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Nonadmissible Genealogical Trees.

GABRIELLA D'ESTE (*)

In the following, K denotes a field, A denotes the free algebra $K\langle x_1, \dots, x_m \rangle$ in $m \geq 2$ non commutative variables, and we always use the term « module » to mean left module. With these hypotheses, we fix the definitions and notations used throughout the paper.

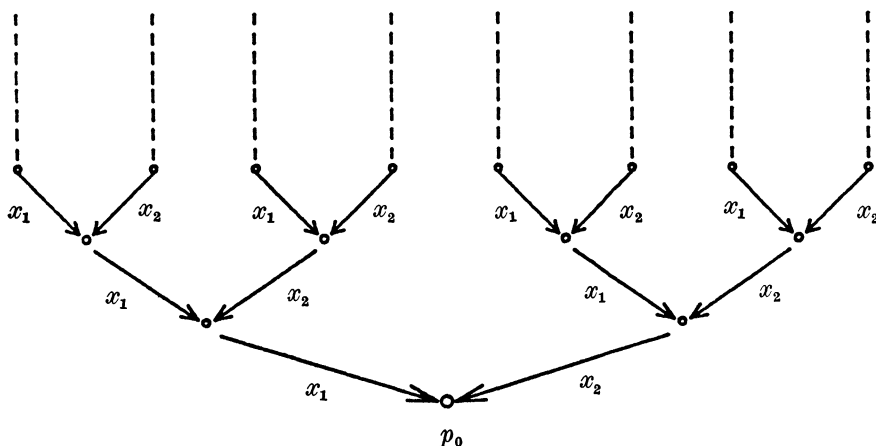
First of all, let T_∞ denote the « genealogical » oriented tree completely determined by the following conditions:

(i) T_∞ has countably many vertices $\{p_n : n \in \mathbb{N}\}$ and countably many arrows of m different types, denoted by $x_1, \dots, x_m$.

(ii) There is no arrow with starting point p_0 , and there is exactly one arrow with starting point p_n for any $n > 0$.

(iii) For any $j = 1, \dots, m$ and any $n \in \mathbb{N}$, there is exactly one arrow of type x_j with ending point p_n .

If $m = 2$, then T_∞ is of the following form.



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Next, let M_∞ denote the A -module defined as follows:

(a) The underlying K -vector space of M_∞ is K^N .

(b) If v is an element of M_∞ of the form $v = (k_n)_{n \in N}$, and $j = 1, \dots, m$, then $x_j(v) = (k_{j,n})_{n \in N}$, where, for any n ,

$$k_{j,n} = k_i \quad \text{if there is an arrow of the form } \underset{p_i}{\circ} \xrightarrow{x_j} \underset{p_n}{\circ}.$$

If v is an element of M_∞ of the form $v = (k_n)_{n \in N}$, then we denote by $\text{supp}(v)$ the set of all $n \in N$ such that $k_n \neq 0$.

For any vertex p of T_∞ , we define an element $w(p) \in A$ as follows: first of all $w(p_0) = 1$; secondly, if $p \neq p_0$ and the path along T_∞ from p to p_0 is of the form

$$\underset{p_0}{\circ} \xleftarrow{z_r} \underset{p}{\circ} \dots \underset{p}{\circ} \xleftarrow{z_1} \underset{p}{\circ}, \quad \text{then } w(p) = z_r \dots z_1.$$

Keeping the notation of [1], and using terminology suggested by [2], we say that a sequence $W = (l_n)_{n \in N}$, with $l_n \in \{x_1, \dots, x_m\}$ for any n , is a *word* in the letters $x_1, \dots, x_m$.

We say that an infinite subtree of T_∞ of the form

$$\underset{p_0}{\circ} \leftarrow \circ \leftarrow \circ \leftarrow \circ \leftarrow \circ \dots$$

is a *branch* of T_∞ . Moreover, if W is the word $(l_n)_{n \in N}$ and the branch B of T_∞ is of the form

$$\underset{p_0}{\circ} \xleftarrow{l_0} \circ \xleftarrow{l_1} \circ \xleftarrow{l_2} \circ \xleftarrow{l_3} \circ \dots,$$

then we say that B is the *branch* of T_∞ corresponding to W .

Finally, if T is a subtree of T_∞ obtained by «glueing together branches of T_∞ », that is with the property that any vertex of T belongs to a branch of T_∞ contained in T , then we briefly say that T is a *genealogical tree*. For any genealogical tree T , we denote by $M(T)$ the A -submodule of M_∞ defined by the formula

$$M(T) = \{v \in M_\infty : p_n \text{ is a vertex of } T \text{ for any } n \in \text{supp}(v)\}.$$

We say that a genealogical tree T is *admissible*, if $\text{soc } M(T)$ is essential and isomorphic to the simple module $A/\langle x_1, \dots, x_m \rangle$.

