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Carter Subgroups and Injectors in a Class of Locally Finite Groups.

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1. Introduction.

The class of $\mathfrak{U}$ of locally finite groups was introduced in [4], where a theory of saturated formations was developed in an arbitrary subclass of $\mathfrak{U}$, closed under subgroups and homomorphic images. Many other results from the theory of finite soluble groups have since been extended to $\mathfrak{U}$, and our main aim here is to develop the basic theory of Fitting classes and their associated injectors.

The class $\mathfrak{U}$ was originally defined as the largest subgroup closed class of locally finite groups satisfying the conditions:

($U1$) *If $G \in \mathfrak{U}$ then G has a series $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ with locally nilpotent factors*,

($U2$) *If $G \in \mathfrak{U}$ and π is any set of primes, then the Sylow* (*that is maximal*) *π-subgroups of G are conjugate in G.*

It was shown in [7] that the first condition is redundant, as it is implied by the second. In fact a much stronger result was obtained Lemma 4.2. of [7] shows that

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LEMMA 1.1. *If $G \in \mathfrak{U}$, then G has a series of normal subgroups*

$$1 \leqslant N \leqslant A \leqslant B \leqslant G$$

with N locally nilpotent, A/N abelian of finite rank, B/A abelian with finite primary components, and G/B finite.

In particular, G/N is hyperfinite, in the sense that it has an ascending series of normal subgroups with finite factors.

We shall need to consider more general series. Let Ω be a totally ordered set. A *series of type Ω* of a group G is a set $(U_\sigma, V_\sigma \colon \sigma \in \Omega)$ of pairs of subgroups of G, indexed by Ω, such that

(i) $V_\sigma \lhd U_\sigma$ for all $\sigma \in \Omega$,

(ii) $U_\tau \leqslant V_\sigma$ if $\tau < \sigma$,

(iii) $G - 1 = \bigcup_{\sigma \in \Omega} (U_\sigma - V_\sigma)$.

Such a series is called a *normal series* of G if the subgroups U_σ, V_σ are all normal in G. If the index set Ω is well ordered, then Ω can be taken to be a set of ordinals and the above reduces to the usual concept of an *ascending series*, with $U_\sigma = V_{\sigma+1}$ and, for limit ordinals σ, $\bigcup_{\tau<\sigma} V_\tau = \bigcup_{\tau<\sigma} U_\tau = V_\sigma$.

A subgroup H of G is said to be *serial* in G (written H ser G) if H is a member of some series of G, and *ascendant* in G (written H asc G) if H is a member of some ascending series of G. If H is serial in a locally finite group G and $N \lhd G$, then HN ser G (see [6], and also [5, Corollary E1]). This makes it easy to see that a serial subgroup of a hyperfinite locally finite group is ascendant, and hence that if G is locally finite, $N \lhd G$, G/N is hyperfinite and H ser G, then HN asc G. This remark will be crucial in the proof of our main result on injectors.

For our results on injectors we work within an arbitrary but fixed subclass $\mathfrak{K}$ of $\mathfrak{U}$ satisfying

($K1$) *$\mathfrak{K}$ is closed under taking subgroups.*

($K2$) *If $G \in \mathfrak{K}$ and C_p is a cyclic group of prime order p, then $G \times C_p \in \mathfrak{K}$.*

(Classes of groups, as usual, are taken to be closed under isomorphisms and to contain the trivial groups.) It follows that $\mathfrak{K}$ contains all cyclic groups of prime order. From now on, $\mathfrak{K}$ denotes a class satisfying these conditions. Among the many possibilities for $\mathfrak{K}$ are the class of all finite soluble groups, the class of all periodic locally soluble groups having a locally nilpotent subgroup of finite index, the class of periodic soluble linear groups, the class of soluble Černikov groups, and $\mathfrak{U}$ itself. A Fitting class of $\mathfrak{K}$-groups (or a $\mathfrak{K}$-Fitting class) is a subclass $\mathfrak{X}$ of $\mathfrak{K}$ such that

($F1$) *Every serial subgroup of an $\mathfrak{X}$-group belongs to $\mathfrak{X}$.*

($F2$) *Every $\mathfrak{K}$-group, generated by serial $\mathfrak{X}$-subgroups, belongs to $\mathfrak{X}$.*

When $\mathfrak{K}$ is the class of all finite soluble groups this coincides with the usual definition as given in [3], and for the class of soluble Černikov groups it agrees with [1].

