A criterion for a rational projectively normal variety to be almost-factorial

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A Criterion for a Rational Projectively Normal Variety to be Almost-Factorial.

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0. Introduction.

In the forthcoming paper «Factorial singularities on rational quartic surfaces of \( P^3 \) », written in collaboration with P. C. Craighero, the properties of such surfaces in connection with their parametric representation on a plane \( P^2 \) have been deeply investigated. In such a research, the curves on the surfaces coming from particular points of the plane, that is the exceptional curves, play a leading role. This fact has suggested the author the investigation between the relation on almost-factoriality of rational surfaces and one of its parametric representation.

This paper presents the answer to the matter. It holds that, if \( \mathcal{F} \) is a rational projectively normal variety, \( \mathcal{F} \) is almost-factorial iff are set-theoretic complete intersection on \( \mathcal{F} \) only a finite subset of sub-

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varieties of codimension 1 of $\mathcal{F}$ which are referred to the parametrization (see def. 5); however, among these there are also the exceptional sub-varieties (see Prop. 1). An immediate affine version of this result is given and some applications to classes of surfaces of $\mathbb{A}^3$ or of $\mathbb{P}^3$ are enunciated.

The result is an extension of the well-known criterion of D. Gallarati for the almost-factoriality which holds for monoid hypersurfaces in $\mathbb{P}^N$. According to it, one can, for example, tackle the question of the classification of the almost-factorial rational surfaces of degree four with only double points.

On the other hand, as it is proved in [1] Prop. 2.11 p. 260, the almost-factoriality is a non local property which is unaffected by isomorphisms in the class of the projectively normal varieties. Since $\mathbb{P}^n$ is factorial (so almost-factorial), for every $n > 0$ one can find very large classes of almost-factorial varieties by means of isomorphisms: for example all the $m$-ple embeddings of $\mathbb{P}^n$ in $\mathbb{P}^N$, $N = \left(\frac{n + m}{n}\right) - 1$, $m > 1$. According to the new result, one can build more models of rational almost-factorial varieties which are not necessarily isomorphic to some $\mathbb{P}^n$ or to some monoid.

At the end of the paper a rational quartic surface with only double points which results to be 12-almostfactorial is examined as a detailed example, and some classes of rational almost-factorial surfaces of $\mathbb{A}^3$ are pointed out as well.

1. Let $k$ be an algebraically closed field of any characteristic. $\mathbb{P}^N$ and $\mathbb{A}^N$ denote respectively the projective and the affine space of dimension $N$ over $k$. Variety on $\mathbb{P}^N$ (or on $\mathbb{A}^N$), will mean always an algebraic irreducible and reduced closed subset on $\mathbb{P}^N$ (or on $\mathbb{A}^N$), $N \geq 3$.

A prime divisor (or more shortly) a prime on a variety $\mathcal{F}$, non singular in codimension 1, will be an irreducible and reduced subvariety on $\mathcal{F}$ of codimension 1. Curves surfaces, hypersurfaces on $\mathbb{P}^N$ (or on $\mathbb{A}^N$) will be varieties of dimension 1, 2, $N - 1$ respectively.

Let $\mathcal{F}$ be a variety on $\mathbb{P}^N$: $J(\mathcal{F})$, $k[\mathcal{F}]$, $k(\mathcal{F})$ denote respectively the (prime and homogeneous) ideal of $\mathcal{F}$ in $k[X_0, \ldots, X_N]$, ($= k[X]$), the quotient ring $k[X]/J(\mathcal{F})$, the field of rational functions on $\mathcal{F}$, ($= \text{field of quotients of elements in } k[\mathcal{F}]$ of same degree).

DEFINITIONS 1. A Variety $\mathcal{F} \subset \mathbb{P}^N$ is called projectively normal if $k[\mathcal{F}]$ is integrally closed.
2. A prime $C$ on a variety $\mathcal{F} \subset \mathbb{P}^n$, is set-theoretic complete intersection (S.T.C.I.) on $\mathcal{F}$ with multiplicity $\lambda$, if exists a hypersurface $\mathcal{G} \subset \mathbb{P}^n$ such that $\mathcal{F} \cdot \mathcal{G} = \lambda C$, i.e. $\lambda C$ is the complete intersection of $\mathcal{F}$ and $\mathcal{G}$.

3. A Variety $\mathcal{F}$ projectively normal is almost-factorial if every prime $C \subset \mathcal{F}$ is S.T.C.I. on $\mathcal{F}$. In particular $\mathcal{F}$ is $g$-almostfactorial if every prime $C$ on $\mathcal{F}$ is S.T.C.I. on $\mathcal{F}$ with multiplicity $\lambda \leq g$.

Let $\mathcal{F}$ be a projective variety of $\mathbb{P}^n$, with $n = \dim \mathcal{F}$. We recall some well known facts (see [4], pp. 107-124) about birational correspondences between projective spaces and in particular between $\mathbb{P}^n$ and $\mathcal{F}$.