According to this definition, a word W is an *admissible word* in the sense of [1] if and only if the branch of T_∞ corresponding to W is an admissible genealogical tree.

The last two definitions used in the sequel deal with nonadmissible words. We say that a word $W = (l_n)_{n \in \mathbb{N}}$ is a *strongly nonadmissible word*, if any word U of the form

$$U = (l_{-r}, \dots, l_{-1}, l_0, l_1, l_2, l_3, \dots),$$

with $r \geq 1$ and $l_{-i} \in \{l_n : n \in \mathbb{N}\}$ for any $i = 1, \dots, r$, is a nonadmissible word. We say that a word W is a *weakly nonadmissible word*, if W is neither admissible nor strongly nonadmissible.

In section 1, we show that a genealogical tree T is not admissible if and only if T contains a sequence of distinct vertices $(q_n)_{n \in \mathbb{N}}$ which are starting points of «connected paths», that is with the property that $w(q_{n+1}) \in Aw(q_n)$ for any n . As we shall see, this characterization of the nonadmissible genealogical trees is the obvious «two-dimensional» version of a characterization, deduced from [1], of the nonadmissible branches of T_∞ . Using this result, we give an example of a nonadmissible genealogical tree formed by countably many admissible branches.

In section 2, we first determine all the strongly nonadmissible words. Roughly speaking, we can say that a word is strongly nonadmissible if and only if it is as «chaotic» as might be expected. Next, we prove that there exist as many as possible admissible words, strongly nonadmissible words and weakly nonadmissible words. Finally, we construct an admissible genealogical tree with $2^{\aleph_0}$ branches.

In section 3, we investigate the structure of the A -module $M(T)$ for what is probably the easiest choice of a nonadmissible genealogical tree, namely that of a tree with exactly one branch corresponding to a word W of the form $W = (x, x, x, x, \dots)$ for some letter x . In this case, $M(T)$ is the direct sum of $|M(T)|$ indecomposable A -modules, running through all the indecomposable injective $K[x]$ -modules.

A first example of a nonadmissible genealogical tree with all admissible branches was announced at the LMS Durham Symposium on Representations of Algebras (July 1985), and I would like to thank the organizers – and in particular Prof. S. Brenner – for the opportunity of taking part in the meeting.

1. We begin with a result on words.

LEMMA 1 ([1] Theorem 1). *A word $W = (l_n)_{n \in \mathbb{N}}$ is admissible if and only if there exists some $d \in \mathbb{N}$ such that $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for any $n > d$.*

Next we formulate a negative version of Lemma 1.

LEMMA 2. *A word $W = (l_n)_{n \in \mathbb{N}}$ is not admissible if and only if there exists a strictly increasing sequence of natural numbers $(d_n)_{n \in \mathbb{N}}$ such that $l_0 \dots l_{d_{n+1}} \in Al_0 \dots l_{d_n}$ for any n .*

Using the terminology fixed in the introduction, we can restate Lemmas 1 and 2 in the following form.

- (*) A branch B of T_∞ is admissible if and only if B contains a vertex $q \neq p_0$ such that $w(p) \notin Aw(q)$ for any vertex p of B different from q .
- (**) A branch B of T_∞ is not admissible if and only if B contains a sequence of distinct vertices $(q_n)_{n \in \mathbb{N}}$ such that $w(q_{n+1}) \in Aw(q_n)$ for any n .

We shall see at the end of this section that the existence of a special vertex q as in (*) does not characterize the admissible genealogical trees. However the next theorem shows that the existence of a sequence of vertices $(q_n)_{n \in \mathbb{N}}$ as in (**) actually characterizes the non-admissible genealogical trees.

THEOREM 3. *Let T be a genealogical tree. Then the following conditions are equivalent:*

- (i) T is a nonadmissible genealogical tree.
- (ii) *There exists a sequence $(q_n)_{n \in \mathbb{N}}$ of distinct vertices of T such that $w(q_{n+1}) \in Aw(q_n)$ for any n .*

PROOF. (i) $\Rightarrow$ (ii). The hypothesis that T is a nonadmissible genealogical tree enables us to find a nonzero vector $v \in M(T)$ such that $(1, 0, 0, 0, 0, 0, \dots) \notin Av$. Consequently, if $f \in A$, then either $f(v) = 0$ or $f(v)$ has infinite support. We claim that, if $n \in \text{supp}(v)$, then there exist infinitely many $i \in \text{supp}(v)$ such that $w(p_i) \in Aw(p_n)$. Indeed, since $w(p_n)(v) \neq 0$, it follows that $w(p_n)(v)$ has infinite support. This implies that the set $\{i \in \text{supp}(v) : w(p_i) \in Aw(p_n)\}$ is infinite, as

claimed. Hence we may immediately construct, by induction, a sequence of distinct vertices $(q_n)_{n \in \mathbb{N}}$ of T with the property that $q_n \in \{p_i : i \in \text{supp}(v)\}$ and that $w(q_{n+1}) \in Aw(q_n)$ for any n . Therefore (ii) holds.

