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On Absolutely Simple Locally Finite Groups.

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1. Introduction.

A well-known result of Kegel [5; pp. 172-173] (or [6; p. 115]) asserts that if G is a countably infinite locally finite simple group then

(1.1) there is an ascending chain $F_1 \subseteq \ldots \subseteq F_n \subseteq \ldots$ of finite subgroups of G satisfying

(a) $\bigcup F_n = G$, and

(b) for each $n > 1$ there is a maximal normal subgroup M_n of F_n such that $F_{n-1} \cap M_n = 1$.

The import of this result lies in the display of finite simple sections of unbounded orders in the finite subgroups of a countably infinite locally finite simple G. In general, the condition (1.1) does not imply simplicity [6; p. 116]. Indeed, there are countably infinite residually finite groups satisfying (1.1).

A minor adaptation of Kegel's arguments can be used to strengthen (1.1) to a condition equivalent to simplicity, and we will give such a condition in Theorems 1 and 2. In these same theorems we give a similar criteria for the absolute simplicity of G.

Recall that G is *absolutely simple* if the only composition series of G is the one consisting of 1 and G only; equivalently, G is absolutely

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simple if the only serial subgroups of G are 1 and G (see [11; I, p. 12, p. 16] or our § 2.1 for the relevant definitions). Obviously, every absolutely simple group is simple and, in general, the absolutely simple groups form a proper subclass of the class of simple groups ([1] or [11; II, 3.4]). However, it is not known whether or not every *locally finite* simple group is absolutely simple. Our Theorems 1 and 2 put the (possible) differences between these two (locally finite) classes in a « local » context.

In the sequel, we frequently encounter ascending chains $F_1 \subseteq \ldots \subseteq F_n \subseteq \ldots$ of finite subgroups of the countably infinite locally finite G with $\bigcup F_n = G$. Such a chain is called an *approximating sequence* of G (caution; this term is used in a different way in [6; p. 116]).

Part of Theorem 1 is stated in terms of subnormal subgroups. Recall that if $M \subseteq G$, the *standard series* of M in G is defined inductively by

$$M(0, G) = G \quad \text{and for } n \leqslant 1, \quad M(n, G) = M^{M(n-1), G)}.$$

We also have occasion to use the subgroups

$$M(\omega, G) = \cap \{M(n, G) \colon n \geqslant 0\}.$$

THEOREM 1. Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite G.

a) If G is simple, there is a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and an approximating sequence $\{F_k\}$ of G satisfying

   i) $F_1 = D_{n_1}$ and for $k > 1$, $F_k = D_{n_{k-1}}(1, D_{n_k}) = D_{n_{k-1}}^{D_{n_k}}$ and

   ii) for $k > 1$, if $V \trianglelefteq_{\neq} F_k$ then there is an $x \in N_{F_{k-1}}(F_k)$ such that $V^x \cap F_{k-1} = 1$.

b) If G is absolutely simple, there is a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and an approximating sequence $\{F_k\}$ of G satisfying

   i) $F_1 = D_{n_1}$, and for $k > 1$, $F_k = D_{n_{k-1}}(\omega, D_{n_k})$, and

   ii) for $k > 1$, if $V \trianglelefteq_{\neq} F_k$, then $V \cap F_{k-1} = 1$.

Obviously, the condition in (ii) of part (*a*) implies (1.1).

A type of converse is provided in

THEOREM 2. Suppose $\{F_k\}$ is an approximating sequence of the countably infinite locally finite G.

a) If $\{F_k\}$ satisfies the property (ii) of Theorem 1(*a*), then G is simple.

b) If $\{F_k\}$ satisfies the property (ii) of Theorem 1(*b*), then G is absolutely simple.

An interesting interplay between Theorems 1 and 2 is that the existence of a single approximating sequence $\{F_k\}$ of G satisfying condition (ii) of Theorem 1(*a*) (or Theorem 1(*b*)) implies that a sequence with similar properties can be extracted from any approximating sequence (as in part *a*(i) of Theorem 1). The conditions *a*(ii) and *b*(ii) of Theorem 1 accentuate the possible differences between the countable « simple » and « absolutely simple » locally finite groups.

We also note that the condition (ii) of Theorem 1(*b*) is equivalent to

(1.2) for all $k\geqslant 1$, $1\neq x\in F_k$ implies $x^{F_{k+1}}=F_{k+1}$.

While the above results are stated for countable groups, they can be extended to higher cardinal ties by employing « countable » local theorems for the classes of simple and absolutely simple groups. The available theorems are recorded in

(1.3) The infinite group G is simple (absolutely simple) if and only if G has a local system of countable simple (absolutely simple) subgroups (see [6; p. 114], [9; p. 190], [7; p. 131] for the simple case and [3; p. 529] for the absolutely simple case).