Examples of subgroup-closed Fitting classes are easily obtained from Fitting classes of finite soluble groups.

LEMMA 1.2. *Let $\mathfrak{X}$ be a subgroup-closed Fitting class of finite soluble groups. Then the class $L\mathfrak{X} \cap \mathfrak{K}$ of all locally-$\mathfrak{X}$ groups in $\mathfrak{K}$ is a Fitting class of $\mathfrak{K}$-groups.*

PROOF. The class $L\mathfrak{X} \cap \mathfrak{K}$ is clearly closed under taking serial subgroups. Let G be a $\mathfrak{K}$-group generated by serial $L\mathfrak{X} \cap \mathfrak{K}$-subgroups H_i $(i \in I)$. If F is a finite subgroup of G, then $F \leqslant \langle F_1, \ldots, F_n \rangle = L$, where F_r is a finite subgroup of H_{i_r} $(1 \leqslant r \leqslant n)$. Thus $F \leqslant L = \langle L \cap H_{i_1}, \ldots, L \cap H_{i_n} \rangle$. Each $L \cap H_{i_r}$ is a subnormal $\mathfrak{X}$-subgroup of L, whence $L \in \mathfrak{X}$, and $F \in \mathfrak{X}$.

In particular, we have the Fitting class $(L\mathfrak{N})^k \cap \mathfrak{K}$ of $\mathfrak{K}$-groups of locally nilpotent length at most k. For examples of Fitting classes that are not subgroup-closed (working relative to the class of soluble Černikov groups) see [1].

If $\mathfrak{X}$ is any Fitting class of $\mathfrak{K}$-groups, and $G \in \mathfrak{K}$, the join $G_{\mathfrak{X}}$ of all serial $\mathfrak{X}$-subgroups of G is a characteristic $\mathfrak{X}$-subgroup $G_{\mathfrak{X}}$, the $\mathfrak{X}$-*radical* of G. A routine argument gives

LEMMA 1.3. *If $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class, $G \in \mathfrak{K}$ and H* ser G, *then $H_{\mathfrak{X}} = H \cap G_{\mathfrak{X}}$.*

If $\mathfrak{X}$ is any class of groups, then an $\mathfrak{X}$-injector of the group G is an $\mathfrak{X}$-subgroup V of G such that $V \cap H$ is a maximal $\mathfrak{X}$-subgroup of H whenever H ser G. This agrees with the definition used for soluble Černikov groups in [1], and in particular is consistent with the finite case. Our main result is

THEOREM. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{K}$-groups. Then every $\mathfrak{K}$-group G has $\mathfrak{X}$-injectors, and any two $\mathfrak{X}$-injectors of G are conjugate.*

The proof is roughly similar to the finite case [3], but several technical difficulties have to be overcome. The proof in the finite case depends on the conjugacy of the self-normalizing nilpotent (or Carter) subgroups of a finite soluble group, and this result, a version of which is known for $\mathfrak{U}$-groups, must first be recast into the appropriate form.

2. Carter subgroups.

If $\mathfrak{X}$ is any class of groups, then an $\mathfrak{X}$-projector of a group G is an $\mathfrak{X}$-subgroup X of G such that $XK = H$ whenever $X \leqslant H \leqslant G$, $K \lhd H$ and $H/K \in \mathfrak{X}$. Though the Carter subgroups of a finite soluble group were originally defined as the self-normalizing nilpotent subgroups [2], they are of course now known to be the nilpotent projectors. In [4], the Carter subgroups of a $\mathfrak{U}$-group G were defined as its locally nilpotent projectors. They were shown to exist and form a conjugacy class, and it was shown [4, Lemma 5.8] that they are the self-normalizing locally nilpotent sugbroups of G, provided that the locally nilpotent subgroups of G are all hypercentral. However they do not have this description in general, since a locally nilpotent group may possess proper self-normalizing subgroups.