(ii) $\Rightarrow$ (i). Let $(q'_n)_{n \in \mathbb{N}}$ be a subsequence of $(q_n)_{n \in \mathbb{N}}$ such that $\deg w(q'_{n+1}) > 2 \deg w(q'_n)$ for any n . Next let u be an element of $M(T)$ such that $\text{supp}(u) = \{i \in \mathbb{N} : p_i = q'_n \text{ for some } n\}$. Then we may write u as an infinite sum of the form $u = \sum_{n \in \mathbb{N}} u_n$, where, for any n , the support of u_n has exactly one element s_n and $s_i < s_j$ if $i < j$. We want to show that $(1, 0, 0, 0, 0, 0, \dots) \notin Au$. To see this, fix any $f \in A$ such that $f(u) \neq 0$. Evidently we can write $f(u)$ as an infinite sum of the form $f(u) = \sum_{n \in \mathbb{N}} f(u_n)$. At this point, let $i = \min \{n \in \mathbb{N} : f(u_n) \neq 0\}$ and choose some $j \geq i$ such that $\deg w(q'_j) > \deg f$. Then all the vectors $f(u_n)$ with $n \geq j$ have nonempty and pairwise disjoint supports. Hence $f(u)$ has infinite support, and so $(1, 0, 0, 0, 0, 0, \dots) \notin Au$. This proves that T is a nonadmissible genealogical tree, as asserted in (i). ■

As an immediate consequence of Theorem 3, we obtain the following corollary.

COROLLARY 4. *Let T be a genealogical tree formed by finitely many admissible branches. Then T is an admissible genealogical tree.*

The next corollary shows that we cannot weaken the hypotheses of Corollary 4.

COROLLARY 5. *There exists a nonadmissible genealogical tree formed by countably many admissible branches.*

PROOF. Let x and y denote two distinct letters from $x_1, \dots, x_m$. Next let $(q_n)_{n \in \mathbb{N}}$ denote the sequence of vertices of T_∞ defined inductively by the formula

$$w(q_n) = \begin{cases} x^2 & \text{if } n = 0 \\ xy^n w(q_{n-1}) & \text{if } n > 0. \end{cases}$$

Finally, for any n , let B_n denote the branch of T_∞ uniquely determined by the following conditions:

- (i) q_n is a vertex of B_n .

(ii) If $n > 0$, then any path along B_n arriving at q_n consists of all arrows denoted by x , while any path along B_0 arriving at q_0 consists of all arrows denoted by y .

Hence the branch B_0 is of the form

$$\circ \xleftarrow{y} \circ \xleftarrow{x} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \dots, \\ p_0 \qquad \qquad \qquad q_0$$

while, for any $n > 0$, the branch B_n is of the form

$$\circ \xleftarrow{x} \circ \xleftarrow{y} \circ \dots \circ \xleftarrow{y} \circ \dots \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \dots. \\ p_0 \qquad \underbrace{\hspace{10em}}_{n \text{ arrows}} \qquad \qquad \qquad q_n$$

At this point, let T denote the genealogical tree obtained by glueing together all the branches B_n 's. Since $(q_n)_{n \in \mathbb{N}}$ is a sequence of vertices of T satisfying condition (ii) of Theorem 3, it follows that T is a non-admissible genealogical tree. On the other hand, let B be a branch of T . Then either $B = B_n$ for some n , or $B = B_\infty$, where B_∞ is the following branch of T_∞ :

$$\circ \xleftarrow{x} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \dots. \\ p_0$$

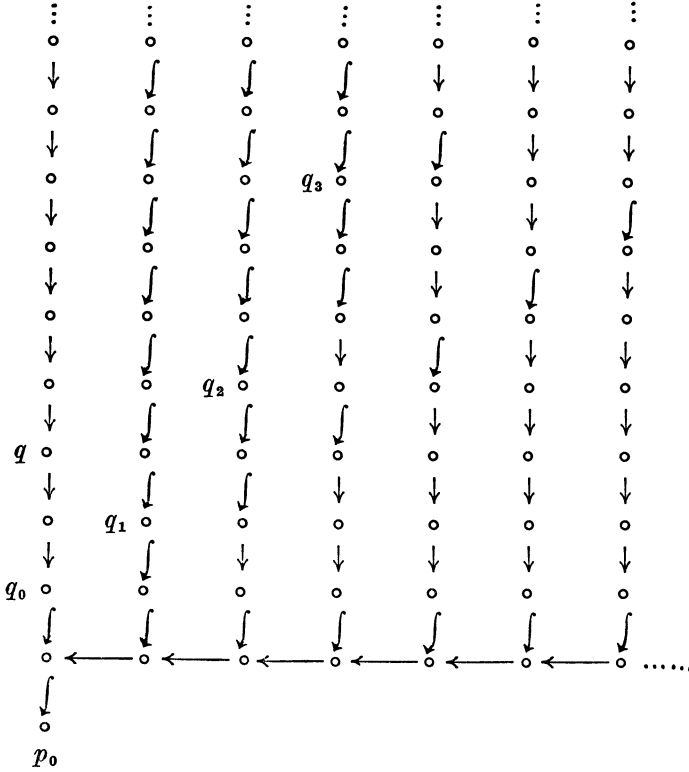
In both cases, Lemma 1 guarantees that B is an admissible branch. This completes the proof of the corollary. ■

