RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

E. GREGORIO

Generalized Morita equivalence for linearly topologized rings

Rendiconti del Seminario Matematico della Università di Padova, tome 79 (1988), p. 221-246

<http://www.numdam.org/item?id=RSMUP_1988__79__221_0>

Rend. Sem. Mat. Univ. Padova, Vol. 79 (1988)

Generalized Morita Equivalence for Linearly Topologized Rings.

E. Gregorio (*)

0. Introduction.

Since the appearance of Morita theory on equivalences between categories of modules, many authors have tried to generalize it and to characterize equivalences between subcategories of modules with suitable closure properties. The most important paper on the subject is Fuller's [3] in which importance was given to topological concepts, namely to the connections between the theory of density and that of equivalence.

In this paper we generalize Morita theory to a similar one on equivalences between categories of modules associated to linearly topologized rings by proving results which extend Morita's and Fuller's.

It should be noted that a parallel generalization of Morita duality already exists in the literature.

The most important result, on which are based all the others, is exposed in Section 1: it is a theorem on representation of equivalences between categories of modules; this representation is obtained by a limit process inspired by MacDonald's paper [5].

In section 2 we apply this theorem to the equivalences between the categories of discrete modules over linearly topologized rings, showing that these equivalences are always induced by a suitable bimodule; recall that a right linearly topologized ring (R, τ) is a topological ring having a local basis (i.e. a basis of neighborhoods of

(*) Indirizzo dell'A.: Dipartimento di Matematica Pura e Applicata, Via Belzoni 7 - 35131 Padova, Italy.

zero) consisting of right ideals and that a module $M \in \text{Mod-}R$ is a discrete module over (R, τ) if and only if $\text{Ann}_R(x) \in \mathcal{F}_\tau$ (the filter of all τ-open right ideals of R) for all $x \in M$. We denote by $\text{Mod-}R_\tau$ the Grothendieck category of all discrete modules over (R, τ). The definition of linearly topologized module is obvious.

Section 3 is devoted to the introduction of a useful functor between the categories of complete linearly topologized modules over a linearly topologized ring induced by a bimodule, which has properties similar to those of the tensor product.

In Section 4 the modules which induce equivalences between the categories of discrete modules over linearly topologized rings are characterized; these modules are studied more deeply in Section 5, where we prove also a density theorem which generalizes Fuller's density theorem in [3].

In Section 6 we give some examples to prove that we have reached an effective generalization of the previous theories.

Finally, in Section 7 we specialize the results to the case of commutative rings.

If (R, τ) is a right linearly topologized ring then denote by $\text{LTC-}R_\tau$ the category of all complete linearly topologized modules over (R, τ); if τ is the discrete topology, then put $\text{LTC-}R_\tau = \text{LTC-}R$. All topological modules and rings considered in this paper are, unless the contrary is stated (e.g. in the definition of tensor product), Hausdorff. Moreover we shall be concerned only with right linearly topologized rings and right modules, with a few exceptions in Section 5. Morphisms of right modules will be written on the left (and morphisms of left modules on the right). Any subcategory of a given category will be closed under isomorphisms. Given two topological modules M_R and N_R we denote by $\text{Chom}_R(M, N)$ the group of all continuous R-morphisms from M to N and by $\text{Cend}\,(M_R)$ the ring of all continuous endomorphisms of M_R.

1. Equivalences between finitely closed categories of modules.

1.1. Let A be a ring and $\mathfrak{D}_A$ a finitely closed subcategory of Mod-A, i.e. full and closed with respect to finite direct sums submodules and homomorphic images. If $\mathcal{F}_A$ is the set of those right ideals of A such that $A/I \in \mathfrak{D}_A$, then it is clear that $\mathcal{F}_A$ is a filter of ideals in A and defines a linear topology on A (let us denote it by σ) which is not, in general, Hausdorff. It is also clear that $\mathfrak{D}_A$ is a subcategory of Mod-A_σ. The Hausdorff completion of (A, σ) is $\hat{A} = \varprojlim_{I\in\mathcal{F}_A} A/I$.

1.2. Let us fix, for the rest of Section 1, two rings A and R and two finitely closed subcategories $\mathfrak{D}_A$ and $\mathcal{G}_R$ of Mod-A and Mod-R respectively. We shall also assume that an equivalence

$$(F\colon \mathfrak{D}_A \to \mathcal{G}_R,\ G\colon \mathcal{G}_R \to \mathfrak{D}_A)$$

is given. We denote by $\mathcal{F}_A$ and $\mathcal{F}_R$ the filters of (right) ideals (and by σ and τ the associated topologies) defined on A and R by $\mathfrak{D}_A$ and $\mathcal{G}_R$.

Lemma ([cf. [5, Lemma 1]). *Let* $P_R = \varprojlim_{I\in\mathcal{F}_A} F(A/I)$, *endowed with the limit topology of the discrete topologies on* $F(A/I)$. *Then* $P \in \mathrm{LTC}\text{-}R_\tau$ *and there exists a canonical ring morphism*

$$\psi\colon A \to \mathrm{Chom}_R(P, P)\,.$$

Proof. For $I \in \mathcal{F}_A$ and a $a \in A$ define

$$\lambda_{a,I}\colon A/(I:a) \to A/I$$

by $\lambda_{a,I}(x + (I:a)) = ax + I$ (where $(I:a) = \{x \in A\colon ax \in I\}$)$:\lambda_{a,I}$ is a morphism, hence there exists a unique morphism $\psi(a)\colon P \to P$ which makes commutative all the diagrams of the form

$$\begin{array}{ccc} P & \xrightarrow{\psi(a)} & P \\ \downarrow & & \downarrow \\ F(A/(I:a)) & \xrightarrow{F(\lambda_{a,I})} & F(A/I) \end{array}$$

as $I \in \mathcal{F}_A$ (here the vertical arrows are the canonical projections); $\psi(a)$ is clearly continuous and one can easily verify that ψ is a ring morphism. $\square$

In this way P becomes on A-R-bimodule; moreover $P_R \in \mathrm{LTC}\text{-}R_\tau$ and A acts on P_R by continuous endomorphisms. If we endow the modules in $\mathcal{G}_R$ with the discrete topology, then we can consider the functor $\mathrm{Chom}_R(P, -)\colon \mathcal{G}_R \to$ Mod-A defined in the obvious way.

1.3 Representation theorem (cf. [5, Lemma 2]). *There is a natural isomorphism*

$$\mu\colon G \to \mathrm{Chom}_R(P, -)\,.$$

Proof. If $M \in \mathcal{G}_R$ we can consider the chain of (group) isomorphisms

$$GM \cong \operatorname{colim} \mathrm{Ann}_{GM}(I) \cong \operatorname{colim} \mathrm{Hom}_A(A/I, GM) \cong$$
$$\cong \operatorname{colim} \mathrm{Hom}_R\big(F(A/I), M\big)$$

(here and in all this proof limits and colimits are taken with I running through $\mathcal{F}_A$); let $s_M\colon GM \to \operatorname{colim} \mathrm{Hom}_R\big(F(A/I), M\big)$ be their composition; denote by

$$b_I\colon \mathrm{Hom}_R\big(F(A/I), M\big) \to \operatorname{colim} \mathrm{Hom}_R\big(F(A/I), M\big)\,, \qquad l_I\colon P \to F(A/I)$$

the canonical morphisms of colimit and limit respectively. There is a unique (group) morphism

$$t_M\colon \operatorname{colim} \mathrm{Hom}_R\big(F(A/I), M\big) \to \mathrm{Hom}_R(P, M)$$

such that $t_M b_I = \mathrm{Hom}_R(l_I, M)$, for any $I \in \mathcal{F}_A$. It follows from [2, p. 57] that t_M is injective: indeed l_I is surjective so that $\mathrm{Hom}_R(l_I, M)$ is injective for all $I \in \mathcal{F}_A$.