To remedy this, let us say that a subgroup H of G is *self-serializing* in G, if H is the only subgroup of G containing H as a serial subgroup. Then the Carter subgroups of a $\mathfrak{U}$-group G have the following characterization, which is important for us.

THEOREM 2.1. *The Carter subgroups of a $\mathfrak{U}$-group G are precisely its self-serializing locally nilpotent subgroups.*

PROOF. Let C be a Carter subgroup of G and suppose that $C \operatorname{ser} K \leqslant G$. If $C < K$, then we have subgroups $C \leqslant V < U \leqslant K$ with $V \lhd U$. By $(U1)$, we can choose W with $V < W \leqslant U$ and W/V locally nilpotent, contradicting the fact that C is a locally nilpotent projector of G. Thus $C = K$ and C is self-serializing.

Conversely, let C be a self-serializing locally nilpotent subgroup of G. We prove that C is a Carter subgroup of G by induction on the locally nilpotent length of G. If G is locally nilpotent, the result is clear, since every subgroup of a locally nilpotent group is serial. We need to consider separately the case $G \in (L\mathcal{N})^2$. In this case, let R be the locally nilpotent radical of G, so that G/R is locally nilpotent. If $\boldsymbol{R} = \{R_p\}$ and $\boldsymbol{C} = \{C_p\}$ are the unique Sylow bases of R and C respectively, then C is contained in the basis normalizer D of the Sylow basis $\{R_p C_p\}$ of RC. However, C is then serial in D, and so $C = D$. By Lemma 1.1, G/R is hyperfinite and so hypercentral, and so if $CR < G$, we have $CR < N = N_G(CR)$. Using the conjugacy of the basis normalizers of CR and the Frattini argument, we have $N = CRN_N(C)$. But C is certainly self-normalizing, whence $N = CR$. This contradiction shows that $CR = G$, and hence C is a Carter subgroup of G [4, Theorem 5.1].

Now let $G \in (L\mathcal{N})^k$, where $k \geqslant 3$, and again let R be the $L\mathcal{N}$-radical of G. Suppose that CR/R ser $H/R \leqslant G/R$. Then CR/R lies in the locally nilpotent radical K/R of H/R. Since C is a self-serializing locally nilpotent subgroup of the $(L\mathcal{N})^2$-group K, C is a Carter subgroup of K as we have seen. By the Frattini argument, $H = KN_H(C) = K$. Thus C is a Carter subgroup of H and, as H/R is locally nilpotent, $H = CR$. This shows that CR/R is a self-serializing locally nilpotent subgroup of G/R, and a Carter subgroup of G/R by induction. As C is also a Carter subgroup of CR, the « Gaschütz Lemma » [4, Lemma 5.3] shows that C is a Carter subgroup of G.

This characterization of Carter subgroups enables us to prove an appropriate form of the main lemma of [3].

LEMMA 2.2. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{K}$-groups. Let $G \in \mathfrak{K}$ and N be a normal subgroup of G such that G/N is locally nilpotent. If U, V are maximal $\mathfrak{X}$-subgroups of G such that $U \cap N = V \cap N$, then U and V are conjugate in G.*

PROOF. We may clearly assume that $G = \langle U, V \rangle$, so that $U \cap N = V \cap N \lhd G$. Let bars denote homomorphic images in $G/U \cap N$, let $\bar{M} = N_{\bar{G}}(\bar{U})$, and let $\boldsymbol{S} = \{\bar{S}_p\}$ be a Sylow basis of $\bar{M}$. Then $\{\bar{U} \cap \bar{S}_p\}$ is a Sylow basis of $\bar{U}$, and, for $q \neq p$, $[\bar{U} \cap \bar{S}_p, \bar{S}_q] \leqslant \bar{U} \cap \bar{N} = 1$. Hence $\bar{U}$ normalizes $\boldsymbol{S}$ and so $\bar{U}$ is contained in a Carter subgroup $\bar{C} = C/(U \cap N)$ of $\bar{M}$ [4, Theorem 5.9]. If C is serial in some subgroup H of G, then U is serial in H and so $U \leqslant H_{\mathfrak{X}}$. By the maximality of U, we have $U = H_{\mathfrak{X}} \lhd H$, and so $\bar{H} \leqslant \bar{M}$, and since $\bar{C}$ is self-serializing in $\bar{M}$, we have $\bar{C} = \bar{H}$. Thus $\bar{C}$ is a self-serializing locally nilpotent subgroup of $\bar{C}$, that is, by Theorem 2.1, a Carter subgroup of $\bar{G}$.