REMARK. We can now justify the observation preceding Theorem 3. Indeed, let T be the genealogical tree constructed in the proof of Corollary 5, and let q be the vertex of T completely determined by the property that $w(q) = x^2y^2$. We claim that, if p is a vertex of T different from q , then $w(p) \notin Ax^2y^2$. To see this, we proceed by induction on the index n of the branch B_n containing p . If either $n = 0$ or $n = 1$, then the assertion is obvious. Hence we may assume that p is a vertex of the branch B_n with $n \geq 2$, and that the assertion holds for all the vertices of the branch B_{n-1} . Now let $(l_n)_{n \in \mathbb{N}}$ denote the word corresponding to the branch B_{n-1} , and let $(l_n^*)_{n \in \mathbb{N}}$ denote the word corresponding to the branch B_n . Then, by the inductive hypothesis, $l_n l_{n+1} l_{n+2} l_{n+3} \neq x^2y^2$ for any n . On the other hand, by the definition of B_n , we may write

$$(l_n^*)_{n \in \mathbb{N}} = (x, \underbrace{y, \dots, y}_n, l_0, l_1, l_2, l_3, \dots).$$

This implies that $l_n^* l_{n+1}^* l_{n+2}^* l_{n+3}^* \neq x^2 y^2$ for any n . Hence $w(p) \notin Ax^2 y^2$, and so the assertion holds for all the vertices of the branch B_n . Consequently, $w(p) \notin Ax^2 y^2$ for any vertex p of T different from q . Since T is a nonadmissible genealogical tree, we conclude that the obvious generalization of (*) does not determine all the admissible genealogical trees.

With the convention that $\circ \longleftarrow \circ$ stands for $\circ \xleftarrow{x} \circ$, and that $\circ \longleftarrow \circ$ stands for $\circ \xleftarrow{y} \circ$, we may visualize the structure of T as follows.



2. The next theorem characterizes strongly nonadmissible words.

THEOREM 6. *Let $W = (l_n)_{n \in \mathbb{N}}$ be a word. Then the following conditions are equivalent:*

- (i) *W is a strongly nonadmissible word.*

(ii) If r is a positive integer and w is a monomial of degree r in the letters $\{l_n: n \in N\}$, then there exists some n such that $w = l_n \dots l_{n+r-1}$.

PROOF. (i) $\Rightarrow$ (ii). Let W be a strongly nonadmissible word, and assume, contrary to (ii), that there is a monomial w of the form $w = l_{-r} \dots l_{-1}$, with $r \geq 1$ and $l_{-i} \in \{l_n: n \in N\}$ for any $i = 1, \dots, r$, such that $w \neq l_n \dots l_{n+r-1}$ for any n . To find a contradiction, let $U = (l'_n)_{n \in N}$ denote the word

$$U = (l'_n)_{n \in N} = (l_{-r}, \dots, l_{-1}, l_0, l_1, l_2, l_3, \dots).$$

Then $l'_0 \dots l'_{r-1} = l_{-r} \dots l_{-1} = w$, while $l'_{n-r+1} \dots l'_n = l_{n-2r+1} \dots l_{n-r} \neq w$ for any $n \geq 2r - 1$. By Lemma 2, this implies that U cannot be a nonadmissible word. Hence W cannot be a strongly nonadmissible word. This contradiction shows that there is some n such that $w = l_n \dots l_{n+r-1}$, and so (ii) holds.

(ii) $\Rightarrow$ (i). Assume that W satisfies condition (ii). Now let $U = (l'_n)_{n \in N}$ be a word of the form

$$U = (l'_n)_{n \in N} = (l_{-r}, \dots, l_{-1}, l_0, l_1, l_2, l_3, \dots),$$

with $r \geq 1$ and $l_{-i} \in \{l_n: n \in N\}$ for any $i = 1, \dots, r$. We claim that U is a nonadmissible word. To see this, let d be a natural number, and let w denote the monomial $w = l'_0 \dots l'_d$. Then, by (ii), we can find some n such that $w = l_n \dots l_{n+d}$. Since $l_n \dots l_{n+d} = l'_{n+r} \dots l'_{n+r+d}$, it follows that $w = l'_0 \dots l'_d = l'_{n+r} \dots l'_{n+r+d}$ with $n + r \geq r > 0$. Consequently, by Lemma 1, U is a nonadmissible word, as claimed. Hence (i) holds, and the theorem is proved. ■

The following corollary gives a «quantitative» result on words.

