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On absolutely simple locally finite groups


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1. Introduction.

A well-known result of Kegel [5; pp. 172-173] (or [6; p. 115]) asserts that if $G$ is a countably infinite locally finite simple group then

\[(1.1) \ \text{there is an ascending chain} \ F_1 \subseteq \ldots \subseteq F_n \subseteq \ldots \ \text{of finite subgroups of} \ G \ \text{satisfying} \]
\[(a) \ \bigcup F_n = G, \ \text{and} \]
\[(b) \ \text{for each} \ n > 1 \ \text{there is a maximal normal subgroup} \ M_n \ \text{of} \ F_n \ \text{such that} \ F_{n-1} \cap M_n = 1.\]

The import of this result lies in the display of finite simple sections of unbounded orders in the finite subgroups of a countably infinite locally finite simple $G$. In general, the condition (1.1) does not imply simplicity [6; p. 116]. Indeed, there are countably infinite residually finite groups satisfying (1.1).

A minor adaptation of Kegel's arguments can be used to strengthen (1.1) to a condition equivalent to simplicity, and we will give such a condition in Theorems 1 and 2. In these same theorems we give a similar criteria for the absolute simplicity of $G$.

Recall that $G$ is absolutely simple if the only composition series of $G$ is the one consisting of 1 and $G$ only; equivalently, $G$ is absolutely

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simple if the only serial subgroups of $G$ are 1 and $G$ (see [11; I, p. 12, p. 16] or our § 2.1 for the relevant definitions). Obviously, every absolutely simple group is simple and, in general, the absolutely simple groups form a proper subclass of the class of simple groups ([1] or [11; II, 3.4]). However, it is not known whether or not every locally finite simple group is absolutely simple. Our Theorems 1 and 2 put the (possible) differences between these two (locally finite) classes in a «local» context.

In the sequel, we frequently encounter ascending chains $F_1 \subseteq \ldots \subseteq F_n \subseteq \ldots$ of finite subgroups of the countably infinite locally finite $G$ with $\bigcup F_n = G$. Such a chain is called an approximating sequence of $G$ (caution; this term is used in a different way in [6; p. 116]).

Part of Theorem 1 is stated in terms of subnormal subgroups. Recall that if $M \subseteq G$, the standard series of $M$ in $G$ is defined inductively by

$$M(0, G) = G \quad \text{and for } n < 1, \quad M(n, G) = M^{M(n-1), G}.$$ 

We also have occasion to use the subgroups

$$M(\omega, G) = \bigcap \{ M(n, G) : n \geq 0 \}.$$ 

**Theorem 1.** Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite $G$.

a) If $G$ is simple, there is a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and an approximating sequence $\{F_k\}$ of $G$ satisfying

i) $\ F_1 = D_{n_1}$ and for $k > 1$, $F_k = D_{n_{k-1}}(1, D_{n_k}) = D_{n_{k-1}}^{D_{n_k}}$ and

ii) for $k > 1$, if $V \not\subseteq F_k$ then there is an $x \in N_{F_{k-1}}(F_k)$ such that $V^x \cap F_{k-1} = 1$.

b) If $G$ is absolutely simple, there is a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and an approximating sequence $\{F_k\}$ of $G$ satisfying

i) $\ F_1 = D_{n_1}$, and for $k > 1$, $F_k = D_{n_{k-1}}(\omega, D_{n_k})$, and

ii) for $k > 1$, if $V \not\subseteq F_k$, then $V \cap F_{k-1} = 1$.

Obviously, the condition in (ii) of part (a) implies (1.1).

A type of converse is provided in
THEOREM 2. Suppose \( \{F_k\} \) is an approximating sequence of the countably infinite locally finite \( G \).

a) If \( \{F_k\} \) satisfies the property (ii) of Theorem 1(a), then \( G \) is simple.

b) If \( \{F_k\} \) satisfies the property (ii) of Theorem 1(b), then \( G \) is absolutely simple.