It follows immediately that Theorems 1 and 2 can be formulated in terms of the countable subgroups of the locally finite G.

Our final result gives a sufficient condition for absolute simplicity.

THEOREM 3. Let G be a countably infinite locally finite simple group and $\{D_n\}$ an approximating sequence of G. If there is a $d\geqslant 0$ such that every perfect subnormal subgroup of D_n has defect at most d in D_n, then G is absolutely simple.

The definition of the defect of a subnormal subgroup is given later in § 2.1; see also [11; I, p. 173].

It is not difficult to see that the perfect subnormal subgroups of a finite group L all have defect d or less if and only if the perfect subnormal subgroups of L/H have defect d or less, where H is the solvable radical of L. Since all known countably infinite locally finite simple groups have approximating sequences $\{D_n\}$ where for $n \geqslant 1$, $D_n/\zeta(D_n)$ is a direct product of non-Abelian simple groups (see [10; p. 385]) we have, as a consequence of Theorem 3 and the above remarks on extensions to higher cardinals,

(1.4) all known locally finite simple groups are absolutely simple.

As a final remark, we point out, that with minor modifications, the groups $\{F_k\}$ of Theorem 1 can always be chosen to be perfect. To see this, let $\{D_n\}$ be an approximating sequence of the simple G and denote by D_n^ω the intersection of the members of the derived series of D_n. Since G cannot be locally solvable [11; p. 154] and $\bigcup \{D_n^\omega : n \geqslant 1\}$ is a normal subgroup of G, we must have $\bigcup \{D_n^\omega : n \geqslant 1\} = G$. Thus, $\{D_n^\omega\}$ is an approximating sequence of perfect subgroups of G. If the groups $\{F_k\}$ are chosen relative to $\{D_n^\omega\}$ (rather than $\{D_n\}$), the F_k's will also be perfect.

2. Proofs.

2.1. *Remarks on serial and subnormal subgroups.*

The standard series of M in G has been defined in § 1. It is frequently easier to work with the commutator form

$$M(n, G) = M[G, nM]$$

where $[G, nM]$ is defined inductively by $[G, 0M] = G$ and for $n \geqslant 1$, $[G, nM] = [[G, (n-1)M], M]$ (c.f. [11; I, p. 173]). The subgroup M of G is subnormal in G written $M \lhd\lhd G$ if and only if $M = M(n, G)$ for some $0 \leqslant n < \omega$. Equivalently, $M \lhd\lhd G$ if and only if $[G, nM] \subseteq M$ for some $n \geqslant 0$. If $M \lhd\lhd G$, the minimal n for which $M = M(n, G)$ is called the *defect* of M in G. The symbol $M \lhd\lhd_n G$ will mean that M is subnormal in G of defect n or less.

Several useful facts are given in

(2.1.1) (*a*) if $N \subseteq M \lhd\lhd_n G$, then $N(n, G) \subseteq M$.

(*b*) If $N \subseteq M \lhd\lhd G$, then $N(\omega, G) \subseteq M$.

(*c*) If G is finite and $M \subseteq G$, then $M^{M(\omega, G)} = M(\omega, G)$.

Our use of the term normal series coincides with the normal systems of Kurosh [8; p. 171] and is essentially equivalent to the series of Robinson [11; I, pp. 9-10]. A subgroup H of G is a serial subgroup of G (written H ser G) if there is a normal series $\mathcal{C}$ of G with $H \in \mathcal{C}$. We will need the following «local» characterization of serial subgroups [4; Theorem 2] (or in the locally finite case [2; Lemma 2]).

(2.1.2) If $H \subseteq G$, then H ser G if and only if for every finitely generated $F \subseteq G$, $F \subseteq H^F$ implies $F \subseteq H$.

An essential lemma for our arguments is

LEMMA 1. Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite G and suppose that for each $n \geqslant 1$ we have a subgroup $M_n \lhd\lhd D_n$ and that $n > m$ implies $M_m \subseteq M_n$. Then $M = \bigcup \{M_n : n \geqslant 1\}$ ser G. Further, if there is a $d \geqslant 0$ such that $M_n \lhd\lhd_d D_n$ for all $n \geqslant 1$ then $M \lhd\lhd_d G$.

PROOF. For the first part, we use the criterion (2.1.2). Let F be a finite subgroup of G and suppose that $F \subseteq M^F$. Then there is an n such that $F \subseteq D_n$ and $F \subseteq M_n^F$. Since $M_n \lhd\lhd \langle M_n, F\rangle \subseteq D_n$ we have $F \subseteq M_n \subseteq M$ as desired.

Suppose now that $M_n \lhd\lhd_d D_n$ for all n. Then, for $n \geqslant 1$, $[D_n, dM_n] \subseteq M_n$ and so

$$[G, dM] = \bigcup \{[D_n, dM_n] : n \geqslant 1\} \subseteq M .$$

Thus, $M \lhd\lhd_d G$ and this completes the proof.