It is clear that a morphism $f\colon P \to M$ is continuous if and only if it factors through l_I, for some I. So it follows plainly that $\mathrm{Chom}_R(P, M) \leqslant \mathrm{Im}\, t_M$; it remains to prove the converse inequality. If $g \in \operatorname{colim} \mathrm{Hom}_R\big(F(A/I), M\big)$, there are $I \in \mathcal{F}_A$ and $h \in \mathrm{Hom}_R\big(F(A/I), M\big)$ such that $g = b_I(h)$; hence

$$t_M(g) = t_M b_I(h) = \mathrm{Hom}_R(l_I, M)(h) = h l_I\,.$$

Finally, if we put $\mu_M = t_M s_M\colon GM \to \mathrm{Chom}_R(P, M)$, we obtain a natural isomorphism $\mu\colon G \to \mathrm{Chom}_R(P, -)$, since it is easy to see that μ_M is a morphism of A-modules. □

We can of course work as before on F, getting $Q = \lim\limits_{J \in \mathcal{F}_R} G(R/J)$, a ring morphism $\varphi\colon R \to \mathrm{Chom}_A(Q, Q)$ and a natural isomorphism $\nu\colon F \to \mathrm{Chom}_A(Q, -)$. Now we look for a relation between P_R and Q_A.

Denote by $(\hat{R}, \hat{\tau})$ and $(\hat{A}, \hat{\sigma})$ the Hausdorff completions of (R, τ) and (A, σ) respectively.

1.4 Proposition. *There exist two canonical isomorphisms*

$$Q \cong \mathrm{Chom}_R\big(P, (\hat{R}, \hat{\tau})\big) \qquad \textit{and} \qquad P = \mathrm{Chom}_A\big(Q, (\hat{A}, \hat{\sigma})\big)\,.$$

Proof. From the definitions and 1.3 we get (limits are taken for $J \in \mathcal{F}_R$):

$$Q \cong \lim G(R/J) \cong \lim \mathrm{Chom}_R(P, R/J)\ .$$

We have the canonical morphisms

$$k_J\colon \lim \mathrm{Chom}_R(P, R/J) \to \mathrm{Chom}_R(P, R/J)$$

and $c_J\colon \hat{R} \to R/J$, as $J \in \mathcal{F}_R$. There is a unique morphism

$$u\colon \mathrm{Chom}_R\big(P, (\hat{R}, \hat{\tau})\big) \to \lim \mathrm{Chom}_R(P, R/J)$$

such that $k_J u = \mathrm{Chom}_R(P, c_J)$, for all $J \in \mathcal{F}_R$. Let us see that u is an isomorphism. If $u(f) = 0$ for $f \in \mathrm{Chom}_R\big(P, (\hat{R}, \hat{\tau})\big)$, then

$$0 = k_J u(f) = \mathrm{Chom}_R(P, c_J)(f) = c_J f$$

for all $J \in \mathcal{F}_R$: hence $\mathrm{Im} f \leqslant \bigcap_{J \in \mathcal{F}_R} \mathrm{Ker}\, c_J = 0$, so that $f = 0$. On the other hand, if $\eta \in \lim \mathrm{Chom}_R(P, R/J)$, then by the properties of limit there exists $g \in \mathrm{Chom}_R\big(P, \hat{R}, \hat{\tau})\big)$ such that $c_J g = k_J(\eta)$, for all $J \in \mathcal{F}_R$, and so $u(g) = \eta$. $\square$

2. Linearly topologized rings.

2.1 Definition. Let (A, σ) and (R, τ) be (right) linearly topologized rings. We say that they are *similar* if there exists an equivalence

$$(F\colon \text{Mod-}A_\sigma \to \text{Mod-}R_\tau,\ G\colon \text{Mod-}R_\tau \to \text{Mod-}A_\sigma)\ .$$

In this case we say also that the pair (F, G) is a *similarity* between (A, σ) and (R, τ).

The following fact is well known.

2.2 PROPOSITION. *Any linearly topologized ring is similar to its Hausdorff completion.*

An obvious application of 1.3 gives the following

2.3 THEOREM. *Let (F, G) be a similarity between (A, σ) and (R, τ). Then there exist modules $P_R \in$ LTC-R_τ and $Q_A \in$ LTC-A_σ and morphisms $\psi\colon A \to \mathrm{Chom}_R(P, P)$ and $\varphi\colon R \to \mathrm{Chom}_A(Q, Q)$ in such a way that*

$$F \cong \mathrm{Chom}_A(Q, -)\,, \qquad G \cong \mathrm{Chom}_R(P, -)\,.$$

At this moment we have «represented» the similarities: in the sequel we shall characterize all bimodules ${}_AP_R$ such that the functor $\mathrm{Chom}_R(P, -)$ defines a similarity between two linearly topologized rings. We need the following

2.4. DEFINITION. Let (R, τ) be a ring and P_R a module, both linearly topologized; denote by $\mathcal{F}(P_R)$ the set of open submodules of P. We say that P is:

(i) *topologically finitely generated* (t.f.g.) if, for any $V \in \mathcal{F}(P_R)$, the quotient P/V is finitely generated;

(ii) *topologically quasi-projective* if, given any $V \in \mathcal{F}(P_R)$ and any continuous morphism $f\colon P \to P/V$ (where P/V has the discrete topology), there exists a continuous R-endomorphism $g\colon P \to P$ such that for all $p \in P$ we have

$$f(p) = g(p) + V;$$

(iii) a *self-generator* if, for any $V \in \mathcal{F}(P_R)$, the closure in P of $\Sigma(V) = \sum\{\mathrm{Im}\, f \mid f \in \mathrm{Chom}_R(P, V)\}$ coincides with V;

(iv) a *τ-generator* if it is a topological module over (R, τ) and, for any non-zero morphism $f\colon M \to N$ in Mod-R, there is a continuous morphism $g\colon P \to M$ such that $fg \neq 0$.

We say that P_R is

(1) a *quasiprogenerator* if it satisfies (i), (ii), (iii) and is complete;

(2) a *τ-progenerator* if it is a quasiprogenerator and a τ-generator.

REMARKS. Let P_R be a discrete module.

(*a*) P_R is a quasiprogenerator if and only if it is a quasi-progenerator in the sense of Fuller [3].

(*b*) Let $\tau = d$ be the discrete topology on R; we shall prove that P_R is a d-progenerator if and only if it is a progenerator in the sense of Morita theory.

3. Chom functors and tensor products.

3.1. Let P_R and M_R be objects in LTC-R_τ: we can endow the group $\mathrm{Chom}_R(P, M)$ with the topology of uniform convergence, which has as a local basis the set of subgroups of the form

$$\mathfrak{I}(V) = \{f \in \mathrm{Chom}_R(P, M) : f(P) \leqslant V\}$$

as V runs through the family $\mathcal{F}(M)$ of all open submodules of M. We shall denote by $\mathrm{Chom}_R^u(P, M)$ the group of continuous R-morphisms from P to M endowed with the topology of uniform convergence. It is almost trivial to see that $\mathrm{Chom}_R^u(P, M)$ is complete.

Consider now the ring $A = \mathrm{Chom}_R(P, P)$, again with the topology σ of uniform convergence: one can easily prove that this is indeed a ring topology and that $\mathrm{Chom}_R^u(P, M) \in$ LTC-A_σ, for all $M \in$ LTC-R_τ.

We have so defined a functor

$$\mathrm{Chom}_R^u(P, -): \text{LTC-}R_\tau \to \text{LTC-}A_\sigma$$

(the action on morphisms being the obvious one), which maps Mod-R_τ into Mod-A_σ.

3.2. In this Section we shall fix two complete linearly topologized rings (A, σ) and (R, τ) and a bimodule ${}_AP_R$ such that

(1) $P_R \in$ LTC-R_τ;

(2) P is faithful on both sides;

(3) A acts on P_R by continuous morphisms and the topology induced on A by the topology of uniform convergence on $\mathrm{Cend}(P_R) = \mathrm{Chom}_R(P, P)$ is coarser than σ.