An interesting interplay between Theorems 1 and 2 is that the existence of a single approximating sequence \( \{F_k\} \) of \( G \) satisfying condition (ii) of Theorem 1(a) (or Theorem 1(b)) implies that a sequence with similar properties can be extracted from any approximating sequence (as in part a(i) of Theorem 1). The conditions a(ii) and b(ii) of Theorem 1 accentuate the possible differences between the countable « simple » and « absolutely simple » locally finite groups.

We also note that the condition (ii) of Theorem 1(b) is equivalent to

\[(1.2) \text{ for all } k \geq 1, \quad 1 \neq x \in F_k \text{ implies } x^{F_{k+1}} = F_{k+1}.\]

While the above results are stated for countable groups, they can be extended to higher cardinalities by employing « countable » local theorems for the classes of simple and absolutely simple groups. The available theorems are recorded in

\[(1.3) \text{ The infinite group } G \text{ is simple (absolutely simple) if and only if } G \text{ has a local system of countable simple (absolutely simple) subgroups (see } \{6; \text{ p. } 114\}, \{9; \text{ p. } 190\}, \{7; \text{ p. } 131\} \text{ for the simple case and } \{3; \text{ p. } 529\} \text{ for the absolutely simple case).}\]

It follows immediately that Theorems 1 and 2 can be formulated in terms of the countable subgroups of the locally finite \( G \).

Our final result gives a sufficient condition for absolute simplicity.

THEOREM 3. Let \( G \) be a countably infinite locally finite simple group and \( \{D_n\} \) an approximating sequence of \( G \). If there is a \( d \geq 0 \) such that every perfect subnormal subgroup of \( D_n \) has defect at most \( d \) in \( D_n \), then \( G \) is absolutely simple.

The definition of the defect of a subnormal subgroup is given later in §2.1; see also \([11; \text{ I, p. } 173]\).

It is not difficult to see that the perfect subnormal subgroups of a finite group \( L \) all have defect \( d \) or less if and only if the perfect subnormal subgroups of \( L/H \) have defect \( d \) or less, where \( H \) is the solvable
radical of \( L \). Since all known countably infinite locally finite simple groups have approximating sequences \( \{D_n\} \) where for \( n \geq 1 \), \( D_n/\zeta(D_n) \) is a direct product of non-Abelian simple groups (see [10; p. 385]) we have, as a consequence of Theorem 3 and the above remarks on extensions to higher cardinals,

\[(1.4) \text{ all known locally finite simple groups are absolutely simple.}\]

As a final remark, we point out, that with minor modifications, the groups \( \{F_n\} \) of Theorem 1 can always be chosen to be perfect. To see this, let \( \{D_n\} \) be an approximating sequence of the simple \( G \) and denote by \( D_n^\circ \) the intersection of the members of the derived series of \( D_n \). Since \( G \) cannot be locally solvable [11; p. 154] and \( \bigcup\{D_n^\circ : n > 1\} \) is a normal subgroup of \( G \), we must have \( \bigcup\{D_n^\circ : n > 1\} = G \). Thus, \( \{D_n^\circ\} \) is an approximating sequence of perfect subgroups of \( G \). If the groups \( \{F_n\} \) are chosen relative to \( \{D_n^\circ\} \) (rather than \( \{D_n\} \)), the \( F_n \)'s will also be perfect.