COROLLARY 7. *There exist $2^{\aleph_0}$ admissible words, strongly nonadmissible words and weakly nonadmissible words.*

PROOF. Let x and y denote two distinct letters from $x_1, \dots, x_m$. We divide the proof in three steps.

Step 1. Let $\alpha = (a_n)_{n \in N}$ be a strictly increasing sequence of positive integers, and let W_α denote the word

$$W_\alpha = (\underbrace{x, \dots, x}_{a_0}, \underbrace{y, x, \dots, x}_{a_1}, \underbrace{y, x, \dots, x}_{a_2}, y, \dots).$$

More precisely, let $W_\alpha = (l_n)_{n \in N}$, where

$$l_n = \begin{cases} y & \text{if } n = i + \sum_{j=0}^i a_j \quad \text{for some } i \in N \\ x & \text{otherwise.} \end{cases}$$

Now let $d = a_0 + a_1 + 1$; then $l_0 \dots l_d = x^{a_0} y x^{a_1} y$. Since α is a strictly increasing sequence, we have $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for any $n > d$. Hence, by Lemma 1, W_α is an admissible word. Since the map $\alpha \mapsto W_\alpha$ is injective, there exist $2^{\aleph_0}$ admissible words.

Step 2. As in Step 1, let $\alpha = (a_n)_{n \in N}$ be a strictly increasing sequence of positive integers. Next, let $\alpha^* = (a_n^*)_{n \in N}$ be the «periodic» sequence defined as follows. First of all, if $n \in N$, then $a_{2^{n+1}-2}^* = a_n$ and $2^{n+1} - 2 = \min \{i \in N : a_i^* = a_n\}$; secondly, we choose $a_1^* = a_0$. Assume now, by induction, that, for some $n \geq 1$, we have already defined all the elements a_i^* with $0 \leq i \leq 2^{n+1} - 2$. Then we define the elements a_i^* with $2^{n+1} - 1 \leq i \leq 2^{n+2} - 3$ by means of the equality $(a_{2^{n+1}-1}^*, \dots, a_{2^{n+2}-3}^*) = (a_0^*, \dots, a_{2^{n+1}-2}^*)$. Now let W_{α^*} denote the word

$$W_{\alpha^*} = (\underbrace{x, \dots, x}_{a_0^*}, y, \underbrace{x, \dots, x}_{a_1^*}, y, \underbrace{x, \dots, x}_{a_1^*}, y, \dots),$$

that is let $W_{\alpha^*} = (l_n)_{n \in N}$, where

$$l_n = \begin{cases} y & \text{if } n = i + \sum_{j=0}^i a_j^* \\ x & \text{otherwise.} \end{cases}$$

Since $l_n l_{n+1} \neq y^2$ for any n , we deduce from Theorem 6 that W_{α^*} cannot be a strongly nonadmissible word. We claim that W_{α^*} is not admissible. Indeed, fix any $d \in N$. Then the definition of α^* enables us to find two natural numbers r and n with $r \geq d$ and $n \geq 1$ such that $l_0 \dots l_d \dots l_r = x^{a_0^*} y \dots y x^{a_{2^{n+1}-2}^*}$. Since

$$(a_0^*, \dots, a_{2^{n+1}-2}^*) = (a_{2^{n+1}-1}^*, \dots, a_{2^{n+2}-3}^*),$$

there exists some $s > 0$ such that $l_0 \dots l_d \dots l_r = l_s \dots l_{s+d} \dots l_{s+r}$. Consequently $l_0 \dots l_d = l_s \dots l_{s+d}$ with $s > 0$. Hence, by Lemma 1, W_{α^*}

is a nonadmissible word. Therefore W_{α^*} is a weakly nonadmissible word. Since the map $\alpha^* \mapsto W_{\alpha^*}$ is injective, we obtain $2^{\aleph_0}$ weakly nonadmissible words.

