therefore finite.

Next, $C_\Gamma(H(F)) \cap C_\Gamma(G/H(F))$ is isomorphic to $\mathrm{Hom}\big(G/H(F), Z(H(F))\big) \leqslant \mathrm{Hom}\,(G/H(F), Z(G))$ via the obvious map. There is no homomorphism $G/H(F) \to Z(H(F))$ with non-periodic image, else there would be elements $a \in G$, $c \in Z(H(F))$ of infinite order and $\gamma \in C_\Gamma(H(F)) \cap C_\Gamma(G/H(F))$ such that $a^\gamma = ac$, and then $\{\langle a^{\gamma^i}\rangle = \langle ac^i\rangle : i \in N\}$ would be an infinite set of images of $\langle a\rangle$. Thus $\mathrm{Hom}\big(G/H(F), Z(H(F))\big)$ is in fact $\mathrm{Hom}\,\big(G/H(F), Z(H(F)) \cap T\big)$, which is finite by 3 and 4.

6) *For every finite subset F of G, there exist a cyclic subgroup $\langle aH(F)\rangle$ of $G/H(F)$ with $a \in Z(G)$, and $\gamma \in C_\Gamma(H(F))$ such that $\langle aH(F)\rangle \nleqslant \langle a^\gamma H(F)\rangle$.*

If not, then $\langle gH(F)\rangle = \langle g^\gamma H(F)\rangle$ for all $g \in G$ (since some power of every element is central) and thus the automorphism group induced by $C_\Gamma(H(F))$ on $G/H(F)$ would be just $\langle -1\rangle$; however, that is impossible since $C_\Gamma(H(F))/\big(C_\Gamma(H(F)) \cap C_\Gamma(G/H(F))\big)$ is infinite.

Next, we define inductively a sequence $\gamma_1, \gamma_2, \dots$ of elements of Γ and a sequence $a_1, a_2, \dots$ of elements of $Z(G)$ as follows:

γ_1 is any non-trivial element of Γ (other than -1 if G is abelian);

a_1 is any element of $Z(G)$ such that $\langle a_1^{\gamma_1}\rangle \neq \langle a_1\rangle$;

if $\gamma_1, \gamma_2, \dots, \gamma_n$ and $a_1, a_2, \dots, a_n$ have been chosen, use 6 to choose $\gamma_{n+1} \in C_\Gamma(H(a_1, \dots, a_n))$, and $a_{n+1} \in Z(G)$ such that $\langle \alpha_{n+1}^{\gamma_{n+1}} H(a_1, \dots, a_n)\rangle \nleqslant \langle a_{n+1} H(a_1, \dots, a_n)\rangle$.

Set $X = \langle a_1, \dots, a_n, \dots\rangle$; X is abelian.

7) *For every n, $X \cap H(a_1, \dots, a_n) \leqslant \langle a_1, \dots, a_n\rangle T$.*

Assume that there is an element $g \notin T$ such that $g \in H(a_1, \ldots, a_n) \cap \cap \langle a_{n+1}, a_{n+2}, \ldots \rangle$, say $gT = a_{n+1}^{k_1} \ldots a_{n+m}^{k_m} T$ with the obvious notation, and $k_m \neq 0$. Then $a_{n+m}^{k_m} \in H(a_1, \ldots, a_n, \ldots, a_{n+m-1})$ so that $a_{n+m} \in \in H(a_1, \ldots, a_{n+m-1})$, which contradicts the choice of a_{n+m}. Thus

$$X \cap H(a_1, \ldots, a_n) \leqslant (\langle a_1, \ldots, a_n \rangle T \langle a_{n+1}, \ldots \rangle) \cap H(a_1, \ldots, a_n) =$$
$$= \langle a_1, \ldots, a_n \rangle T\big(\langle a_{n+1}, \ldots \rangle \cap H(a_1, \ldots, a_n)\big) = \langle a_1, \ldots, a_n \rangle T .$$

We are now ready for the final step. Let i, j be integers such that $1 < i < j$. Then

$$X^{\gamma_j} \cap H(a_1, \ldots, a_i) = \big(X \cap H(a_1, \ldots, a_i)\big)^{\gamma_j} =$$
$$= X \cap H(a_1, \ldots, a_i) \leqslant \langle a_1, \ldots, a_i \rangle T ,$$

by 7. On the other hand, $a_i^{\gamma_i} \in X^{\gamma_i} \cap H(a_1, \ldots, a_i)$, whereas $a_i^{\gamma_i} \notin \notin \langle a_1, \ldots, a_i \rangle T$ by our choice of γ_i. Thus $X^{\gamma_i} \cap H(a_1, \ldots, a_i) \neq X^{\gamma_j} \cap \cap H(a_1, \ldots, a_i)$, so that $X^{\gamma_i} \neq X^{\gamma_j}$ and X has infinitely many images under Γ.

4. Proofs of Theorems C and D.

To prove Theorem C, we note that a group G in which the endomorphism classes of abelian subgroups are finite must have one of the structures described in (iii) of Theorem B. Since it is clear that G cannot be the direct product of infinitely many nontrivial groups, if it is periodic it must be of the type required in Theorem C.

To show that G is periodic, proceed like this. With $n := |G : Z(G)|$, the transfer map to the centre is just $g \mapsto g^n$ [5, 10.1.3]. It follows that every endomorphism of the abelian group G^n extends to one of G. If G has an element of infinite order, so does G^n; if a is such an element, the endomorphism class of $\langle a \rangle$ under the powers of the endomorphism $x \mapsto x^2$ of G^n, and thus of its extension to an endomorphism of G, is infinite.

Conversely, suppose that G is a finite central extension of a direct product C of groups of type Z_{p^∞} for different primes p, let H be a subgroup of G and α an endomorphism of G. Then $H^\alpha C/C$ is one of boundedly finitely many subgroups of G/C, and $H^\alpha C/C \simeq H^\alpha / H^\alpha \cap C$; but $H^\alpha \cap C \geqslant (H \cap C)^\alpha$, and since C is fully invariant and a direct product of different Z_{p^∞}'s, $(H \cap C)^\alpha$ is one of boundedly many subgroups. Thus H^α is one of boundedly many subgroups, as required.

Finally, we prove Theorem D. Let G be a group in which the endomorphism classes have cardinal at most k (finite). In particular, for any elements x, y of G there is an integer $r \leqslant k$ such that $x^r \in C(y)$, and it follows that $G/Z(G)$ has finite exponent m_1, say. Further, G' is finite since the conjugacy classes of elements are boundedly finite [6], of order m_2 say. Let m be any positive integer divisible by m_1 and m_2. For all x, $y \in G$ we have $(xy)^m = x^m y^m c$ for some $c \in G' \cap Z(G)$, so that $(xy)^{m^2} = x^{m^2} y^{m^2}$ and the map $x \to x^{m^2}$ is an endomorphism of G. It follows that G has finite exponent.

Now every periodic group with finite commutator subgroup is locally finite, and thus it has an infinite abelian subgroup if it is infinite. One could refer to [3] for this result, though it is very easy to prove it directly in our case. Suppose that our group G is infinite and let A be an infinite abelian subgroup. Then A contains an infinite elementary abelian p-subgroup P say, whence G/G' must have a non-trivial cyclic direct factor $\langle xG' \rangle$ of p-power order. For each $a \in P$, the map $x \mapsto a$ extends to an endomorphism of G, and thus x has infinite endomorphism class. This contradiction proves that G is finite, as required.

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