Under these hypotheses, the functor $\mathrm{Chom}^u_R(P, -)$ is again a functor $\mathrm{LTC}\text{-}R_\tau \to \mathrm{LTC}\text{-}A_\sigma$, which maps $\mathrm{Mod}\text{-}R_\tau$ into $\mathrm{Mod}\text{-}A_\sigma$.

3.3. Let $N \in \mathrm{LTC}\text{-}A_\sigma$: we can endow $N \otimes_A P$ with the greatest linear topology that makes continuous all morphisms $P \to N \otimes_A P$ of the form

$$n\otimes : p \mapsto n \otimes p$$

for $n \in N$. In this way $N \otimes_A P$ becomes a linearly topologized module over (R, τ), though in general it is not Hausdorff; a local basis for it is the set of R-submodules U of $N \otimes_A P$ such that

$$n\otimes^{-1}(U) = \{p \in P : n \otimes p \in U\}$$

is open for all $n \in N$.

DEFINITION. $N \hat{\otimes}_A P$ is the Hausdorff completion of $N \otimes_A P$. We shall denote by c_N the canonical morphism $N \otimes_A P \to N \hat{\otimes}_A P$.

PROPOSITION. $- \hat{\otimes}_A P$ *is a functor* $\mathrm{LTC}\text{-}A_\sigma \to \mathrm{LTC}\text{-}R_\tau$.

PROOF. It suffices to define the action of morphism $f : M \to N$. Consider $f \otimes_A P : M \otimes_A P \to N \otimes_A P$: this is continuous and so there exists a unique continuous morphism g which makes the diagram

$$\begin{tikzcd}
M \otimes_A P \arrow[r, "f"] \arrow[d, "c_M"'] & N \otimes_A P \arrow[d, "c_N"'] \\
M \hat{\otimes}_A P \arrow[r, "g"] & N \hat{\otimes}_A P
\end{tikzcd}$$

commute. Since this is clearly functorial, we get the desired morphism by putting $f \hat{\otimes}_A P = g$. □

3.4. Let $N \in LTC\text{-}A$, $M \in \mathrm{LTC}\text{-}R$ and $f \in \mathrm{Hom}_A\big(N, \mathrm{Chom}_R(P, M)\big)$: then

$$f'(n \otimes p) = f(n)(p) \qquad (n \in N, p \in P)$$

defines a continuous morphism $f' : N \otimes_A P \to M$: indeed, if V is an open submodule of M and $n \in N$, then $n\otimes^{-1}\big(f'^{-1}(V)\big) = f(n)^{-1}(V)$, and this is open in P by assumption. Hence there is a unique morphism $\zeta(f)\colon N\hat{\otimes}_A P \to M$ making commutative the diagram

$$\begin{tikzcd} N\otimes_A P \arrow[d, "c_N"'] \arrow[dr, "f'"] & \\ N\hat{\otimes}_A P \arrow[r, "\zeta(f)"] & M \end{tikzcd}$$

and we have obtained a group morphism

$$\zeta\colon \mathrm{Hom}_A\big(N, \mathrm{Chom}_R(P, M)\big) \to \mathrm{Chom}_R(N\hat{\otimes}_A P, M)$$

natural in N and M.

3.5. We want to see that the morphism ζ of 3.4 is an isomorphism: if $g\in \mathrm{Chom}_R(N\otimes_A P, M)$, then we can define $g'\colon N \to \mathrm{Hom}_R(P, M)$ by

$$g'(n)(p) = g\big(c_N(n\otimes p)\big);$$

for all $n\in N$, the map $g'(n)$ is continuous: if $V\in \mathcal{F}(M)$, we have

$$g'(n)^{-1}(V) = n\otimes^{-1}\big((gc_N)^{-1}(V)\big)$$

and this is open by assumption.

We now state a sufficient condition to assure that ζ is an isomorphism from $\mathrm{Chom}_A\big(N, \mathrm{Chom}^u_R(P, M)\big)$ onto $\mathrm{Chom}_R(N\hat{\otimes}_A P, M)$.

DEFINITION. Let U be an open submodule of $N\otimes_A P$: we put

$$N[U] = \{n\in N\colon \forall p\in P,\ n\otimes p\in U\}\ .$$

We denote by $\mathcal{B}_P$ the class consisting of the modules $N\in \text{LTC-}A_\sigma$ such that $N[U]$ is open for any $U\in\mathcal{F}(N\otimes_A P)$. We have that $\text{Mod-}A_\sigma\subseteq\mathcal{B}_P$ and that $(A,\sigma)\in\mathcal{B}_P$, since $A\hat{\otimes}_A P$ is topologically isomorphic to P.

3.6. Let $N\in\text{LTC-}A_\sigma$ and $M\in\text{LTC-}R_\tau$: consider the two natural morphisms

$$\alpha_N\colon N\to \mathrm{Chom}^u_R(P, N\hat{\otimes}_A P)$$

$$\beta_M\colon \mathrm{Chom}^u_R(P, M)\hat{\otimes}_A P\to M$$

defined for $n \in N$, $f \in \mathrm{Chom}_R(P, M) = N'$ and $p \in P$ by

$$\alpha_N(n)(p) = c_N(n \otimes p)$$
$$\beta_M(c_{N'}(f \otimes p)) = f(p) .$$

Proposition. *(a) β_M is continuous for all $M \in \mathrm{LTC}\text{-}R_\tau$;*

(b) α_N is continuous for all $N \in \mathfrak{B}_P$;

(c) if $N \in \mathfrak{B}_P$ and $M = N \hat{\otimes}_A P$, then β_M is surjective.

Proof. (a) β_M is just the unique continuous morphism that makes commutative the diagram

$$\begin{tikzcd} \mathrm{Chom}_R^u(P, M) \otimes_A P \arrow[r, "e_M"] \arrow[d] & M \\ \mathrm{Chom}_R^u(P, M) \hat{\otimes}_A P \arrow[ur, "\beta_M"'] & \end{tikzcd}$$

where e_M is the valuation (it is easily seen to be continuous).

(b) Trivial.

(c) Let $N \in \mathfrak{B}_P$, $M = N \hat{\otimes}_A P$ and $f = \beta_M(\alpha_N \hat{\otimes}_A P)\colon M \to M$. If $n \in N$ and $p \in P$ we have clearly $f(c_N(n \otimes p)) = c_N(n \otimes p)$, so that $f = 1_M$. □

3.7. Put $Q = \mathrm{Chom}_R^u(P, (R, \tau))$: $Q_A \in \mathrm{LTC}\text{-}A_\sigma$ and R acts continuously on Q. Assume that the bimodule ${}_RQ_A$ satisfies the same hypotheses as P with A and R interchanged; then we can define the functor

$$- \hat{\otimes}_R Q\colon \mathrm{LTC}\text{-}R_\tau \to \mathrm{LTC}\text{-}A_\sigma .$$

Proposition. *There exist two natural morphisms*

$$\gamma\colon 1_{\mathrm{LTC}\text{-}R_\tau} \to \mathrm{Chom}_R^u(Q, \mathrm{Chom}_R^u(P, -))$$
$$\chi\colon - \hat{\otimes}_R Q \to \mathrm{Chom}_R^u(P, -) .$$

Proof. For $M \in \mathrm{LTC}\text{-}R_\tau$, $m \in M$ and $q \in Q$ we define a continuous morphism $[m, q]\colon P \to M$ by

$$[m, q](p) = mq(p);$$

it is easy to see that the mapping $m \mapsto [m, -]$ is a continuous R-morphism from M to $\mathrm{Chom}_R^u(Q, \mathrm{Chom}_R^u(P, M))$: this we take as γ_M. In order to find χ we observe that $f_M\colon M \otimes_R Q \to \mathrm{Chom}_R(P, M)$ defined by $f_M(m \otimes q) = [m, q]$ is a continuous morphism, so that there is a unique χ_M such that $\chi_M c_M = f_M$. $\square$

4. Similarities between linearly topologized rings.

We shall divide this Section in two parts: in the first one we shall prove that if $(F\colon \text{Mod-}A_\sigma \to \text{Mod-}R_\tau,\ G\colon \text{Mod-}R_\tau \to \text{Mod-}A_\sigma)$ is a similarity between the two complete linearly topologized rings (A, σ) and (R, τ) and ${}_AP_R$ is the bimodule of 2.3, then P_R is a τ-progenerator and (A, σ) is topologically isomorphic to $\mathrm{Chom}_R(P, P)$ with the topology of uniform convergence.