2. Proofs.

2.1. Remarks on serial and subnormal subgroups.

The standard series of \( M \) in \( G \) has been defined in § 1. It is frequently easier to work with the commutator form

\[ M(n, G) = M[G, nM] \]

where \([G, nM]\) is defined inductively by \([G, 0M] = G\) and for \( n > 1 \),

\[ [G, nM] = [[G, (n-1)M], M] \]

(c.f. [11; I, p. 173]). The subgroup \( M \) of \( G \) is subnormal in \( G \) written \( M \triangleleft\triangleleft G \) if and only if \( M = M(n, G) \) for some \( 0 < n < \omega \). Equivalently, \( M \triangleleft\triangleleft G \) if and only if \([G, nM] \subseteq M\) for some \( n > 0 \). If \( M \triangleleft\triangleleft G \), the minimal \( n \) for which \( M = M(n, G) \) is called the defect of \( M \) in \( G \). The symbol \( M \triangleleft\triangleleft_n G \) will mean that \( M \) is subnormal in \( G \) of defect \( n \) or less.

Several useful facts are given in

\[(2.1.1) \]

(a) If \( N \subseteq M \triangleleft\triangleleft_n G \), then \( N(n, G) \subseteq M \).

(b) If \( N \subseteq M \triangleleft\triangleleft G \), then \( N(\omega, G) \subseteq M \).

(c) If \( G \) is finite and \( M \subseteq G \), then \( M^{M(\omega, G)} = M(\omega, G) \).
Our use of the term normal series coincides with the normal systems of Kurosh [8; p. 171] and is essentially equivalent to the series of Robinson [11; I, pp. 9-10]. A subgroup \( H \) of \( G \) is a serial subgroup of \( G \) (written \( \text{ser} \ G \)) if there is a normal series \( C \) of \( G \) with \( H \in C \). We will need the following «local» characterization of serial subgroups [4; Theorem 2] (or in the locally finite case [2; Lemma 2]).

\[(2.1.2) \quad \text{If} \ H \subseteq G, \text{ then } H \text{ ser } G \text{ if and only if for every finitely generated } F \subseteq G, \text{ } F \subseteq H^p \text{ implies } F \subseteq H.\]

An essential lemma for our arguments is

**Lemma 1.** Let \( \{D_n\} \) be an approximating sequence of the countably infinite locally finite \( G \) and suppose that for each \( n > 1 \) we have a subgroup \( M_n \triangleleft D_n \) and that \( n > m \) implies \( M_n \subseteq M_n \). Then \( M = \bigcup \{M_n: n > 1\} \) ser \( G \). Further, if there is a \( d > 0 \) such that \( M_n \triangleleft \triangleleft d \ D_n \) for all \( n > 1 \) then \( M \triangleleft \triangleleft d G \).

**Proof.** For the first part, we use the criterion \((2.1.2)\). Let \( F \) be a finite subgroup of \( G \) and suppose that \( F \subseteq M^p \). Then there is an \( n \) such that \( F \subseteq D_n \) and \( F \subseteq M_n^p \). Since \( M_n \triangleleft \triangleleft \langle M_n, F \rangle \subseteq D_n \) we have \( F \subseteq M_n \subseteq M \) as desired.

Suppose now that \( M_n \triangleleft \triangleleft d \ D_n \) for all \( n \). Then, for \( n > 1 \), \( [D_n, dM_n] \subseteq M_n \) and so

\[ [G, dM] = \bigcup \{[D_n, dM_n]: n > 1\} \subseteq M. \]

Thus, \( M \triangleleft \triangleleft d G \) and this completes the proof.

2.2. For the proof of Theorem 1 we require the following lemma. The proof follows the lines of argument given in [6; pp.112-114].

**Lemma 2.** Let \( \{D_n\} \) be an approximating sequence of the countably infinite locally finite \( G \) and put \( D = D_1 \).

- **a)** If \( G \) is simple and \( d > 0 \) then
  
  i) \( \{D(d, D_n): n > 1\} \) is an approximating sequence of \( G \), and
  
  ii) there is a positive integer \( j \) such that for \( n > j \), \( Y \triangleleft \triangleleft d \ D_n \) implies \( Y \cap D \in \{1, D\} \).

- **b)** If \( G \) is absolutely simple, then
  
  i) \( \{D(\omega, D_n): n > 1\} \) is an approximating sequence of \( G \), and
ii) there is a positive integer } such that for } \geq j, Y \triangleleft \triangleleft D_n \text{ implies } Y \cap D \in \{1, D\}.