2.2. For the proof of Theorem 1 we require the following lemma. The proof follows the lines of argument given in [6; pp. 112-114].

LEMMA 2. Let $\{D_n\}$ be an approximating sequence of the countably infinite locally finite G and put $D = D_1$.

a) If G is simple and $d \geqslant 0$ then

i) $\{D(d, D_n) : n \geqslant 1\}$ is an approximating sequence of G, and

ii) there is a positive integer j such that for $n \geqslant j$, $Y \lhd\lhd_d D_n$ implies $Y \cap D \in \{1, D\}$.

b) If G is absolutely simple, then

i) $\{D(\omega, D_n) : n \geqslant 1\}$ is an approximating sequence of G, and

ii) there is a positive integer j such that for $n \geqslant j$, $Y \lhd\lhd D_n$ implies $Y \cap D \in \{1, D\}$.

PROOF. For the proof of (i) of part (a), note first that for $n \geqslant 1$, $D(d, D_n) \subseteq D(d, D_{n+1})$. From Lemma 1 we have $V = \bigcup \{D(d, D_n): n \geqslant 1\} \lhd\lhd_d G$ and the simplicity of G forces $V = G$.

Part (i) of (b) follows similarly; in this case we have $V = \bigcup \{D(\omega, D_n): n \geqslant 1\}$ ser G (by Lemma 1) and since G is absolutely simple, $V = G$.

Proceeding to part (ii) of (a), suppose that there is no j with the asserted property. There is then an approximating sequence $\{P_n\} \subseteq \{D_n\}$ and subgroups $Y_n \lhd\lhd_d P_n$ such that $Y_n \cap D \notin \{1, D\}$. Since D is finite there is a subgroup M of D with $M \notin \{1, D\}$ and an approximating sequence $\{E_n\} \subseteq \{P_n\}$ such that for $n \geqslant 1$ there are subgroups $X_n \lhd\lhd_d E_n$ with $X_n \cap D = M$. Now for $n \geqslant 1$, $M(d, E_n) \subseteq X_n$ (by (2.1.1)(a)) and so $M = D \cap M(d, E_n)$. From part (i) we have $G = \bigcup \{M(d, E_n): n \geqslant 1\}$ and the contradiction $M = D \cap G$ now follows.

The proof of b(ii) is identical with that of a(ii); in the same manner we arrive at an approximating sequence $\{E_n\} \subseteq \{D_n\}$ and subgroups $X_n \lhd\lhd E_n$ with $M \notin \{1, D\}$. The fact that $\bigcup \{M(\omega, E_n): n \geqslant 1\} = G$ (part (i) of (b)) together with $M = D \cap M(\omega, E_n)$ for $n \geqslant 1$ again yields the contradiction $D = M$.

2.3 PROOF OF THEOREM 1. Let $\{D_n\}$ be an approximating sequence of G and suppose G is simple. If F, S are finite subgroups of G there is, by Lemma 2(a) a positive integer $j = j(F, S)$ such that $\langle F, S\rangle \subseteq F(1, D_j) = \mu(F, S)$ and $Y \lhd\lhd_2 D_j$ implies $Y \cap F \in \{1, F\}$. If $V \underset{\neq}{\lhd} \mu(F, S)$ then $V \lhd\lhd_2 D_j$ and so $V \cap F \in \{1, F\}$. Further, if $F \subseteq V$ and $L = \mathrm{Core}_{D_j}(V)$ then $L \cap F \in \{1, F\}$. If $F \subseteq L$, then $L = F^{D_j} = \mu(F, S)$ which contradicts the fact that $V \neq \mu(F, S)$. Thus, $L \cap F = 1$ and since for every $x \in D_j$ we have $V^x \cap F \in \{1, F\}$, there must be an $x \in D_j$ with $V^x \cap F = 1$.

Now for the construction of the desired subsequence $\{F_k\}$. Put $F_1 = D_1$, $F_2 = \mu(F_1, D_2)$ and $F_3 = \mu(F_2, D_{j_3})$ where $j_3 = \max\{3, j(F_1, D_2)\}$; for $k > 3$, let $F_k = \mu(F_{k-1}, D_{j_k})$ where $j_k = \max\big(k, j(F_{k-1}, D_{j_{k-1}})\}$. One checks easily that $\{F_k\}$ has the properties listed in Theorem 1(a).