**Definition 4.** A parametrization $p$ on $\mathbb{P}^n$ of a projective variety $\mathcal{F}$ is a set

$$H_0, \ldots, H_N \in k[U_0, \ldots, U_n], \quad (= k[U])$$

of homogeneous polynomials (forms) of the same degree such that

1. substituting

$$X_i \rightarrow H_i \quad i = 0, \ldots, N,$$

for every $P \in J(\mathcal{F})$ it follows $P(H_0, \ldots, H_N) = 0$;

2. if $g$ denotes the image of $G$ in the canonical projection $k[X] \rightarrow k[\mathcal{F}] = k[x_0, \ldots, x_N]$, for every $G \in k[X]$, there exists a set

$$F_0, \ldots, F_n \in k[X]$$

of forms of the same degree such that

$$MU_j = f_j(H_0, \ldots, H_N) \quad j = 0, \ldots, n,$$

for a suitable $M \in k[U]$, $M \neq 0$.

We note that in 2) the forms $F_0, \ldots, F_n$ are chosen in $k[X]$ only mod $J(\mathcal{F})$; thus the polynomial $M$ in (2) can change with the set $F_0, \ldots, F_n$; however it must be homogeneous from (2) itself. Moreover (1) and (2) together imply there exists a $k$-homomorphism between the fields $k(\mathcal{F})$ and $k(U)$ of rational functions on $\mathcal{F}$ and $\mathbb{P}^n$ respectively. By this, there must exist a suitable (homogeneous) polynomial $Q \in$
$\in k[X]$, $Q \notin J(\mathcal{F})$ such that

$$\text{(3)} \quad q_{\alpha_i} = H_i(f_0, \ldots, f_n) \quad j = 0, \ldots, N$$

(see also [4] p. 116). We remark also that for every point $A \in \mathbb{P}^n - \{M = 0\}$ is $M(A) \neq 0$, if exists $j$, $0 \leq j \leq N$, for which $U_j \neq 0$ and, by (2)

$$0 \neq M(A) U_j(A) = f_j(H_0(A), \ldots, H_n(A)) \Rightarrow H_j(A) \neq 0$$

for at least one $i$, $0 \leq i \leq n$. This proves that the parametrization $p$ rises, by means of (1), to a map

$$\sigma_\alpha: \mathbb{P}^n \to \mathcal{F} \quad (\subset \mathbb{P}^N)$$

which is regular in $\mathbb{P}^n - \{M = 0\}$ and $\sigma_\alpha$ is invertible by (3).

As above one sees that $\sigma_\alpha^{-1}: \mathcal{F} \to \mathbb{P}^n$ is surely regular in $\mathcal{F} - \mathcal{F} \cap \{Q = 0\}$, but in general nothing can be said for the points of the set $\mathcal{F} \cap \{Q = 0\}$. If we consider another parametrization $q$ on $\mathbb{P}^n$ of $\mathcal{F}$, being $\mathcal{F}$ irreducible, we have that the map $\sigma_\varphi: \mathbb{P}^n \to \mathcal{F}$ rised by $q$ coincides with $\sigma_\alpha$ in an open suitable subset of $\mathbb{P}^n$. By this they give a birational map $\sigma: \mathbb{P}^n \to \mathcal{F}(\subset \mathbb{P}^N)$ which may be biregular too. For example it is what happens in the $m$-ple embedding $\mathbb{P}^n \to \mathcal{F} \subset \mathbb{P}^N$ with $N = \binom{n + m}{n} - 1$.

2. In the following we suppose $\mathcal{F} \subset \mathbb{P}^N$ to be a rational variety (i.e. every variety $\mathcal{F}$ for which exists a birational map $\mathbb{P}^n \to \mathcal{F}$, $n = \dim \mathcal{F}$) and let $p$ a parametrization of $\mathcal{F}$ on $\mathbb{P}^n$.

We are precisely concerned with the particular map $\sigma_\alpha: \mathbb{P}^n \to \mathcal{F}$ gived by $p$ and the sets $\mathbb{P}^n - \{M = 0\}$ and $\mathcal{F} - \mathcal{F} \cap \{Q = 0\}$ which depend from $p$ according to the previous notations, and the said situation.

DEFINITIONS 5. Given a parametrization $p$ on $\mathbb{P}^n$ of a projective variety $\mathcal{F}$, we call referred to $p$ all the subvarieties of codimension 1 in $\mathcal{F}$ which belong to $\{Q = 0\}$.

6. Rational variety of $\mathbb{A}^N$ will be a variety of $\mathbb{A}^N$ whose projective closure is a rational variety in $\mathbb{P}^N$. 

The properties of the map $\sigma$ in the present hypothesis are well known; we recall someone of them in the

**Proposition 1.a)** To every subvariety $\mathcal{V} \subset \mathcal{F}$ not referred to $p$, it corresponds a subvariety $\sigma_p^{-1}(\mathcal{V})$ such that $\dim \sigma_p^{-1}(\mathcal{V}) = \dim \mathcal{V}$ and $\sigma_p^{-1}(\mathcal{V}) \not\subseteq \{M = 0\}$ (see [4], Satz VI, p. 120).

b) The restriction of $\sigma_p^{-1}$ to the set of non-singular points of $\mathcal{W} = \mathcal{V} \cap \{Q = 0\}$ is bijective, moreover to every non-singular point corresponds a non-singular point (see [4] Korollar, p. 121).