Similarly, we have a Carter subgroup $\bar{D} = D/U \cap N$ of $\bar{C}$ with

$V \leqslant D$. The subgroups C and D are conjugate, and so $\langle U, V^x \rangle \leqslant C$ for some $x \in G$. But U and V^x are serial $\mathfrak{X}$-subgroups of C, which belongs to $\mathfrak{K}$, and so $\langle U, V^x \rangle \in \mathfrak{X}$. Finally, the maximality of U and V^x gives $U = \langle U, V^x \rangle = V^x$.

COROLLARY 2.3. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{K}$-groups and $G \in \mathfrak{K}$. Let N, M be normal subgroups of G such that $N \leqslant M$ and G/N is locally nilpotent, and assume that each of M and N has a unique conjugacy class of $\mathfrak{X}$-injectors. Let U be an $\mathfrak{X}$-injector of N and let V be any maximal $\mathfrak{X}$-subgroup of G containing U. Then $V \cap M$ is an $\mathfrak{X}$-injector of M.*

PROOF. The hypotheses imply easily that U is contained in an $\mathfrak{X}$-injector W_0 of M. Now if we form any tower of $\mathfrak{X}$-subgroups of G containing W_0, its union is generated by serial $\mathfrak{X}$-subgroups and so belongs to $\mathfrak{X}$. Hence, by Zorn's Lemma, W_0 is contained in a maximal $\mathfrak{X}$-subgroup W of G. Now $V \cap N = U = W \cap N$ and so by Lemma 2.2, $V = W^x$ for some $x \in G$. Therefore $V \cap M = W^x \cap M = W_0^x$, as required.

COROLLARY 2.4. *If $\mathfrak{X}$ is a Fitting class of $\mathfrak{K}$-groups and $G \in \mathfrak{K}$, then any two $\mathfrak{X}$-injectors of G are conjugate in G.*

PROOF. This follows by using Lemma 2.2 and induction on the $L\mathfrak{N}$-length, exactly as in the finite case [3].

3. Injectors.

Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{K}$-groups. Then $C(\mathfrak{X})$, the characteristic of $\mathfrak{X}$, is defined to be the set of primes p such that $\mathfrak{X}$ contains a cyclic group of order p. Standard arguments show that if $C(\mathfrak{X}) = \pi$, then every $\mathfrak{X}$-group is a π-group, and $\mathfrak{X}$ contains every locally nilpotent π-group in $\mathfrak{K}$. Details of these arguments can be found in [1]. They are similar to the finite case, and it is for them that $(K2)$ is needed.

In the rest of the paper, $\mathfrak{X}$ denotes a Fitting class of $\mathfrak{K}$-groups, $\pi = C(\mathfrak{X})$ and $R = G_{L\mathfrak{N}}$. By the above remarks, $O_\pi(R) \leqslant G_{\mathfrak{X}}$, so $RG_{\mathfrak{X}}/G_{\mathfrak{X}}$ is a π'-group, and $G/RG_{\mathfrak{X}}$ is hyperfinite, by Lemma 1.1. By the remarks in the introduction, if H ser G, then $HRG_{\mathfrak{X}}$ asc G, and much of our proof of the main theorem will consist of an induction argument on an ascending series from $HRG_{\mathfrak{X}}$ to G. Limit ordinals are dealt with by the following, in which $\mathfrak{U}$-group properties are not involved.

LEMMA 3.1. *Let G be the union $G = \bigcup_{\lambda \in \Lambda} G_\lambda$ of a set of serial subgroups G_λ $(\lambda \in \Lambda)$. Then V is an $\mathfrak{X}$-injector of G if and only if $V \cap G_\lambda$ is an $\mathfrak{X}$-injector of G_λ, for each $\lambda \in \Lambda$.*

PROOF. If V is an $\mathfrak{X}$-injector of G, the definition shows that $V \cap G_\lambda$ is an $\mathfrak{X}$-injector of G_λ, for each $\lambda \in \Lambda$.