\textbf{Proof.} For the proof of (i) of part (a), note first that for } n \geq 1, D(d, D_n) \subseteq D(d, D_{n+1}). \text{ From Lemma 1 we have } V = \bigcup \{D(d, D_n): n \geq 1\} \triangleleft \triangleleft_4 G \text{ and the simplicity of } G \text{ forces } V = G.

Part (i) of (b) follows similarly; in this case we have } V = \bigcup \{D(\omega, D_n): n \geq 1\} \text{ ser } G \text{ (by Lemma 1) and since } G \text{ is absolutely simple, } V = G.

Proceeding to part (ii) of (a), suppose that there is no } j \text{ with the asserted property. There is then an approximating sequence } \{D_n\} \text{ of } D \text{ and subgroups } Y_n \triangleleft \triangleleft D_n \text{ such that } Y_n \cap D \notin \{1, D\}. \text{ Since } D \text{ is finite there is a subgroup } M \text{ of } D \text{ with } M \notin \{1, D\} \text{ and an approximating sequence } \{E_n\} \subseteq \{P_n\} \text{ such that for } n \geq 1 \text{ there are subgroups } X_n \triangleleft \triangleleft E_n \text{ with } X_n \cap D = M. \text{ Now for } n \geq 1, M(d, E_n) \subseteq X_n \text{ (by (2.1.1)(a)) and so } M = D \cap M(d, E_n). \text{ From part (i) we have } G = \bigcup \{M(d, E_n): n \geq 1\} \text{ and the contradiction now follows.}

The proof of b(ii) is identical with that of a(ii); in the same manner we arrive at an approximating sequence } \{E_n\} \subseteq \{D_n\} \text{ and subgroups } X_n \triangleleft \triangleleft E_n \text{ with } M \notin \{1, D\}. \text{ The fact that } \bigcup \{M(\omega, E_n): n \geq 1\} = G \text{ (part (i) of (b)) together with } M = D \cap M(\omega, E_n) \text{ for } n \geq 1 \text{ again yields the contradiction } D = M.

\textbf{2.3 Proof of Theorem 1.} Let } \{D_n\} \text{ be an approximating sequence of } G \text{ and suppose } G \text{ is simple. If } F, S \text{ are finite subgroups of } G \text{ there is, by Lemma 2(a) a positive integer } j = j(F, S) \text{ such that } \langle F, S \rangle \leq F(1, D_j) = \mu(F, S) \text{ and } Y \triangleleft \triangleleft D_j \text{ implies } Y \cap F \in \{1, F\}. \text{ If } V \triangleleft \triangleleft \mu(F, S) \text{ then } V \triangleleft \triangleleft_2 D_j \text{ and so } V \cap F \in \{1, F\}. \text{ Further, if } F \subseteq V \text{ and } L = \text{Core}_{D_j}(V) \text{ then } L \cap F \subseteq \{1, F\}. \text{ If } F \subseteq L, \text{ then } L = F^{D_j} = \mu(F, S) \text{ which contradicts the fact that } V \neq \mu(F, S). \text{ Thus, } L \cap F = 1 \text{ and since for every } x \in D, \text{ we have } V^x \cap F \subseteq \{1, F\}, \text{ there must be an } x \in D_j \text{ with } V^x \cap F = 1.