Let $\mathcal{F}'$ and $\mathcal{F}$ be two varieties of dimension $n$ and $\tau$ a birational map $\tau: \mathcal{F}' \to \mathcal{F}$. We recall that a prime $\mathcal{V} \subset \mathcal{F}$ is said exceptional for $\tau$ if $\dim \tau^{-1}(\mathcal{V}) < n - 1$. The rational variety $\mathcal{F}$ can have only a finite number of exceptional primes for the birational map $\mathbb{P}^n \to \mathcal{F}$; indeed they can belong among the maximal components of $\mathcal{F} \cap \{Q = 0\}$ for every parametrization $p$ on $\mathbb{P}^n$ of $\mathcal{F}$; they are then referred to every parametrization $p$. On the other hand, if the birational map $\mathbb{P}^n \to \mathcal{F}$ is biregular, no prime of $\mathcal{F}$ is exceptional for it.

**Lemma 1.** Let $\mathcal{F}$ be a rational variety, $\mathcal{V} \subset \mathbb{P}^n$, and $p$ be a parametrization on $\mathbb{P}^n$ of $\mathcal{F}$. For each prime $\mathcal{V} \subset \mathcal{F}$ not referred to $p$, it exists at least an irreducible form $\Psi \in k[U]$ such that

$$\Psi(F_0, \ldots, F_n) \in J(\mathcal{V}).$$

**Proof.** Let $C_1, \ldots, C_s \in k[X]$ such that $J(\mathcal{V}) = (C_1, \ldots, C_s)$ and let be

$$D_i = C_i(H_0, \ldots, H_n) \quad i = 1, \ldots, s.$$ 

Let us denote always with $\sigma_p: \mathbb{P}^n \to \mathcal{F}$ the rational mapping raised by $p$ and $Q \in [X]$, $M \in k[U]$ the forms in the (3) and (2) respectively. First we have

$$\sigma_p^{-1}(\mathcal{V}) = \{D_1 = \ldots = D_s = 0\}.$$ 

Indeed obviously $\sigma_p^{-1}(\mathcal{V}) \subseteq \{D_1 = \ldots = D_s = 0\}$; on the other hand, by Prop. 1.a), $\sigma_p^{-1}(\mathcal{V})$ is irreducible and of codimension 1 in $\mathbb{P}^n$, so it must be a hypersurface of $\mathbb{P}^n$. From this $\sigma_p^{-1}(\mathcal{V}) \supset \{D_1 = \ldots = D_s = 0\}$ and $J(\sigma_p^{-1}(\mathcal{V}))$ will be a principal ideal generated by $\Psi = \text{G.C.D.} \{D_1, \ldots, D_s\}$ and $\Psi$ will be irreducible and $\Psi$ does not divide $M$. Later,
being the ring \( k[U] \) U.F.D. and \( \mathcal{U} = \text{G.C.D}\{D_1, \ldots, D_s\} \), there exist \( A_i, B_i \in k[U] \), \( i = 1, \ldots, s \), such that

\[
D_i = \mathcal{U}A_i \quad \text{for } i = 1, \ldots, s; \quad 1 = \sum_{i=1}^{s} A_i B_i
\]

by which \( \mathcal{U} = \mathcal{U} \sum_{i=1}^{s} A_i B_i = \sum_{i=1}^{s} D_i B_i = \sum_{i=1}^{s} C_i(H_0, \ldots, H_N) B_i \).

Let us consider now the form \( \mathcal{U}(F_0, \ldots, F_n) \in k[X] \) obtained by substituting \( F_i \) to place of \( U_i \), \( i = 0, \ldots, n \). It results

\[
\mathcal{U}(F_0, \ldots, F_n) = \sum_{i=1}^{s} C_i(H_0(F_0, \ldots, F_n), \ldots, H_N(F_0, \ldots, F_n)) B_i(F_0, \ldots, F_n).
\]

Now we want to calculate its image \( \mathcal{U}(f_0, \ldots, f_n) \) in \( k[F] \). From (3) it is

\[
C_i(H_0(f_0, \ldots, f_n), \ldots, H_N(f_0, \ldots, f_n)) = q^{a_{x_i}} C_i(x_0, \ldots, x_N),
\]

by this, from (\#) one gets

\[
\mathcal{U}(f_0, \ldots, f_n) = \sum_{i=1}^{s} q^{a_{x_i}} C_i(x_0, \ldots, x_N) B_i(f_0, \ldots, f_n)
\]

which belongs to the image of \( J(U) \) in \( k[F] \), whence

\[
\mathcal{U}(F_0, \ldots, F_n) \in J(U).
\]

**Proposition 2.** Let \( F \subset \mathbb{P}^N \) be a rational projectively normal variety of dim \( F = n \) and \( p \) a parametrization of \( F \) on \( \mathbb{P}^n \). The following are equivalent:

a) \( F \) is almost-factorial;

b) every prime on \( F \) referred to \( p \) is set-theoretic complete intersection on \( F \).