Step 3. Let σ be a sequence of the form $\sigma = (w_n)_{n \in \mathbb{N}}$, where w_n runs through all the monomials of positive degree in the letters x and y . For any n , let d_n denote the degree of w_n . Next, let

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ d_0 + \dots + d_{n-1} & \text{if } n > 0, \end{cases}$$

and let $b_n = d_0 + \dots + d_n - 1$ for any n . Finally, let W_σ denote the word $(l_n)_{n \in \mathbb{N}}$ defined by glueing together all the monomials w_n , as illustrated in the following picture.

$$\boxed{w_0} \boxed{w_1} \boxed{w_2} \boxed{w_3} \dots \rightsquigarrow W_\sigma.$$

More precisely, let W_σ denote the word $(l_n)_{n \in \mathbb{N}}$ uniquely determined by the condition that $l_{a_n} \dots l_{b_n} = w_n$ for any n . Then W_σ obviously satisfies condition (ii) of Theorem 6, and so W_σ is a strongly nonadmissible word. Since the map $\sigma \mapsto W_\sigma$ is injective, there exist $2^{\aleph_0}$ strongly nonadmissible words.

The corollary now follows from Steps 1, 2 and 3. ■

Finally, we give an example of a very large admissible genealogical tree.

COROLLARY 8. *There exists an admissible genealogical tree formed by $2^{\aleph_0}$ branches.*

PROOF. Let T be the genealogical tree obtained by glueing together all the branches of T_∞ corresponding to the words W_α constructed in Step 1 of the proof of Corollary 7. Hence any W_α is of the form

$$W_\alpha = (\underbrace{x, \dots, x}_{a_0}, y, \underbrace{x, \dots, x}_{a_1}, y, \underbrace{x, \dots, x}_{a_2}, \dots),$$

where $\alpha = (a_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive integers. We claim that T is an admissible genealogical tree. Suppose the contrary, and let $(q_n)_{n \in \mathbb{N}}$ be a sequence of vertices of T satisfying condition (ii) of Theorem 3. Then we can find two natural numbers i

and j with $i > 0$ such that $w(q_n) \in Ayx^i yA$ for any $n \geq j$. This implies that the set

$$S = \{p: p \text{ vertex of } T, w(p) \in Ayx^i y\}$$

is infinite. To see that this is impossible, fix any $p \in S$. Then, by the definition of T , there exists a strictly increasing sequence of positive integers $\alpha = (a_n)_{n \in \mathbb{N}}$ such that $i = a_r$ for some $r \in \mathbb{N}$ and $w(p) = x^{a_0} y \dots y x^{a_r} y$. Since $r + 1 \leq i$, it follows that

$$\underbrace{a_0 + \dots + a_r}_{r+1} + r + 1 \leq \underbrace{i + \dots + i}_i + i = i(i + 1).$$

and so $\deg w(p) \leq i(i + 1)$. This means that S is finite, and this is the desired contradiction. This contradiction shows that T is an admissible genealogical tree, as claimed. Since T has $2^{\aleph_0}$ branches, the corollary is proved. ■

REMARK. The above proof shows that the assertion of Corollary 8 holds for $m = 2$, and so for any $m \geq 2$. However, if $m \geq 3$, then it is even easier to construct an admissible genealogical tree with $2^{\aleph_0}$ branches. In fact, let T be the genealogical tree obtained by glueing together all the branches B of T_∞ corresponding to words W of the form $W = (l_n)_{n \in \mathbb{N}}$ with $l_0 = x_3$ and $l_n \in \{x_1, x_2\}$ for any $n > 0$. Then evidently T does not satisfy condition (ii) of Theorem 3, and so T is an admissible genealogical tree with $2^{\aleph_0}$ branches.

3. We do not know the structure of an A -module of the form $M(T)$ with T a nonadmissible genealogical tree. However, the next proposition shows that, if T is a nonadmissible genealogical tree, then $M(T)$ may be very far from being indecomposable.

PROPOSITION 9. *There exists a genealogical tree T such that $M(T)$ is the direct sum of $|M(T)|$ indecomposable A -modules, running through all the indecomposable injective $K[x]$ -modules.*

PROOF. Let x be a letter from $x_1, \dots, x_m$, and let T be the genealogical tree of the form

$$\begin{array}{c} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \dots \\ v_0 \end{array}$$

More precisely, let T be the branch of T_∞ corresponding to the word

$(l_n)_{n \in \mathbb{N}}$ with $l_n = x$ for any n . We claim that T satisfies the hypotheses of Proposition 9. To see this, we divide the proof in three steps. Throughout the proof, we denote by M the module $M(T)$ and by $t(M)$ the torsion submodule of M .

Step 1. M is an injective $K[x]$ -module.