For the proof of (b), let $\{D_n\}$ be an approximating sequence of the absolutely simple G and F and S finite subgroups of G. Using Lemma 2(b) there is a positive integer $j = j(F, S)$ such that

$\langle F, S\rangle \subseteq F(\omega, D_j) = \mu(F, S)$ and $Y \lhd\lhd D_j$ implies $Y \cap F \in \{1, F\}$. If $V \underset{\neq}{\lhd} \mu(F, S)$ and $F \subseteq V$ we have $F \subseteq V \lhd\lhd \mu(F, S)$ which forces $\mu(F, S) = V$ (by (2.1.1)(b)). From this we conclude that $F \cap V = 1$.

The sequence $\{F_k\}$ satisfying (i) and (ii) of Theorem 1(b) may now be constructed as follows:

$$F_1 = D_1, \dots, \qquad F_n = \mu(F_{n-1}, D_n), \dots .$$

2.4 PROOF OF THEOREM 2. Suppose G has an approximating sequence $\{F_k\}$ satisfying the property (ii) of Theorem 1(a) and let $1 \neq H \lhd G$. Then for some k_0, $k \geqslant k_0$ implies $H \cap F_k \neq 1$. For any such k, $f_k \cap (F_{k+1} \cap H)^x = F_k \cap H = 1$ for any $x \in N(F_{k+1})$. Thus, $F_{k+1} \cap H = F_{k+1}$ and this forces $H = G$. We have shown that G is simple and this concludes the proof of part (a).

For part (b), suppose the approximating sequence $\{F_k\}$ satisfies the property (ii) of Theorem 1(b) and that $1 \neq H$ ser G. As above, there is a k_0 such that $k \geqslant k_0$ implies $H \cap F_k \neq 1$. Thus, if $k \geqslant k_0$, $1 \neq H \cap F_k \lhd\lhd F_k$ and $(H \cap F_k) \cap (H \cap F_{k+1})^{F_{k+1}} \neq 1$. This gives $(H \cap F_{k+1})^{F_{k+1}} = F_{k+1}$ and we conclude that $(H \cap F_{k+1}) = F_{k+1}$. It follows that $H = G$ and that G is absolutely simple.

2.5. Prior to our proof of Theorem 3, we need

(2.5.1) If H is a serial locally solvable subgroup of a locally finite G, then H^G is also locally solvable.

The proof of (2.5.1) is straightforward and will not be given here. For the proof of Theorem 3, let G be a countable simple locally finite group and $\{D_n\}$ an approximating sequence such that for eacn n, the perfect subnormal subgroups of D_n are of defect d or less. Now let H ser G with $1 \neq H$; from (2.5.1) and the fact that simple locally solvable groups are finite [11; I, p. 154], we see that H is not locally solvable. Thus, there is an n_0 such that for $n \geqslant n_0$, $H \cap D_n$ is not solvable. Consequently, if $n \geqslant n_0$, the subgroup $(H \cap D_n)^{\omega}$, the intersection of the terms of the derived series of $H \cap D_n$, is a non-trivial perfect subnormal subgroup of D_n. From our assumptions, we have $(H \cap D_n)^{\omega} \lhd\lhd_d D_n$. Lemma 1 now implies that $V =$ $= \bigcup \{(H \cap D_n)^{\omega} : n \geqslant 1\} \lhd\lhd_d G$ and the simplicity of G forces $V = G$. Since $V \subseteq H$, we have $H = G$ also, and G is absolutely simple.

REFERENCES

[1] S. CHEHATA, *An algebraically simple ordered group*, Proc. London Math. Soc., (3) **2** (1952), pp. 183-197.
[2] B. HARTLEY, *Serial subgroups of locally finite groups*, Proc. Cambridge Philos. Soc., **71** (1972), pp. 199-201.
[3] K. HICKIN, *Countable type local theorems in algebra*, J. of Algebra, **27** (1973), pp. 523-537.
[4] K. HICKIN - R. E. PHILLIPS, *On classes of groups defined by systems of subgroups*, Arch. der Math., **24** (1973), pp. 346-350.
[5] O. KEGEL, *Über einfache, lokal endliche Gruppen*, Math. Z., **95** (1967), pp. 169-195.
[6] O. KEGEL - B. A. F. WEHRFRITZ, *Locally Finite Groups*, North Holland, London (1973).
[7] R. D. KOPPERMAN - A. R. D. MATHIAS, *Some problems in group theory*, « The Syntax and Semantics of Infinitary Languages », Lecture Notes in Math., **72**, Springer-Verlag, New York (1968).
[8] A. G. KUROSH, *The Theory of Groups*, Vol. II, Chelsea, New York (1956).
[9] R. E. PHILLIPS, *Countably recognizable classes of groups*, Rocky Mt. J. of Math., **1** (1971), pp. 489-497.
[10] R. E. PHILLIPS, *Existentially closed locally finite central extensions; multipliers and local systems*, Math. Z., **487** (1984), pp. 383-392.
[11] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups*, Vol. I and II, Springer-Verlag, New York (1972).

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