More precisely if \( C_1, \ldots, C_t \) are the primes on \( F \) referred to \( p \) and \( \lambda_i C_i \) is the complete intersection \( F \) with a suitable hypersurface \( \mathcal{G}_i \subset \mathbb{P}^N \), \( i = 1, \ldots, t \), then for every prime \( C \subset F \) it exists a hypersurface \( \mathcal{G} \subset \mathbb{P}^N \)
such that

\[ \mathcal{F} \cdot \mathcal{G} = \lambda \mathcal{C} \quad \text{where} \quad \lambda = \text{L.C.M.}\{\lambda_1, \ldots, \lambda_t\}, \]

that is \( \mathcal{F} \) is \( \lambda \)-almostfactorial.

**Proof.** \( a) \Rightarrow b) \) is obvious. So we have only to prove \( b) \Rightarrow a) \)
Let \( C_1, \ldots, C_t \) be all the primes of \( \mathcal{F} \) referred to \( p \). By hypothesis \( b) \) there exist hypersurfaces \( \mathcal{L}_i = \{L_i = 0\} \subset P^N \) and \( \lambda_i > 0, \; i = 1, \ldots, t, \) such that

\[ \mathcal{F} \cdot \mathcal{L}_i = \lambda_i C_i \quad i = 1, \ldots, t. \]

Let \( \lambda = \text{L.C.M.}\{\lambda_1, \ldots, \lambda_t\} \) and let \( n_1, \ldots, n_t \) be positive integers such that

\[ \lambda = n_i \lambda_i \quad i = 1, \ldots, t. \]

For every prime \( D \subset \mathcal{F} \) we denote with \( C(D) \) the affine cone of \( D \) and let \( C(\mathcal{F}) \) be the affine cone of \( \mathcal{F} \), both in \( A^{n+1} \). Obviously \( C(D) \) has codimension 1 in \( C(\mathcal{F}) \) for every prime \( D \subset \mathcal{F} \). Moreover the ring \( k[\mathcal{F}] \) can be considered as the ring of the regular functions on \( C(\mathcal{F}) \). Let \( K \) be the quotient field of \( k[\mathcal{F}] \). For every prime \( C \subset \mathcal{F} \) not referred to \( p \) it exists, by Lemma 1, an irreducible polynomial \( \Psi \in k[U] \) such that

\[ \Psi(F_0, \ldots, F_n) \in J(\mathcal{V}). \]

Let be \( H = \Psi(F_0, \ldots, F_n) \in k[X] \) and let \( h \) be its projection in \( k[\mathcal{F}] \). We have

\[ \{H = 0\} \cdot \mathcal{F} = \text{div} \ (h) = \mu C + \nu_1 D_1 + \ldots + \nu_r D_r, \]

\[ \mu > 0, \; \nu_i > 0, \; i = 1, \ldots, r, \]

where \( \mu = 1 \) by Prop. 1.b) because \( C \) is not referred to \( p \), and \( D_1, \ldots, D_r \) are distinct primes on \( \mathcal{F} \), different from \( C \), which are necessarily referred to \( p \). Indeed, if \( D_i \) is one of \( D_1, \ldots, D_r \), it is or exceptional for \( p \) (and then referred to \( p \)), or, if it would not be referred to \( p \), the ideal \( J(\sigma^{-1}_p(D_i)) \), by Lemma 1 is a principal ideal which contains \( \Psi \) itself. Since \( \Psi \) is irreducible, then \( J(\sigma^{-1}_p(D_i)) = (\Psi) = J(\sigma^{-1}(C)) \). On the other hand \( \{\Psi = 0\} \not\subset \{M = 0\} \) and \( \sigma_p \) is regular in \( P^n - \{M = 0\} \) so we would have \( D_i = C \). So \( D_i, \; i = 1, \ldots, r \), is in any case referred to \( p \). Of course \( r < t \), being \( t \) the number of all primes of \( \mathcal{F} \) referred
to \( p \) and among them there are \( D_1, \ldots, D_r \). We can suppose that the primes of \( \mathcal{F} \) referred to \( p \) in (6) to be \( D_1 = C_1, \ldots, D_r = C_r \).

Let us consider now the polynomials \( L_i^m \in k[X] \), \( i = 1, \ldots, r \) and we denote with \( p_i \) their images in \( k[\mathcal{F}] \). Since is

\[
g = k^1 / (p_1 \ldots p_r) \in K.
\]

it results, by (4), (5) and (6),

\[
(7) \quad \text{div} (g) = \lambda C + \lambda v_1 C_1 + \\
+ \ldots + \lambda v_r C_r - [v_1 \lambda_1 n_1 C_1 + \ldots + v_r \lambda_r n_r C_r] = \lambda C.
\]

Note (7) means that for every valuation \( v \), of the field \( K \) centered in the subvariety \( \mathcal{E} \) of codimension 1 in \( C(\mathcal{F}) \) is

\[
v_e(g) = 0 \text{ if } \mathcal{E} \neq C(C) \quad \text{and} \quad v_e(g) = \lambda \text{ if } \mathcal{E} = C(C).
\]