The second part will be concerned with the inverse of the above result, namely that if we are given a complete linearly topologized ring (R, τ) and a τ-progenerator P_R, then the functor $\mathrm{Chom}_R(P, -)$ defines an equivalence between $\text{Mod-}R_\tau$ and $\text{Mod-}A_\sigma$, where (A, σ) is the ring $\mathrm{Chom}_R(P, P)$ with the topology of uniform convergence.

Part I.

In this part we assume that we are given two complete linearly topologized rings (A, σ) and (R, τ) and a similarity between them, $(F\colon \text{Mod-}A_\sigma \to \text{Mod-}R_\tau,\ G\colon \text{Mod-}R_\tau \to \text{Mod-}A_\sigma)$. With the notations of 2.3 we have

$$G \cong \mathrm{Chom}_R(P, -)\,, \qquad F \cong \mathrm{Chom}_A(Q, -)$$

where $Q = \mathrm{Chom}_R(P, (R, \tau))$ and $P = \mathrm{Chom}_A(Q, (A, \sigma))$ algebraically (1.4).

4.1 Lemma. *If the notations are as before, then* $Q \cong \mathrm{Chom}_R^u(P, (R, \tau))$ *and* $P \cong \mathrm{Chom}_A^u(Q, (A, \sigma))$.

Proof. Obviously we can prove this lemma only for Q.

All the limits in this proof will be taken for $J \in \mathcal{F}_\tau$.

By the completeness of (R, τ) we have that $R \cong \lim R/J$: if $\lim R/J$ is identified as usual with a submodule of the direct product of the

R/J, then, if I is an open right ideal of (R, τ), I is identified with $\{(r_J): r_I = 0\} = \{f \in \lim R/J: \pi_I(f) = 0\}$, where π_I is the canonical projection from the direct product.

A local basis for $\mathrm{Chom}_R^u(P, (R, \tau))$ is the set of submodules of the form $\mathfrak{J}(I)$ as $I \in \mathcal{F}_\tau$.

Identify Q, as usual, with a submodule of the direct product $\prod_{J \in \mathcal{F}_\tau} \mathrm{Chom}_R(P, R/J)$; then an element of Q is a family $f = (f_J: P \to R/J)$ of morphisms; if h_I denotes the projection from the product, $h_I(f) = f_I$ and a local basis for Q is the set of submodules of the form $Q \cap \ker h_I$.

If $\varphi: Q \to \mathrm{Chom}_R(P, \lim R/J)$ is the algebraic isomorphism given in 1.4, then, for $g: P \to R$, $\varphi^{-1}(g) = (\pi_J g)$; if $g \in \mathfrak{J}(I)$, then $\varphi^{-1}(g)$ is in $\ker h_I$. Conversely, if $f = (f_J) \in \ker h_I$ and $p \in P$, then $\varphi(f)(p) = (f_J(p))$ and this belongs to I since $f_I(p) = 0$. □

4.2 Lemma. *A is algebraically isomorphic to* $\mathrm{Chom}_R(P, P)$ *and* σ *is finer than the topology of uniform convergence.*

Proof. Recalling that A is complete, we get the isomorphisms

$$A \cong \lim A/I \cong \lim GF(A/I) \cong \lim \mathrm{Chom}_R(P, F(A/I))$$

and we get the conclusion by using arguments similar to those of 1.4 (limits are taken for $I \in \mathcal{F}_\sigma$).

Recall that P is defined as $\lim F(A/I)$; denote by $l_I: P \to F(A/I)$ the canonical map of the limit. If $V \in \mathcal{F}(P)$ then there is $I \in \mathcal{F}_\sigma$ such that $V \geqslant \mathrm{Ker}\, l_I$ and $\mathfrak{J}(V) \geqslant \mathfrak{J}(\mathrm{Ker}\, l_I)$. On the other hand it is plain that $I \leqslant \mathfrak{J}(\mathrm{Ker}\, l_I)$, since, if $i \in I$ and $p \in P$, then $(I:i) = A$ and $l_I(ip) = F(\lambda_{I,i}) l_{(I:i)}(p) = 0$ $(\lambda_{I,i}: A/(I:i) \to A/I$ is defined in 1.2). Hence $\mathfrak{J}(V) \geqslant I$ and is open with respect to σ. □

4.3. Since $P \cong \mathrm{Chom}_A^u(Q, (A, \sigma))$, there exists by 3.7 a natural morphism $\chi: - \widehat{\otimes}_A P \to \mathrm{Chom}_A^u(Q, -)$. For $N \in \mathcal{B}_P$ put $M = N \widehat{\otimes}_A P$ and $(\alpha_N)_* = \mathrm{Chom}_A^u(Q, \alpha_N)$; then

$$(\alpha_N)_* \circ \chi_N: N \widehat{\otimes}_A P \to \mathrm{Chom}_A^u(Q, \mathrm{Chom}_R^u(P, M));$$

if $n \in N$, $p \in P$ and $t = c_N(n \otimes p)$ we have

$$(\alpha_N)_* \circ \chi_N(t) = (\alpha_N)_*([n, p]) = \alpha_N \circ [n, p]$$

and, for $q \in Q$, $x \in P$,

$$(\alpha_N \circ [n, p])(q)(x) = \alpha_N(np(q))(x) = c_N(np(q) \otimes x) = c_N(n \otimes p(q)x) .$$

If we identify Q_A with $\mathrm{Chom}^u_R(P, (R, \tau))$ then

$$\gamma_M(t)(q)(x) = [t, q](x) = tq(x) = c_N(n \otimes pq(x)) .$$

But the elements $pq(x) \in P$ and $p(q)x \in P = \mathrm{Chom}^u_A(Q, A)$ are the same with respect to the identifications we made: indeed the isomorphism $P \to \mathrm{Chom}^u_A(Q, A)$ is defined by $p \mapsto \tilde{p}$, where $p(q)$ is the only element in A such that for all $x \in P$

$$(\tilde{p}(q))x = pq(x) .$$

Hence $(\alpha_N)_* \circ \chi_N$ is an isomorphism and χ_N is injective.

From the above it follows immediately that if N is discrete, then also $N \hat{\otimes}_A P$ is. Indeed χ_N is a continuous injective morphism from $N \hat{\otimes}_A P$ into $\mathrm{Chom}^u_A(Q, N)$ which is discrete.

4.4 THEOREM. *F is naturally isomorphic to* $- \hat{\otimes}_A P$.

PROOF. Since Mod-A_σ is contained in $\mathcal{B}_P$, it follows from 3.4 and 3.5 that $- \otimes_A P$ is a left adjoint of $\mathrm{Chom}_R(P, -)\colon \text{Mod-}R_\tau \to \text{Mod-}A_\sigma$; by the uniqueness of adjoints we get the conclusion. □

4.5 COROLLARY. *If $I \in \mathcal{F}_\sigma$, then*

$$F(A/I) = P/\overline{IP}$$

where $\overline{IP}$ is the closure of IP in P.

PROOF. It is well known that $A/I \otimes_A P \simeq P/IP$, by means of $f\colon (a + I) \otimes p \mapsto ap + IP$ $(a \in A, p \in P)$. Let us verify that f is a homeomorphism, if we endow P/IP with the quotient topology.

(i) If $V \in \mathcal{F}(P)$ and $V \geqslant IP$ then, for any $a \in A$ we have

$$(a + I) \otimes^{-1}(f^{-1}(V/IP)) = \{p \in P\colon ap \in V\} .$$

(ii) Let $U \in \mathcal{F}(A/I \otimes_A P)$: then $V = (1 + I)\otimes^{-1}(U) \in \mathcal{F}(P)$; but $p \in V$ implies $p \in f(U)$, hence $V \geqslant f(U)$, so that f is open.