Conversely, suppose $V \cap G_\lambda$ is an $\mathfrak{X}$-injector of G_λ, for each $\lambda \in \Lambda$, and let H ser G. Then $H \cap G_\lambda$ ser G_λ, and so $V \cap H \cap G_\lambda$ is maximal among the $\mathfrak{X}$-subgroups of G_λ. If $V \cap H \leqslant W \leqslant H$ and $W \in \mathfrak{X}$, then $W \cap G_\lambda \in \mathfrak{X}$, whence we find $V \cap H \cap G_\lambda = W \cap G_\lambda$, and $W = \bigcup_{\lambda \in \Lambda} (V \cap H \cap G_\lambda) = V \cap H$. Hence $V \cap H$ is a maximal $\mathfrak{X}$-subgroup of H.

The following is useful in dealing with serial subgroups not containing $G_{\mathfrak{X}}$.

LEMMA 3.2. *If W is an $\mathfrak{X}$-subgroup of the serial subgroup H of G and $W \geqslant H_{\mathfrak{X}}$, then $WG_{\mathfrak{X}} \in \mathfrak{X}$.*

PROOF. Let $(U_\sigma, V_\sigma \colon \sigma \in \Omega)$ be a series from H to G. Since $G_{\mathfrak{X}} \cap H = H_{\mathfrak{X}} \leqslant W \leqslant H$, we have $[W, G_{\mathfrak{X}} \cap U_\sigma] \leqslant G_{\mathfrak{X}} \cap V_\sigma$, and so

$$WG_{\mathfrak{X}} \cap V_\sigma = W(G_{\mathfrak{X}} \cap V_\sigma) \lhd W(G_{\mathfrak{X}} \cap U_\sigma) = WG_{\mathfrak{X}} \cap U_\sigma .$$

Thus, intersecting with $WG_{\mathfrak{X}}$ gives a series from $H \cap WG_{\mathfrak{X}} = W(H \cap G_{\mathfrak{X}}) = WH_{\mathfrak{X}} = W$ to $WG_{\mathfrak{X}}$. Therefore W is a serial $\mathfrak{X}$-subgroup of $WG_{\mathfrak{X}}$ and hence $WG_{\mathfrak{X}} \in \mathfrak{X}$.

The main lemma is as follows.

LEMMA 3.3. *Let M be a normal subgroup of finite index of G containing $G_{\mathfrak{X}}$. If M has an $\mathfrak{X}$-injector, then G has an $\mathfrak{X}$-injector.*

PROOF. By induction on $|G/M|$, we may assume that M has prime index p. Taking account of Corollary 2.4, our hypothesis implies that every serial subgroup of M has a unique conjugacy class of $\mathfrak{X}$-injectors.

Let U be an $\mathfrak{X}$-injector of M and V be maximal among the $\mathfrak{X}$-subgroups of G containing U. Since U has index at most p in any $\mathfrak{X}$-subgroup of G containing it, the existence of V is clear. We shall show that V is an $\mathfrak{X}$-injector of G. If H ser G, then certainly $V \cap H \in \mathfrak{X}$; the problem is to show that $V \cap H$ is a maximal $\mathfrak{X}$-subgroup of H. Since $V \cap M = U$, an $\mathfrak{X}$-injector of M, we have that $V \cap H \cap M = U \cap H$ is an $\mathfrak{X}$-injector of $H \cap M$.

CASE (i) $HG_{\mathfrak{X}} = G$. Let W be an $\mathfrak{X}$-subgroup of H containing $V \cap H$. Now since $M \geqslant G_{\mathfrak{X}}$, we have $U \geqslant G_{\mathfrak{X}}$, and so $V \cap H \geqslant U \cap \cap H \geqslant G_{\mathfrak{X}} \cap H = H_{\mathfrak{X}}$, by Lemma 1.3. By Lemma 3.2, $WG_{\mathfrak{X}} \in \mathfrak{X}$. But $WG_{\mathfrak{X}} \geqslant (V \cap H)G_{\mathfrak{X}} = V$ and so, by the maximality of V, we deduce that $WG_{\mathfrak{X}} = V$. Hence $W \leqslant V$, as required.