By this \( g \) is an element which belongs to the integral closure of \( k[\mathcal{F}] \), by the structure theorem of noetherian integrally closed domains. On the other hand, being \( k[\mathcal{F}] \) normal because \( \mathcal{F} \) is projectively normal (see d.\,f. 1), \( g \in k[\mathcal{F}] \). It exists then at least a homogeneous \( G \in k[X] \) such that its projection in \( k[\mathcal{F}] \) is \( g \). Moreover we get

\[
(8) \quad \mathcal{F} \cdot G = \lambda C.
\]

We note that the integer \( \lambda \) in (8) does not depend on \( C \) but only on all the primes referred to \( p \). So \( \mathcal{F} \) is \( \lambda \)-almostfactorial.

**Remark 1.** Prop. 2 is an extension of a well known criterion of D. Gallarati on the monoid hypersurfaces \( \mathcal{M} \subset \mathbb{P}^v \) (see [3], cap. III, 17, p. 38, and also [7], Prop. 1):

*Every prime of \( \mathcal{M} \) is set-theoretic complete intersection of \( \mathcal{M} \) iff all the primes of the cone of the straight lines passing through the vertex of \( \mathcal{M} \) are set-theoretic complete intersection.*

Indeed the projection from the vertex \( V \) of \( \mathcal{M} \) onto a hyperplane not passing through \( V \), gives a parametrization of \( \mathcal{M} \) on that hyperplane. The primes of \( \mathcal{M} \) referred to this parametrization are just the primes of the cone of straight lines of \( \mathcal{M} \) passing through \( V \). They are all exceptional too.
Remark 2. Prop. 2 also shows that the image $\mathcal{F}$ of a $m$-ple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$, $N = \binom{n+m}{n} - 1$, is $m$-almostfactorial (fact well known). Indeed in the usually parametrization $p$ of $\mathcal{F}$ on $\mathbb{P}^n$ (see [4], pag. 124 for example) is referred to $p$ a prime $C \subset \mathcal{F}$ which can be supposed to be the image of a hyperplane of $\mathbb{P}^n$. It results that $mC$ is just the complete intersection of $\mathcal{F}$ with a suitable hyperplane in $\mathbb{P}^N$. Since $\mathcal{F}$ is projectively normal one gets that $\mathcal{F}$ is $m$-almostfactorial.

We can formulate an affine version of Prop. 2 in the following

**Proposition 3.** Let $\mathcal{E} \subset \mathbb{A}^N$ be a rational normal variety. The following facts are equivalent:

a) $\mathcal{E}$ is almost-factorial;

b) every prime $C_a$ on $\mathcal{E}$ referred to a parametrization $p$ of the projective closure $\mathcal{E}$ is S.T.C.I. on $\mathcal{E}$.

**Proof.** Let $\mathcal{F} = \mathcal{E}$ be the projective closure of $\mathcal{E}$ and $C_a \subset \mathcal{E}$ be a prime whose projective closure $C = C_a$ is not referred to the parametrization of $\mathcal{F}$. To obtain relation (7) we argue as in Prop. 2. (7) induces on $\mathcal{E}$

$$\text{div} (g_a) = \lambda C_a,$$

where $g_a$ now belongs to the quotient field of $\mathcal{E}$. Since $\mathcal{E}$ is normal, the ring $k[\mathcal{E}]$ coincides with its integral closure. The arguments as at the end of the proof of Prop. 2 prove that $g_a \in k[\mathcal{E}]$. So it exists a suitable polynomial $G$ for which it results

$$\mathcal{E} \cdot \{ G = 0 \} = \lambda C_a.$$

This proves $b) \Rightarrow a)$, while $a) \Rightarrow b)$ is obvious.

3. Applications and examples.

It is well known that a hypersurface $\mathcal{F} \subset \mathbb{P}^n$ is projectively normal iff is non singular in codimension 1 (see [6] Prop. 1 p. 389 and Prop 2 p. 391) arguing, in the projective case, on the affine cone of $\mathcal{F}$. In the case of surfaces of $\mathbb{P}^3$ we can apply Prop. 2 to state the
COROLLARY 1. A rational non singular surface $\mathcal{F} \in \mathbb{P}^3$ is almost-factorial only if it is a plane. (So it is factorial).

PROOF. Let $d$ be the degree of $\mathcal{F}$. Since $\mathcal{F}$ is non singular and rational, the geometric genus

$$p_g(\mathcal{F}) = p_g(\mathbb{P}^2) = \frac{(d-1)(d-2)(d-3)}{6} = 0.$$ 

So, if $\mathcal{F}$ is not a plane, $\mathcal{F}$ is a non singular quadric or cubic. But these surfaces are not almost-factorial (see [3], or [7]).