The Hausdorff space associated to P/IP is $P/\overline{IP}$; moreover $A/I \hat{\otimes}_A P$ is the Hausdorff completion of $P/\overline{IP}$ and so there exists a topological imbedding $P/IP \to A/I \otimes_A P$. By 4.4, $A/I \hat{\otimes}_A P$ is discrete, so that the domain is discrete too, hence complete. □

4.6 THEOREM. *P_R is a τ-progenerator and* $(A, \sigma) = \mathrm{Chom}^u_R(P, P)$.

PROOF. Let $V \in \mathcal{F}(P)$: then $V \geqslant \mathrm{Ker}\, l_I$ for some open right ideal I of A (where $P = \varprojlim_{I \in \mathcal{F}_\sigma} F(A/I)$ and, for $I \in \mathcal{F}_\sigma$, l_I is the canonical map of the limit). Then there is an epimorphism $F(A/I) \to P/V$ and, by applying G, we get an epimorphism $A/I \to G(P/V)$; hence V is of the form $\mathrm{Ker}\, l_J$, for some open ideal J of (A, σ).

(i) *P_R is topologically finitely generated.* It is sufficient to prove that $F(A/I)$ is finitely generated for all open right ideals I of (A, σ); but this follows from the fact that F has a right adjoint, so that it preserves colimits.

(ii) *P_R is topologically quasi-projective.* If $V \in \mathcal{F}(P)$ then we can consider $P/V = F(A/I)$; if $f\colon P \to P/V$ is a continuous morphism then we have $f_* = \mathrm{Chom}^u_R(P, f)\colon \mathrm{Chom}^u_R(P, P) \to A/I$, and $f_*(1) = a + I$ for some $a \in A$.

(iii) *P_R is a self-generator.* Let $V \in \mathcal{F}(P)$: then $P/V = F(A/I)$ and $IP \leqslant V$, so that also $\overline{IP} \leqslant V$. Then we have

$$P/V \cong F(A/I) \cong A/I \hat{\otimes}_A P \cong P/\overline{IP} \to P/V$$

and the composition is the identity, so that $\overline{IP} = V$.

(iv) *P_R is a τ-generator.* Indeed P_R is clearly a module over R_τ; if f is a non-zero morphism in Mod-R_τ then $G(f) \neq 0$ in Mod-A_σ so that there are an open right ideal I in (A, σ) and a morphism g with domain A/I such that $G(f)g \neq 0$.

It remains to see that σ is coarser than the topology of uniform convergence. Let I be an open ideal in (A, σ): then $I \leqslant \mathfrak{I}(\overline{IP})$ and we have

$$\mathrm{Chom}_R(P, P/\overline{IP}) \cong G(A/I \hat{\otimes}_A P) \cong GF(A/I) \cong A/I$$

and a monomorphism of $A/\mathfrak{J}(\overline{IP})$ into $\mathrm{Chom}_R(P, P/\overline{IP})$. Therefore $I = \mathfrak{J}(IP)$ is open in the topology of uniform convergence. $\square$

Part II.

In this part of Section 4 we fix a linearly topologized ring (R, τ). First we need a definition and a lemma.

DEFINITION. Let $P_R \in \mathrm{LTC}\text{-}R_\tau$ and $M \in \mathrm{Mod}\text{-}R$: we say that P_R is (*topologically*) *M-projective* if for all submodules L of M_R and all continuous morphisms $f\colon P \to M/L$ there exists a continuous morphism $g\colon P \to M$ such that the following diagram is commutative (M and M/L are endowed with the discrete topology; the row is the projection)

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "f"] & \\ M \arrow[r] & M/L \arrow[r] & 0 \end{tikzcd}$$

We denote by $\mathcal{T}(P_R)$ the class of those R-modules M such that P is M-projective.

4.7 LEMMA. *$\mathcal{T}(P_R)$ is closed under*

(i) *homomorphic images*;

(ii) *submodules*;

(iii) *finite direct sums.*

If P_R is t.f.g. then (P_R) is also closed under infinite direct sums.

The proof is the same as the proof of Proposition 16.12 in [1].

4.8 THEOREM. *If P_R is a τ-progenerator, $A = \mathrm{Chom}_R(P, P)$ and σ is the topology of uniform convergence on A, then the pair*

$$(\mathrm{Chom}_R(P, -), -\hat{\otimes}_A P)$$

is an equivalence between $\mathrm{Mod}\text{-}R_\tau$ and $\mathrm{Mod}\text{-}A_\sigma$.

We divide the proof into several steps.

(4.8.1) DEFINITION. $\mathfrak{C}_P$ is the class consisting of those modules N in $\mathrm{Mod}\text{-}A_\sigma$ such that $N \hat{\otimes}_A P$ is discrete.

If $N \in \text{Mod-}A_\sigma$, then we denote by $K(N)$ the kernel of the topology of $N \otimes_A P$: it is obvious that $N \in \mathcal{C}_P$ iff $K(N)$ is open in $N \otimes_A P$.

(4.8.2) *$\mathcal{C}_P$ is closed with respect to direct sums.*

Let $(N_\lambda)_{\lambda \in \Lambda}$ be a family in $\mathcal{C}_P$ and $N = \bigoplus\limits_\lambda N_\lambda$: then $N \otimes_A P \cong \bigoplus\limits_\lambda (N_\lambda \otimes_A P)$ algebraically, so we can identify them.

Let $x = (x_\lambda)_{\lambda \in \Lambda} \in \bigoplus N_\lambda$: if we take for all $\lambda \in \Lambda$ an open submodule U of $N \otimes_A P$, then

$$x \otimes^{-1} \Big(\sum_\lambda U_\lambda\Big) = \{p \in P\colon x_\lambda \otimes p \in U_\lambda, \forall \lambda\} = \bigcap_{\lambda \in \Lambda} x_\lambda \otimes^{-1}(U_\lambda)$$

which is open in P, because $\mathcal{F}(P)$ is closed under finite intersections and $x_\lambda \otimes^{-1}(U_\lambda) = P$ for almost all λ. This shows that any submodule of $N \otimes_A P$ of the form $\sum\limits_\lambda U_\lambda$ (U_λ open in $N_\lambda \otimes_A P$) is open. Let now V be an open submodule of $N \otimes_A P$: put $V_\lambda = V \cap (N_\lambda \otimes_A P)$ and take $n_\lambda \in N_\lambda$. We have

$$n_\lambda \otimes^{-1} (V_\lambda) = \{p \in P\colon n_\lambda \otimes p \in V_\lambda\} = \{p \in P\colon n_\lambda \otimes p \in V\} = n_\lambda \otimes^{-1}(V)$$

(n_λ is considered first in N_λ and then in N). Hence V contains a submodule of the form $\sum\limits_\lambda U_\lambda$, with each U_λ open in $N_\lambda \otimes_A P$. Therefore $K(N) \geqslant \sum\limits_\lambda K(N_\lambda)$ and N is in $\mathcal{C}_P$.

(4.8.3) *$\mathcal{C}_P$ is closed with respect to homomorphic images.*

Let $N \in \mathcal{C}_P$ and L be a submodule of N_A. If $\pi\colon N \to N/L$ is the canonical projection and $f = \pi \otimes_A P$, one can easily verify that f is open and surjective and that the open submodules of $N/L \otimes_A P$ are precisely those of the form $f(U)$, with U open in $N \otimes_A P$ and $U \geqslant \operatorname{Ker} f$. Let $n \in N$: then $(n + L) \otimes^{-1}(K(N/L)) \geqslant n \otimes^{-1}(K(N))$.

(4.8.4) *$\mathcal{C}_P$ contains a family of generators of* Mod-A_σ.

It suffices to show that if $V \in \mathcal{F}(P)$ and $I = \mathfrak{I}(V)$, then $A/I \in \mathcal{C}_P$: The canonical isomorphism $A/I \otimes_A P \to P/IP$ is a homeomorphism and the Hausdorff space associated to P/IP is $P/\overline{IP}$ (see 4.5). Since P is a self-generator, $\overline{IP} = V$ and so $P/\overline{IP}$ is discrete.