PROOF. Let f be an element of $K[x]$ of the form $f = x^i + a_{i-1} \cdot x^{i-1} + \dots + a_1 x + a_0$ for some $i > 0$, and let v be a vector of M of the form $v = (k_n)_{n \in \mathbb{N}}$. Now let $\bar{v}$ denote the vector $\bar{v} = (\bar{k}_n)_{n \in \mathbb{N}}$ defined inductively by the formula

$$\bar{k}_n = \begin{cases} 0 & \text{if } 0 \leq n \leq i-1, \\ k_{n-i} - (a_{i-1} \bar{k}_{n-1} + \dots + a_0 \bar{k}_{n-i}) & \text{if } n \geq i. \end{cases}$$

Then, for any $n \geq i$, we obtain

$$\begin{aligned} \bar{k}_n + a_{i-1} \bar{k}_{n-1} + \dots + a_0 \bar{k}_{n-i} &= \\ &= (k_{n-i} - (a_{i-1} \bar{k}_{n-1} + \dots + a_0 \bar{k}_{n-i})) + a_{i-1} \bar{k}_{n-1} + \dots + a_0 \bar{k}_{n-i} = k_{n-i}. \end{aligned}$$

It follows that $f(\bar{v}) = v$. Therefore M is a divisible $K[x]$ -module, and so, by ([3] Theorem 2.8), M is an injective $K[x]$ -module.

Step 2. $\text{soc } t(M) \cong \bigoplus_v K[x]/(p)$ with p running through all the monic and irreducible polynomials of $K[x]$.

PROOF. Let f be an element of $K[x]$ of the form $f = x^i + a_{i-1} \cdot x^{i-1} + \dots + a_1 x + a_0$ for some $i > 0$, and let $V_f = \{v \in t(M) : f(v) = 0\}$. We shall prove that V_f is a cyclic $K[x]$ -module isomorphic to $K[x]/(f)$. To this end, we first note that, if $v \in V_f$ and v is of the form $v = (k_n)_{n \in \mathbb{N}}$, then

$$k_{n+i} + a_{i-1} k_{n+i-1} + \dots + a_1 k_{n+1} + a_0 k_n = 0 \quad \text{for any } n.$$

Consequently, for any element $(c_0, \dots, c_{i-1}) \in K^i$ there exists a unique element $(c_n^*)_{n \in \mathbb{N}} \in V_f$ satisfying $c_n^* = c_n$ for any $n = 0, \dots, i-1$. This means that the canonical projection $\pi: M \rightarrow K^i$, such that

$$\pi((k_n)_{n \in \mathbb{N}}) = (k_0, \dots, k_{i-1}) \quad \text{for any } (k_n)_{n \in \mathbb{N}} \in M,$$

induces a K -vector space isomorphism between V_f and K^i . Next, let v_f denote the element of V_f uniquely determined by the condition that $\pi(v_f) = (0, \dots, 0, 1)$. Then $\{\pi(x^j(v_f)): j = 0, \dots, i-1\}$ is obviously a base of the K -vector space K^i . This proves that $\{x^j(v_f): j = 0, \dots, i-1\}$ is a base of the K -vector space V_f ; hence $V_f = K[x]v_f$. Since $\dim_K(V_f) = i$ and $f \in \text{ann}_{K[x]}(v_f)$, it follows that $\text{ann}_{K[x]}(v_f) = (f)$. Thus V_f is isomorphic to $K[x]/(f)$, as claimed. We also note that $t(M) = \sum_f V_f$ with f running through all the monic polynomials of $K[x]$ of positive degree. Therefore

$$(*) \quad |t(M)| = \max \{|K|, \aleph_0\} = |K(x)|$$

and $\text{soc } t(M) = \sum_p V_p = \bigoplus_p V_p$, where p ranges over all the monic and irreducible polynomials of $K[x]$.

Step 3. M has a decomposition of the form $M = t(M) \oplus V_0$, where V_0 is a vector space over $K(x)$ and $\dim_{K(x)}(V_0) = |M|$.

PROOF. The existence of a $K(x)$ -vector space V_0 such that $M = t(M) \oplus V_0$ follows from Step 1 and ([3] Theorem 4.4 and Corollary to Theorem 2.32). Now let v be the following element of M :

$$v = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, \dots).$$

More precisely, let $v = (k_n)_{n \in N}$, where

$$k_n = \begin{cases} 1 & \text{if either } n = 0 \text{ or } n = \sum_{j=2}^i j \text{ for some } i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the vectors $\{x^n(v): n \in N\}$ are K -linearly independent. Indeed, suppose, by contradiction, that this is not true. Then, for some $i \in N$, we may write $x^{i+1}(v) = \sum_{j=0}^i t_j x^j(v)$ with $t_j \in K$ for any $j = 0, \dots, i$. On the other hand, by the definition of v , we can find some n such that $n + i + 1 \in \text{supp}(v)$ and $n + j \notin \text{supp}(v)$ for any $j = 0, \dots, i$. Consequently $n \in \text{supp}(x^{i+1}(v))$ and $n \notin \text{supp}(x^j(v))$ for any $j = 0, \dots, i$. Hence $x^{i+1}(v) \neq \sum_{j=0}^i t_j x^j(v)$, contrary to the hypothesis.