COROLLARY 2. A rational surface $\mathcal{F} \subset \mathbb{P}^3$ of degree $d > 1$ is almost-factorial if only if it has a positive (finite) number of singular points, and has a parametrization $p$ on $\mathbb{P}^2$ such that every curve on $\mathcal{F}$ which is referred to $p$ is S.T.C.I. of $\mathcal{F}$.

PROOF. It follows from Prop. 2 and from what we have recalled about the condition for a hypersurface of $\mathbb{P}^n$ to be projectively normal.

EXAMPLE 1. A quartic surface in $\mathbb{P}^3$ with only two double singular points. Let us denote with $\{T, X, Y, Z\}$ the coordinates in $\mathbb{P}^3$. Let $\mathcal{F}$ be:

$$\mathcal{F} = \{T^2X^2 + TY^3 - Z^4 = 0\}.$$

The surface $\mathcal{F}$ is singular only in the double points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$.

A parametrization $p$ of $\mathcal{F}$ is, for example, given by

$$H_0 = W^2(W^2 - U^2)^2, \quad H_1 = V^6 U, \quad H_2 = V^4 W(W^2 - U^2),$$

$$H_3 = V^2 W^3(W^2 - U^2) \in k[W, U, V]$$

because we have, first

$$H_0^3 H_1^3 + H_0 H_2^3 - H_3^4 = 0,$$

secondly, let us choose $F_0 = Z^2$, $F_1 = TX$, $F_2 = YZ \in k[T, X, Y, Z]$. We can consider then the rational map

$$\sigma: (W, U, V) \rightarrow (T = H_0, X = H_1, Y = H_2, Z = H_3) \subset \mathcal{F} \subset \mathbb{P}^3.$$
The restriction on \( \mathcal{F} \), of the map
\[
\pi: (T, X, Y, Z) \to (W = F_0, U = F_1, V = F_2) \subset \mathbb{P}^2
\]
is just \( \sigma^{-1} \). Indeed for every \( F \in k[T, X, Y, Z] \) let \( f \) be the canonical projection of \( F \) in
\[
k[T, X, Y, Z]/J(\mathcal{F}) = k[\mathcal{F}] = k[t, x, y, z];
\]
one gets the relations
\[
H_0(f_0, f_1, f_2) = \ldots = (ty^a z^a) t, \quad H_1(f_0, f_1, f_2) = \ldots = (ty^a z^a) x, \\
H_2(f_0, f_1, f_2) = \ldots = (ty^a z^a) y, \quad H_3(f_0, f_1, f_3) = \ldots = (ty^a z^a) z
\]
from which it results \( Q = TY^a Z^a \). Moreover the product-map
\[
(t, x, y, z) \to (W = f_0, U = f_1, V = f_2) \to
\]
\[
\to (t = H_0, x = H_1, y = H_2, z = H_3)
\]
is the identity on \( \mathcal{F} \), while the identities
\[
F_0(H_0, \ldots, H_3) = H_3^2 = [V^a W^a (W^2 - U^2)^2] W, \\
F_1(H_0, \ldots, H_3) = H_0 H_3 = [V^a W^a (W^2 - U^2)^2] U, \\
F_2(H_0, \ldots, H_3) = H_1 H_3 = [V^a W^a (W^2 - U^2)^2] V,
\]
show that one must assume \( M = V^a W^a (W^2 - U^2)^2 \) and they prove that the product-map
\[
(W, U, V) \to (t = H_0, x = H_1, y = H_2, z = H_3) \to
\]
\[
\to (W = f_0, U = f_1, V = f_2)
\]
is the identity on \( \mathbb{P}^2 \). The components of \( \mathcal{F} \cap \{Q = 0\} \) are
\[
\mathcal{R} = \{T = Z = 0\}, \quad \mathcal{C}_3 = \{Z = TX^a + Y^2 = 0\}, \\
\mathcal{C}_2 = \{Y = TX - Z^2 = 0\}, \quad \mathcal{C}_2' = \{Y = TX + Z^2 = 0\}
\]
and they are the curves on \( \mathcal{F} \) referred to \( p \).
Now we have to prove that every such curves is S.T.C.I. of \( \mathcal{F} \). Clearly is

\[
\mathcal{F} \cdot \{ T = 0 \} = 4 \mathcal{R} , \quad \mathcal{F} \cdot \{ TX^2 + Y^2 = 0 \} = 4 \mathcal{C}_3 , \\
\mathcal{F} \cdot \{ TX - Z^2 = 0 \} = 3 \mathcal{C}_2 + 2 \mathcal{R} , \quad \mathcal{F} \cdot \{ TX + Z^2 = 0 \} = 3 \mathcal{C}_2' + 2 \mathcal{R} .
\]

Let us consider on \( \mathcal{F} \) the divisor defined by the quotient

\[
\frac{[(tx - z^2)^2]}{t}.
\]

It is just the divisor of \( 2tx^2 - 2xz^2 + y^3 \) because in \( k[\mathcal{F}] \) equality \( z^4 = t(tx^2 + y^3) \) holds. From this, it follows

\[
\mathcal{F} \cdot \{ 2TX^2 - 2XZ^2 + Y^2 = 0 \} = 6 \mathcal{C}_3.
\]

By the same arguments one gets

\[
\mathcal{F} \cdot \{ 2TX^2 + 2XZ^2 + Y^2 = 0 \} = 6 \mathcal{C}_3'.
\]

Since every curve of \( \mathcal{F} \) referred to \( p \) is S.T.C.I. on \( \mathcal{F} \) we can apply to \( \mathcal{F} \) Prop. 2: \( \mathcal{F} \) is then almost-factorial; more precisely \( \mathcal{F} \) is 12-almostfactorial.