(4.8.5) $\mathcal{C}_P = \text{Mod-}A_\sigma$.

Follows easily from the arguments above.

(4.8.6) *The functor* $-\hat{\otimes}_A P$: Mod-$A_\sigma \to$ Mod-R_τ *is left adjoint to the functor* $\mathrm{Chom}_R(P, -)$: Mod-$R_\tau \to$ Mod-A_σ. *Consequently it is right exact and preserves colimits.*

This comes from 3.3 and 3.4.

(4.8.7) $\mathrm{Chom}_R(P, -)$ *commutes with direct sums in* Mod-R_τ.

Since P_R is t.f.g. the result is almost obvious.

Consider, for $N \in$ Mod-A_σ and $M \in$ Mod-R_τ, the morphisms of 3.5

$$\alpha_N\colon N \to \mathrm{Chom}_R^u(P, N \hat{\otimes}_A P)$$

$$\beta_M\colon \mathrm{Chom}_R^u(P, M) \hat{\otimes}_A P \to M$$

(4.8.8) *If* $(M_\lambda)_{\lambda\in\Lambda}$ *is a family in* Mod-R_τ *such that* β_{M_λ} *is an isomorphism for all* λ *and* $M = \bigoplus_\lambda M_\lambda$, *then* β_M *is an isomorphism too.*

It suffices to consider the chain of isomorphisms

$$\mathrm{Chom}_R(P, M) \hat{\otimes}_A P \cong \big(\bigoplus_\lambda \mathrm{Chom}_R(P, M_\lambda)\big) \hat{\otimes}_A P \cong$$
$$\cong \bigoplus_\lambda \big(\mathrm{Chom}_R(P, M \hat{\otimes})_A P\big) \cong M$$

(where the last morphism is $\bigoplus_\lambda \beta_{M_\lambda}$); the resulting morphism is easily seen to be β_M.

(4.8.9) *If* $V \in \mathcal{F}(P)$ *then* $\beta_{P/V}$ *is an isomorphism. Therefore, if* Γ_R *denotes the direct sum of all* P/V *as* V *runs through* $\mathcal{F}(P)$, *then* Γ_R *is a generator of* Mod-R_τ *such that, if* X *is any set and* $M = \Gamma^{(X)}$, *then* β_M *is an isomorphism.*

From the fact that P_R is quasi-projective it follows that $\mathrm{Chom}_R(P, P/V) \cong A/\mathfrak{J}(V)$. Hence the first statement follows from the proof of 4.8.4, while the second one is an application of 4.8.8 and the fact that P is a τ-generator.

(4.8.10) β_M *is an isomorphism for all* $M \in$ Mod-R_τ.

Let $M \in$ Mod-R_τ: there exists a Γ-resolution of M, i.e. an exact sequence of the form

$$\Gamma^{(X)} \to \Gamma^{(Y)} \to M \to 0 .$$

From this (and applying 4.7) we get the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Chom}_R(P, \Gamma^{(X)}) \hat{\otimes}_A P \arrow[r] \arrow[d, "\beta_{\Gamma(X)}"] & \mathrm{Chom}_R(P, \Gamma^{(Y)}) \hat{\otimes}_A P \arrow[r] \arrow[d, "\beta_{\Gamma(Y)}"] & \mathrm{Chom}_R(P, M) \hat{\otimes}_A P \arrow[r] \arrow[d, "\beta_M"] & 0 \\
\Gamma^{(X)} \arrow[r] & \Gamma^{(Y)} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

and β_M is an isomorphism by the Five Lemma.

(4.8.11) *α_N is an isomorphism for all $N \in \text{Mod-}A_\sigma$.*
Let $V \in \mathcal{F}(P)$: from the isomorphisms

$$\mathrm{Chom}_R(P, A/\mathfrak{J}(V) \hat{\otimes}_A P) \cong \mathrm{Chom}_R(P, P/V) \cong A/\mathfrak{J}(V)$$

one deduces easily that $\alpha_{A/\mathfrak{J}(V)}$ is an isomorphism. Moreover, denoting by Δ_A the direct sum of all $A/\mathfrak{J}(V)$ as V runs through $\mathcal{F}(P)$, it is clear that Δ is a generator of Mod-A_σ, such that, if X is a set and $N = \Delta^{(X)}$, α_N is an isomorphism. By reasoning as in 4.8.10 we get the conclusion. □

We are now in the position to state our main

4.9 Theorem. *Let (A, σ) and (R, τ) be complete linearly topologized rings and $(F\colon \text{Mod-}A_\sigma \to \text{Mod-}R_\tau,\ G\colon \text{Mod-}R_\tau \to \text{Mod-}A_\sigma)$ a similarity. Then there exists a τ-progenerator P_R such that:*

(a) (A, σ) is topologically isomorphic to $\mathrm{Chom}_R(P, P)$ *endowed with the topology of uniform convergence;*

(b) $G \cong \mathrm{Chom}_R(P, -)$;

(c) $F \cong - \hat{\otimes}_A P$.

Conversely, let (R, τ) be *a linearly topologized ring, P_R a τ-progenerator and $(A, \sigma) =$* $\mathrm{Cend}^u(P_R)$*: then we have the similarity*

$$(- \hat{\otimes}_A P\colon \text{Mod-}A_\sigma \to \text{Mod-}R_\tau,\ \mathrm{Chom}_R(P, -)\colon \text{Mod-}R_\tau \to \text{Mod-}A_\sigma).$$

5. Quasiprogenerators.

5.1 Definition. Let $P \in \text{LTC-}R$: we denote by

(i) Gen (P_R) the full subcategory of Mod-R generated by the set

$\{P/V\colon V \in \mathcal{F}(P)\}$, i.e. the smallest full subcategory of Mod-R containing each P/V $(V \in \mathcal{F}(P))$ and closed under direct sums and homomorphic images;

(ii) $\overline{\mathrm{Gen}}\,(P_R)$ the smallest full subcategory of Mod-R containing $\mathrm{Gen}\,(P_R)$ and closed under submodules.

If $M \in$ LTC-R we denote by $\mathrm{Tr}_M\,(P)$ the submodule of M generated by $\mathrm{Im}\,f$ as $f \in \mathrm{Chom}_R\,(P, M)$.

The P-topology τ_P on R is the linear topology having as a local basis the set of right ideals J of R such that $R/J \in \overline{\mathrm{Gen}}\,(P_R)$.

5.2 Remarks. If P is discrete, then these notations coincide with those of Fuller in [3]. It is clear that P is a topological module over (R, τ_P).

5.3 Lemma. *If P_R is a quasiprogenerator then* $\mathrm{Gen}\,(P_R) = \overline{\mathrm{Gen}}\,(P_R)$.

Proof. The proof runs exactly as that of Lemma 2.2 in [3]. The only thing to show is that P generates all the submodules of P/V, for $V \in \mathcal{F}(P)$. Assume then that $V \in \mathcal{F}(P)$ and that $W \geqslant V$; if $w \in W$ then there exists a net (w_δ) in $\mathrm{Tr}_W\,(P)$ convergent to w (recall that P is a self-generator). But then there exists $\bar{\delta}$ such that $w - w_{\bar{\delta}} \in V$, so, that $w + V = w_{\bar{\delta}} + V \in \mathrm{Tr}_{P/V}(P)$. □

5.4 Theorem. *Let $P_R \in$ LTC-R and put* $(A, \sigma) = \mathrm{Cend}^u\,(P_R)$. *The following conditions are equivalent*:

(*a*) $\mathrm{Chom}_R(P, -)$ *is an equivalence between* $\overline{\mathrm{Gen}}\,(P_R)$ *and* Mod-A_σ;

(*b*) P_R *is a quasiprogenerator.*

If these conditions are fulfilled then the inverse equivalence is $- \hat{\otimes}_A P$ (cf. [3, Theorem 2.6]).