This contradiction shows that the vectors $\{x^n(v): n \in N\}$ are K -linearly independent. Therefore $V_0 \neq 0$, and we deduce from (*) that

$$(**) \quad |V_0| = |K(x)| \dim_{K(x)}(V_0) = |t(M)| \dim_{K(x)}(V_0).$$

To end the proof, we distinguish two cases.

Suppose first that $|t(M)| < |M|$. Then we clearly have $|V_0| = |M|$. Moreover, by (**), $|V_0| = \dim_{K(x)}(V_0)$. Consequently $\dim_{K(x)}(V_0) = |M|$, as desired.

Assume now that $|t(M)| = |M|$. In the case, we first note that, by (*), we have

$$(1) \quad |M| = |K| > \aleph_0.$$

Next, let $\mathcal{B}$ denote a base of the $K(x)$ -vector space V_0 , and let E denote the smallest subfield of K with the property that, if $v \in \mathcal{B}$ and $v = (k_n)_{n \in N}$, then $k_n \in E$ for any n . Then evidently

$$(2) \quad |E| \leq \max \{ \dim_{K(x)}(V_0), \aleph_0 \}.$$

Suppose, by contradiction, that $|E| < |K|$. Then, by (1), $\text{tr deg}(K/E)$, the transcendence degree of K over E , is infinite. Hence we may choose a vector $v^* \in M$ of the form $v^* = (k_n^*)_{n \in N}$ such that

$$(3) \quad k_{n+1}^* \text{ is not algebraic over } E(k_0^*, \dots, k_n^*) \text{ for any } n.$$

To find a contradiction, we write v^* in the form $v^* = v' + v''$ with $v' = (k'_n)_{n \in N} \in t(M)$ and $v'' = (k''_n)_{n \in N} \in V_0$. Then there exist $f', f'' \in K[x]$, of the form

$$f' = x^i + a_{i-1}x^{i-1} + \dots + a_1x + a_0$$

and

$$f'' = x^j + b_{j-1}x^{j-1} + \dots + b_1x + b_0$$

with $i, j > 0$, such that $f'(v') = 0$ and $f''(v'') = \sum_{i=1}^r f_i(v_i)$ with $r \geq 1$, $f_i \in K[x]$ and $v_i \in \mathcal{B}$ for any $i = 1, \dots, r$. Let $c_1, \dots, c_h$ denote the coefficients of $f_1, \dots, f_r$, and let F denote the following subfield of K :

$$F = E(a_0, \dots, a_{i-1}, b_0, \dots, b_{j-1}, c_1, \dots, c_h).$$

At this point, the hypothesis that $f'(v') = 0$ guarantees that

$$k'_{n+i} + a_{i-1} k'_{n+i-1} + \dots + a_1 k'_{n+1} + a_0 k'_n = 0 \quad \text{for any } n.$$

Consequently

$$(4) \quad k'_n \in F(k'_0, \dots, k'_{i-1}) \quad \text{for any } n.$$

On the other hand, the hypothesis that $f''(v'') = \sum_{i=1}^r f_i(v_i)$ implies that

$$k''_{n+j} + b_{j-1} k''_{n+j-1} + \dots + b_1 k''_{n+1} + b_0 k''_n \in F \quad \text{for any } n.$$

Therefore

$$(5) \quad k''_n \in F(k''_0, \dots, k''_{j-1}) \quad \text{for any } n.$$

Putting (4) and (5) together, we conclude that

$$(6) \quad k_n^* = k'_n + k''_n \in F(k'_0, \dots, k'_{i-1}, k''_0, \dots, k''_{j-1}) \quad \text{for any } n.$$

Finally, let F^* denote the field $F^* = F(k_n^*: n \in N)$. Then (6) implies that $\text{tr deg}(F^*/F)$ is finite. Since $\text{tr deg}(F/E)$ is obviously finite, it follows that also $\text{tr deg}(F^*/E)$ is finite, contrary to the hypothesis that v^* satisfies (3). This contradiction shows that $|E| = |K|$. By (1) and (2), this implies that $\dim_{K(x)}(V_0) = |M|$.

Combining Steps 1, 2 and 3, we see that the genealogical tree T satisfies the hypotheses of Proposition 9. ■

The A -module M constructed in the proof of Proposition 9 gives a «concrete» example of an injective cogenerator for the category of all $K[x]$ -modules. We also note that the indecomposable summands of M are, in a sense, the «smallest» possible indecomposable summands of an A -module of the form $M(B)$ with B a branch of T_∞ . In fact, we have the following corollary.

COROLLARY 10. *Let B be a branch of T_∞ and let $z = x_1 + \dots + x_m$. Then $M(B)$, regarded as a $K[z]$ -module, is the direct sum of $|M(B)|$ injective $K[z]$ -modules.*

PROOF. Replace x by z in the proof of Proposition 9. ■

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