As example let us consider the rational curve

\[
\mathcal{C}_2 = \{ t = (1 - s)^2, x = 4s^4(1 - 2s), y = 4s^3(1 - s), z = 2s^2(1 - s) \}.
\]

\( \mathcal{C}_s \) belongs to \( \mathcal{F} \). The curve \( \mathcal{C}_s \) belongs even to the surfaces \( \{ Z^3 + TXZ - TY^2 = 0 \}, \{ Y^2Z + TXY + 2TXZ - TY^2 = 0 \} \) etc., and even to the quadric \( Q = \{ TX + YZ - Z^2 = 0 \} \). The images on \( \mathbb{P}^2 \), by means of \( \sigma^{-1} \), of the intersections with \( \mathcal{F} \) of such surfaces are curves of which a common component, not component of \( \{ M = 0 \} = \{ V^sW^2(W^2 - U^2)^2 = 0 \} \), is the line \( \{ U + V - W = 0 \} \). By this we can suppose \( \mathcal{Y}(W, U, V) = U + V - W \), so \( \mathcal{Y}(F_0, F_1, F_2) = TX + + YZ - Z^2 \).

On the other hand it is

\[
\mathcal{F} \cdot Q = \mathcal{C}_2 + \mathcal{C}_2 + \mathcal{R}.
\]

Now \( \mathcal{C}_2 \) and \( \mathcal{R} \) are S.T.C.I. on \( \mathcal{F} \), so it is also \( \mathcal{C}_s \) with multeplicity at most 12. It is enough to consider on \( \mathcal{F} \) the divisor defined by the
Indeed in $k[\mathcal{F}]$ the identities

$$ty^2 = -(tx - z^2)(tx + z^2), \quad z^4 = t(tx^2 + y^3);$$

hold, which give, first

$$[tx - z^2 + yz]^3/(tx - z^2) =$$

$$= [(tx - z^2)^3 + 3(tx - z^2)yz + 3y^2z^2 + y^3z^3]/(tx - z^2) =$$

$$= 2tx^2 - 2tx^2 + ty^3 + 3txyz - 3yz^3 + 3y^2z^2 - xy^3 - tx^2z - xz^3 =$$

$$= tL + zN$$

where we have assumed, for example,

$$L = 2x^2t - 2txz + y^3 + 3xyz - x^2z, \quad N = -y^3 - 3yz^3 + 3y^2z - xz^2.$$

Secondly, the divisor of the quotient

$$(tL + zN)^4/t$$

coincides with the divisor of

$$g = t^3L^4 + 4t^2zL^3N + 6tz^2L^2N^2 + 4z^3LN^3 + (tx^2 + y^3)N^4.$$ 

It then exists in $k[X]$ a polynomial $G$, homogeneous of degree 15, such that its image on $k[\mathcal{F}]$ is $g$; then we get $\{G = 0\} \cdot \mathcal{F} = 12C_5$.

**Example 2. Classes of rational surfaces with an affine part isomorphic to a plane.** It is easy to determine a class of surfaces $\mathcal{F}_n \subset \mathbb{A}^3$ of degree $n$, for every $n > 0$, which are isomorphic to a plane and having projective closure non singular in codimension 1 and $n$-almostfactorial.

P. C. Craighero has pointed this example to me in the case $n = 4$.

Let $a(T), b(T), c(T)$ be arbitrary polynomials of $k[T]$ of degree $r, s, m$ respectively for which is

$$rs + 1 = m = n \quad \text{or} \quad rs = n = m + 1.$$
Let us consider the two isomorphisms of $\mathbb{A}^3$

$$\eta: (U, V, W) \rightarrow (U' = U + a(W), \ V' = V, \ W' = W)$$

and

$$\chi: (U', V', W') \rightarrow (U'' = U', \ V'' = V', \ W'' = W' + b(U') + c(V'))$$

and their product

$$\chi \circ \eta: (U, V, W) \rightarrow$$

$$\rightarrow (U'' = U + a(W), \ V'' = V, \ W'' = W + b(U + a(W)) + c(V)).$$

The affine surfaces of $\mathbb{A}^3$:

$$\mathcal{F}_n = \{ Z + b(X + a(Z)) + c(Y) = 0 \}$$

are isomorphic to a plane (and more precisely to the plane $\{ W'' = 0 \}$) by means of $\chi \circ \eta$ and are of degree $n$, and, for example, they admit the parametrization

$$X = U - a[-b(U) - c(V)], \quad Y = V, \quad Z = -b(U) - c(V)$$

whose inverse is $(U = X + a(Z), \ V = Y)$, with $Z + b(X + a(Z)) + c(Y) = 0$. Such surfaces are then rational and factorial. Their projective closure $\overline{\mathcal{F}}_n$ is non singular in codimension 1; the section of such surfaces with the plane at the infinity is a straight line which is just the complete intersection of such two surfaces. Such a line is the only curve on the surface $\overline{\mathcal{F}}_n$ which is referred to the parametrization of $\overline{\mathcal{F}}_n$; by this $\overline{\mathcal{F}}_n$ is $n$-almostfactorial.