Proof. If $(\hat{R}, \hat{\tau}_P)$ denotes the Hausdorff competion of (R, τ_P) then it is clear by 5.3 that P_R is a quasiprogenerator if and only if it is a $\hat{\tau}_P$-progenerator. The second part of the theorem comes then directly from 4.9. □

In analogy with 3 we want to prove a density theorem for quasiprogerators. From now on P_R will be a quasiprogenerator and $(A, \sigma) = \mathrm{Cend}^u\,(P_R)$.

5.5 Lemma. *Let $n \in \boldsymbol{N}$ and give P^n the product topology. If U is an open submodule of P^n and we put $\mathfrak{J}(U) = \{f \in \mathrm{Chom}_R(P, P^n)\colon \mathrm{Im}\, f \leqslant U\}$, then $\overline{\mathfrak{J}(U)P} = U$; in other words P_R generates (topologically) all open submodules of P^n.*

Proof. It is clear that $\mathrm{Chom}_R^u(P, P^n) = A^n$ with the product topology and that $\mathfrak{J}(U)$ is open in A^n. Consider the canonical morphisms

$$\varphi\colon A^n/\mathfrak{J}(U) \to \mathrm{Chom}_R(P, P^n/U)$$

$$\psi\colon A^n/\mathfrak{J}(U) \otimes_A P \to P^n/\mathfrak{J}(U)P;$$

the first one is injective, while the second is a topological isomorphism (it is defined like the f in the proof of 4.5).

Since $A^n/\mathfrak{J}(U) \in \text{Mod-}A_\sigma$, the Hausdorff completion of $P^n/\mathfrak{J}(U)P$ is discrete, so that $\overline{\mathfrak{J}(U)P}$ is open (and contained in U). We have the canonical epimorphism $h\colon P^n/\overline{\mathfrak{J}(U)P} \to P^n/U$: if we apply $\mathrm{Chom}_R(P, -)$ and recall that $P^n/\overline{\mathfrak{J}(U)P} = A^n/\mathfrak{J}(U) \hat{\otimes}_A P$ we get an epimorphism $A^n/\mathfrak{J}(0) \to \mathrm{Chom}_R(P, P^n/U)$ which is easily seen to be φ. Thus h is an isomorphism and $\overline{\mathfrak{J}(U)P} = U$. $\square$

5.6 Consider the ring $B = \mathrm{Cend}({}_AP)$ of all continuous A-endomorphisms of P and on B the topology of pointwise convergence, which has as a local basis the family of sets of the form

$$W(x_1, \ldots, x_n;\, V) = \{\xi \in B\colon (x_i)\xi \in V,\, l \leqslant i \leqslant n\}$$

for $n \in \boldsymbol{N}$, $x_i \in P$ and $V \in \mathcal{F}(P)$. Let $\psi\colon R \to B$ be the canonical morphism: if we give R the P-topology τ_P, then ψ is continuous. Indeed if $x \in P$ and $V \in \mathcal{F}(P)$ we have that $\psi^{-1}(W(x;\, V)) = (V\!:\!x)$, and this is an open ideal of (R, τ_P). It is obvious then that $\psi(R)$ is dense in B if and only if P satisfies the following condition

(*D*) *for all $b \in B$, for each choice of $x_1, \ldots, x_n$ in P and for all $V \in \mathcal{F}(P)$, there exists $r \in R$ such that, for $i = 1, \ldots, n$, $x_i r \in x_i b + V$.*

We now see that if P_R is a quasiprogenerator then it satisfies condition (*D*).

Let U be an open submodule of P^n: by 5.5, $\overline{\mathfrak{J}(U)P} = U$. Hence

$$UB = \overline{(\mathfrak{J}(U)P)}B \leqslant \overline{(\mathfrak{J}(U)P)B} = \overline{\mathfrak{J}(U)(PB)} = \overline{\mathfrak{J}(U)P} = U\,.$$

Now, if $(x_1, \ldots, x_n) \in P^n$ and $U = V^n$ (for $V \in \mathcal{F}(P)$), we have

$$((x_1, \ldots, x_n) + V^n)B \leqslant (x_1, \ldots, x_n)R + V^n$$

and we are done.

5.7 Density theorem for quasiprogenerators. *Let $P_R \in$ LTC-R be a quasiprogenerator, $(A, \sigma) =$ Cend^u (P_R), $B =$ Cend $({}_AP)$ and give B the topology β of pointwise convergence. The canonical morphism*

$$\psi\colon (R, \tau_P) \to (B, \beta)$$

is continuous, open on its image and $\psi(R)$ is dense in (B, β).

Proof. We have only to prove that ψ is open on its image: let J be an open right ideal of R. Since $R/J \in \overline{\text{Gen}}\,(P_R) = \text{Gen}\,(P_R)$, there are an open submodule V of P_R and an epimorphism $f\colon (P/V)^n \to R/J$. If $f(x_1 + V, \ldots, x_n + V) = 1 + J$, it is clear that $J \supseteq \bigcap_{i=1}^{n} (V\colon x_i)$, so that $\psi(J)$ contains $W(x_1, \ldots, x_n; V) \cap \psi(R)$. $\square$

Assume that P_R is strictly linearly compact (i.e. topologically artinian and complete) and that (R, τ_P) is complete: then (R, τ_P) is strictly linearly compact too and so $\psi(R) = B$. Hence the topology induced on B by R is strictly linearly compact and therefore minimal, so that it coincides with the topology of pointwise convergence, which is Hausdorff. (For a detailed treatment of strictly linearly compact modules and rings see e.g. [4]).

5.8 Corollary. *Let P_R be a strictly linearly compact quasiprogenerator, τ_P the P-topology on R and $A =$ Cend (P_R). Then the Hausdorff completion of (R, τ_P) is $B =$ Chom$_A$ (P, P) with the topology of pointwise convergence.*

6. Examples.

The theory developed so far is very similar to that of 3. We shall now see that we have reached an effective generalization by giving some examples.

Example of a non discrete quasiprogenerator.

Let R be a ring, $(S_\gamma)_{\gamma\in\Gamma}$ a family of pairwise non-isomorphic simple R-modules and P_R the product of all S_γ, endowed with the product topology of the discrete topologies. Moreover let $G = \bigoplus_\gamma S_\gamma$, with the relative topology: G is dense in P. Finally set $(A, \sigma) = \mathrm{Chom}^u_R(P, P)$ and, for all $\gamma \in \Gamma$, $D_\gamma = \mathrm{End}(S_\gamma)$; we put on each D_γ the discrete topology.

6.1 THEOREM. *With the notations as above, P_R is a quasiprogenerator, which is not discrete if Γ is infinite.*

PROOF. (*a*) *(A, σ) is topologically isomorphic to $\prod_\gamma D_\gamma$.*

For all $\gamma \in \Gamma$, we have that $\mathrm{Hom}_R(G, S_\gamma) = \mathrm{Chom}_R(G, S_\gamma)$. Hence we have the algebraic isomorphisms

$$A = \mathrm{Chom}_R(P, P) \cong \prod_\gamma \mathrm{Chom}_R(P, S_\gamma) \cong \prod_\gamma \mathrm{Chom}_R(G, S_\gamma) \cong$$
$$\cong \prod_\gamma \mathrm{Hom}_R(G, S_\gamma) = \prod_\gamma \prod_\delta \mathrm{Hom}_R(S_\delta, S_\gamma) \cong \prod_\gamma D_\gamma;$$

if $d = (d_\gamma)_{\gamma\in\Gamma} \in \prod_\gamma D_\gamma$, its action on $x = (x_\gamma)_{\gamma\in\Gamma} \in P$ is given by $dx = (d_\gamma x_\gamma)_{\gamma\in\Gamma}$. A local basis of P is the family of sets

$$W(F) = \{(x_\gamma)_{\gamma\in\Gamma} \in P : x_\gamma = 0 \text{ if } \gamma \in F\}$$

as F runs through the finite subsets of Γ. Then, identifying A with $\prod_\gamma D_\gamma$, one has

$$\mathfrak{J}(W(F)) = \{(d_\gamma)_{\gamma\in\Gamma} \in \prod_\gamma D_\gamma : d_\gamma = 0 \text{ if } \gamma \in F\},$$

so that the topology of uniform convergence coincides with the product topology of the discrete topologies.