Now for the construction of the desired subsequence } \{F_k\}. \text{ Put } F_1 = D_1, F_2 = \mu(F_1, D_2) \text{ and } F_3 = \mu(F_2, D_3) \text{ where } j_3 = \max \{3, j(F_1, D_3)\}; \text{ for } k > 3, \text{ let } F_k = \mu(F_{k-1}, D_{j_k}) \text{ where } j_k = \max \{k, j(F_{k-1}, D_{j_{k-1}})\}. \text{ One checks easily that } \{F_k\} \text{ has the properties listed in Theorem 1(a).}

For the proof of (b), let } \{D_n\} \text{ be an approximating sequence of the absolutely simple } G \text{ and } F \text{ and } S \text{ finite subgroups of } G. \text{ Using Lemma 2(b) there is a positive integer } j = j(F, S) \text{ such that}
\( \langle F, \mathcal{S} \rangle \subseteq F(\omega, D) = \mu(F, \mathcal{S}) \) and \( Y \triangleleft \triangleleft D \) implies \( Y \cap F \in \{1, F\} \). If \( V \subseteq \mu(F, \mathcal{S}) \) and \( F \triangleright V \) we have \( F \subseteq V \triangleleft \mu(F, \mathcal{S}) \) which forces \( \mu(F, \mathcal{S}) = V \) (by (2.1.1)(b)). From this we conclude that \( F \cap V = 1 \).

The sequence \( \{F_n\} \) satisfying (i) and (ii) of Theorem 1(b) may now be constructed as follows:

\[
F_1 = D_1, \ldots, \quad F_n = \mu(F_{n-1}, D_n), \ldots.
\]

2.4 PROOF OF THEOREM 2. Suppose \( G \) has an approximating sequence \( \{F_k\} \) satisfying the property (ii) of Theorem 1(a) and let \( 1 \neq H < G \). Then for some \( k_0 \), \( k > k_0 \) implies \( H \cap F_k \neq 1 \). For any such \( k \), \( f_k \cap (F_{k+1} \cap H) = F_k \cap H = 1 \) for any \( x \in N(F_{k+1}) \). Thus, \( F_{k+1} \cap H = F_{k+1} \) and this forces \( H = G \). We have shown that \( G \) is simple and this concludes the proof of part (a).

For part (b), suppose the approximating sequence \( \{F_k\} \) satisfies the property (ii) of Theorem 1(b) and that \( G \) is simple. As above, there is a \( k_0 \) such that \( k > k_0 \) implies \( H \cap F_k \neq 1 \). Thus, if \( k > k_0 \), \( 1 \neq H \cap F_k \triangleleft F_k \) and \( (H \cap F_k) \cap (H \cap F_{k+1})^{F_{k+1}} = 1 \). This gives \( (H \cap F_{k+1})^{F_{k+1}} = F_{k+1} \) and we conclude that \( (H \cap F_{k+1}) = F_{k+1} \). It follows that \( H = G \) and that \( G \) is absolutely simple.

2.5. Prior to our proof of Theorem 3, we need

(2.5.1) If \( H \) is a serial locally solvable subgroup of a locally finite \( G \), then \( H^o \) is also locally solvable.

The proof of (2.5.1) is straightforward and will not be given here. For the proof of Theorem 3, let \( G \) be a countable simple locally finite group and \( \{D_n\} \) an approximating sequence such that for each \( n \), the perfect subnormal subgroups of \( D_n \) are of defect \( d \) or less. Now let \( H \triangleright G \) with \( 1 \neq H \); from (2.5.1) and the fact that simple locally solvable groups are finite \([11; I, p. 154]\), we see that \( H \) is not locally solvable. Thus, there is an \( n_0 \) such that for \( n > n_0 \), \( H \cap D_n \) is not solvable. Consequently, if \( n > n_0 \), the subgroup \( (H \cap D_n)^{G} \), the intersection of the terms of the derived series of \( H \cap D_n \), is a non-trivial perfect subnormal subgroup of \( D_n \). From our assumptions, we have \( (H \cap D_n)^{G} \triangleleft \leq D_n \). Lemma 1 now implies that \( V = \bigcup \{(H \cap D_n)^{G}: n > 1\} \triangleleft \leq G \) and the simplicity of \( G \) forces \( V = G \). Since \( V \leq H \), we have \( H = G \) also, and \( G \) is absolutely simple.
REFERENCES


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