If the polynomial $b(T)$ is linear ($s = 1$) such surfaces are monoids; then one can apply Gallarati’s criterion to them with the same conclusions.

**Example 3.** Classes of trinomial rational surfaces (of the kind $\{X^m + Y^n = Z^2\}$). Let $m, n$ be coprime positive integers, with $m < n$. Let $(r_o, s_o)$ a integer solution of the diophantine equation

$$(\alpha) \quad \quad \quad \quad \quad \quad x^m - yn = 1.$$
Every other integer solution \((r, s)\) of such equation is

\[
(\beta_1) \quad r = r_0 + tn,
\]

\[
(\beta_2) \quad s = s_0 + tm \quad \text{for every } t \in \mathbb{Z}.
\]

For every integer \(r\) in \((\beta_1)\), we consider the classes of affine surfaces of \(\mathbb{A}^3\)

\[
\mathcal{E}_{m,n,r} = \{X^m + Y^n = Z^{m-1}\}, \quad \mathcal{F}_{m,n,r} = \{X^m + Y^n = Z^r m\} \quad \text{for } r > 0,
\]

\[
\mathcal{G}_{m,n,r} = \{X^m + Y^n = Z^{-rm}\}, \quad \mathcal{J}_{m,n,r} = \{X^m + Y^n = Z^{1-rm}\} \quad \text{for } r < 0.
\]

One gets a parametrization of such surfaces assuming

\[
\begin{align*}
x &= z^r u, & y &= z^s v \quad \text{with } z = (1 - v^n)/u^m & \quad \text{for the } \mathcal{E}_{m,n,r},
\end{align*}
\]

\[
\begin{align*}
x &= z^r u, & y &= z^s v \quad \text{with } z = v^n/(1 - u^m) & \quad \text{for the } \mathcal{F}_{m,n,r},
\end{align*}
\]

\[
\begin{align*}
x &= z^{-r} u, & y &= z^{-s} v \quad \text{with } z = (1 - u^m)/v^n & \quad \text{for the } \mathcal{G}_{m,n,r},
\end{align*}
\]

\[
\begin{align*}
x &= z^{-r} u, & y &= z^{-s} v \quad \text{with } z = u^m/(1 - v^n) & \quad \text{for the } \mathcal{J}_{m,n,r},
\end{align*}
\]

while for the inverse map we have to assume for the \(\mathcal{E}_{m,n,r}\) and \(\mathcal{F}_{m,n,r}\):

\[
u = x/z^r, \quad v = y/z^s
\]

and \(u = xz^r, \quad v = yz^s\) for the \(\mathcal{G}_{m,n,r}\) and for the \(\mathcal{J}_{m,n,r}\) respectively.

Every surface \(\mathcal{E}_{m,n,r}\) and \(\mathcal{J}_{m,n,r}\) is \(m\)-almostfactorial, while the surfaces \(\mathcal{F}_{m,n,r}\) and \(\mathcal{G}_{m,n,r}\) are \(n\)-almostfactorial, by Prop. 2 and Prop. 3.

There are other classes of affine rational surfaces which are trinomial and almost-factorial. For example, for every pair of positive integers \((m, n)\) the affine surfaces in \(\mathbb{A}^3:\)

\[
\mathcal{B}_{m,n} = \{X^m + Y^n = Z^{mn+1}\} \quad \text{and} \quad \mathcal{C}_{m,n} = \{X^m + Y^n = Z^{mn-1}\}.
\]

\(\mathcal{B}_{m,n}\) admits a parametrization \(x = z^m u, \quad y = z^n v, \quad z = u^m + v^n\) whose inverse is \(u = x/z^n, \quad v = y/z^m, \quad \mathcal{C}_{m,n}\) admits a parametrization \(x = z^n u, \quad y = z^m v, \quad z = 1/(u^m + v^n)\) whose inverse is \(u = x/z^n, \quad v = y/z^m\). The surfaces \(\mathcal{B}_{m,n}\) and \(\mathcal{C}_{m,n}\) are factorial if \(m, n\) are coprimes, \(\mathcal{B}_{m,n}\) is \((mn + 1)\)-almostfactorial and \(\mathcal{C}_{m,n}\) is \((mn - 1)\)-almostfactorial otherwise.
Note the surface $C_{2,3}$ which belongs to the family $C_{m,n}$. By interchanging $-z$ with $z$, one gets the well known surface $\{x^2 + y^3 + z^5 = 0\}$ which is factorial even if it has a singular point in $0 = (0, 0, 0)$ (cfr. [5] Example 5.8 p. 420).

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