(*b*) *P_R is t.f.g.; any open submodule of P_R is of the form $W(F)$, for some finite subset F of Γ.*

Let $V \in \mathcal{F}(P)$: there exists $F \subseteq \Gamma$, finite, such that $V \geqslant W(F)$. Then we have an epimorphism $P/W(F) \to P/V$, which splits, since $P/W(F) \cong \prod_{\gamma\in F} S_\gamma$. The conclusion is now obvious.

(*c*) *P_R is topologically quasi-projective.*

Let F be a finite subset of Γ and $f\colon P \to P/W(F)$ be a continuous morphism. Then by (*b*) the kernel of f is of the form $W(F')$ for some $F' \subseteq \Gamma$, finite. Hence we can consider $f'\colon \prod_{\gamma\in F'} S_\gamma \to \prod_{\gamma\in F} S_\gamma$: extending this by zero, we obtain the desired $g\colon P \to P$.

(*d*) *P_R is a self-generator.*

Obvious from (*a*) and (*b*).

If the set Γ is not finite, then the topology on P is not discrete. □

Example of a non discrete τ-progenerator.

6.2 THEOREM. *Let (R, τ) be a complete linearly topologized ring and X a non empty set; consider the direct product R^X with the product topology. Then R^X is a τ-progenerator.*

PROOF. (*a*) *R^X is topologically finitely generated.*

A local basis for R^X is the set of submodules of the form $\prod_{x\in X} I_x$ where $I_x \in \mathcal{F}_\tau$ for all $x \in X$ and $I_x = R$ for all but a finite number of x in X. Take $V = \prod_{x\in X} I_x \leqslant R^X$ to be one of them: if $F = \{x \in X\colon I_x \neq R\}$ then $R^X/V = \bigoplus_{\gamma\in F} R/I_x$, which is finitely generated.

(*b*) *R^X is a τ-generator.*

Let $f\colon M \to N$ be a non zero morphism in Mod-R_τ. Fix $m \in M$ with $f(m) \neq 0$ and $y \in X$: we can define a continuous morphism $g\colon R^X \to M$ by

$$g\big((r_x)_{x\in X}\big) = mr_y$$

and $fg \neq 0$.

(*c*) *R^X is τ-projective.*

For $y \in X$, $e_y = (r_x)_{x\in X} \in R^X$, where $r_x = 0$ if $x \neq y$ and $r_y = 1$.

Suppose we are given an epimorphism $f\colon M \to N$ in Mod-R_τ and a continuous morphism $g\colon R^X \to N$. The kernel of g is open so that it contains a submodule of the form $V = \prod_{x\in X} I_x$ of the type mentioned above. Let $F = \{x \in X\colon I_x \neq R\}$: then F is finite and for $x \notin F$ we have $e_x \in V$ and $g(e_x) = 0$. If we put $n_x = g(e_x)$, these are almost all zero.

Take, for $x \in X$, $m_x \in M$ such that $f(m_x) = n_x$, with $m_x = 0$ if $x \notin F$. Now we get a continuous morphism $h\colon R^X \to M$ by putting

$$h\big((r_x)_{x\in X}\big) = \sum_{x\in X} m_x r_x$$

(where we sum only the finitely many non zero terms), since

$$\ker h \geqslant \prod_{x\in X} \operatorname{Ann}_R(m_x)$$

which is open in R^X. It is easy to see that fh and g coincide on the direct sum $R^{(X)}$, and this one is dense in R^X.

(*d*) *R^X is quasiprojective.*

Let V be an open submodule of R^X, $\pi\colon R^X \to R^X/V$ the projection and $f\colon R^X \to R^X/V$ a continuous morphism; for $x \in X$ take $m_x \in R^X$ such that $\pi(m_x) = f(e_x)$, with the condition that $m_x = 0$ whenever $f(e_x) = 0$: by reasoning as above we can see that the set $\{x \in X\colon m_x \neq 0\}$ is finite.

Define $g_x\colon R \to R^X$ by $g_x(r) = m_x r$: thus we have the codiagonal morphism $g\colon R^{(X)} \to R^X$, which is continuous if we endow $R^{(X)}$ with the relative topology of R^X, as it is easily proved by taking into account the fact that almost all the g_x's are zero. Hence there is a unique extension of g to a continuous endomorphism g of R^X. It is plain that g coincides with f on $R^{(X)}$ and so $\pi g = f$.

(*e*) *R^X is a self-generator.*

Let V be an open submodule of R^X and let $\prod_{x\in X} I_x$ be an open basic neighbourhood of zero of the type mentioned above; if $F = \{x \in X\colon I_x \neq R\}$ then F is finite. Let $X(F)$ denote the directed set of all finite subsets of X which contain F.

If Y is a finite subset of X, we may define a continuous endomorphism $f_Y\colon R^X \to R^X$ by

$$f_Y\big((r_x)_{x\in X}\big) = (s_x)_{x\in X}\,,$$

where $s_x = r_x$ if $x \in Y$ and $s_x = 0$ if $x \notin Y$. It is clear that for all $Y \in X(F)$ we have $f_Y(V) \leqslant V$, so that any element r of V is the limit in V of the net $\big(f_Y(r)\big)_{Y\in X(F)}$.

Now we are done, because it is clear that any element of V, having at most finitely many non zero components, belongs to the image of a continuous morphism $R^X \to V$. $\square$

7. Commutative rings.

DEFINITION. (*a*) Let R be a ring and τ, τ' right linear topologies on it: we say that τ and τ' are *equivalent* if any right ideal of R closed for one topology is closed also for the other.

(*b*) If (R, τ) and (A, σ) are linearly topologized rings, we say that τ and σ are *equivalent* if R and A are isomorphic and τ and σ become equivalent topologies when we identify R and A.

7.1 THEOREM. *Let (R, τ) and (A, σ) be linearly topologized rings and assume that they are commutative, complete, and similar: then*

(i) *R and A are (algebraically) isomorphic;*

(ii) *τ and σ are equivalent topologies.*

PROOF. Let P_R be the τ-progenerator which gives the similarity: there exists an injective ring morphism $\psi\colon R \to \operatorname{Cend}(P_R) = A$. We want to show that for any open ideal I of A, $\psi^{-1}(I)$ is closed in R; assume then that $I = \mathfrak{I}(V)$, with $V \in \mathcal{F}(P)$:

$$\psi^{-1}(\mathfrak{I}(V)) = \{r \in R\colon Pr \leqslant V\} = (V:P) = \bigcap_{p \in P} (V:p)$$

which is closed in R.

By the commutativity of A there is an injective ring morphism $\varphi\colon A \to \operatorname{Cend}(Q_A) = R$, where $Q = \operatorname{Chom}_R(P, (R, \tau))$; it is not difficult to convince oneself that $\psi\varphi = 1_A$ and $\varphi\psi = 1_R$. $\square$

7.2 COROLLARY. *Let (R, τ) and (A, σ) be as in* 7.1. *Assume also that τ and σ are minimal. Then (R, τ) and (A, σ) are topologically isomorphic. This is in particular true when they are strictly linearly compact rings.*

REFERENCES

[1] F. W. Anderson - K. R. Fuller, *Rings and Categories of Modules*, Springer, Berlin, Heidelberg, New York, 1974.
[2] L. Fuchs, *Infinite Abelian Groups*, Academic Press, New York, 1970.
[3] K. R. Fuller, *Density and Equivalence*, J. Algebra, **29** (1974), pp. 528-550.
[4] H. Leptin, *Linear Kompakten Moduln und Ringe*, Math. Z., **62** (1955), pp. 241-267.
[5] R. N. S. MacDonald, *Representable dualities between finitely closed subcategories of modules*, Can. J. Math., **31** (1979), pp. 465-475.
[6] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sec. A, **6** (1958), pp. 85-142.

Manoscritto pervenuto in redazione il 13 aprile 1987.