CASE (ii) $HG_{\mathfrak{X}}R = G$. Recall that $\pi = C(\mathfrak{X})$ and $G_{\mathfrak{X}}R/G_{\mathfrak{X}}$ is a π'-group. Let S be a Sylow π-subgroup of G containing V. Then $S \cap HG_{\mathfrak{X}}$ is a Sylow π-subgroup of $HG_{\mathfrak{X}}$, since this subgroup is serial [5, Theorem E], and since $G/G_{\mathfrak{X}}$ is the product of $HG_{\mathfrak{X}}/G_{\mathfrak{X}}$ and the normal π'-subgroup $G_{\mathfrak{X}}R/G_{\mathfrak{X}}$, it follows that $S \cap HG_{\mathfrak{X}}/G_{\mathfrak{X}}$ is also a Sylow π-subgroup of $G/G_{\mathfrak{X}}$. This gives $V \leqslant S \leqslant HG_{\mathfrak{X}}$. Now we need only apply Case (i) to $HG_{\mathfrak{X}}$.

CASE (iii) $HG_{\mathfrak{X}}R < G$. By the remarks in the introduction, there is an ascending series

$$HG_{\mathfrak{X}}R = H_0 \lhd \ldots \lhd H_\alpha \lhd \ldots \lhd H_\varrho = G$$

and after refining if necessary, we may assume that each factor is finite abelian. Let λ be minimal such that $V \cap H_\lambda$ is a maximal $\mathfrak{X}$-subgroup of H_λ. If $\lambda = 0$, then $V \cap HG_{\mathfrak{X}}R$ is a maximal $\mathfrak{X}$-subgroup of $HG_{\mathfrak{X}}R$ containing the $\mathfrak{X}$-injector $V \cap M \cap HG_{\mathfrak{X}}R$ of $M \cap \cap HG_{\mathfrak{X}}R$. By Case (ii) applied to $HG_{\mathfrak{X}}R$, we obtain that $V \cap H$ is a maximal $\mathfrak{X}$-subgroup of H.

Thus we may assume that $\lambda > 0$, so that $V \cap H_\lambda$ is a maximal $\mathfrak{X}$-subgroup of H while $V \cap H_\alpha$ is not a maximal $\mathfrak{X}$-subgroup of H_α if $\alpha < \lambda$.

CASE (iii*a*) *$\lambda - 1$ exist.* Put $L = H_{\lambda-1} \cap M$, and note that H_λ/L is finite abelian. We have $V \cap L = U \cap H_{\lambda-1}$, which is a maximal $\mathfrak{X}$-subgroup of L. Let W_0 be a maximal $\mathfrak{X}$-subgroup of $H_{\lambda-1}$ containing $U \cap H_{\lambda-1}$, and W be a maximal $\mathfrak{X}$-subgroup of H_λ containing W_0. Then $W \cap L = U \cap H_{\lambda-1} = V \cap L$, and so by Lemma 2.2, W is conjugate to $V \cap H_\lambda$ in H_λ: Therefore $V \cap H_{\lambda-1} = W^x \cap H_{\lambda-1}$ for some $x \in H_\lambda$, contrary to the fact that $V \cap H_{\lambda-1}$ is not a maximal $\mathfrak{X}$-subgroup of $H_{\lambda-1}$.

CASE (iii*b*) *λ is a limit ordinal.* Now $U \cap H_0$ is certainly not a maximal $\mathfrak{X}$-subgroup of H_0, but it is a maximal $\mathfrak{X}$-subgroup of $H_0 \cap M$, so we have $U \cap H_0 < W_0$ for some maximal $\mathfrak{X}$-subgroup W_0 of H_0.

We now construct subgroups W_α $(\alpha \leqslant \lambda)$ such that $W_\alpha \leqslant W_\beta$ for $\alpha \leqslant \beta$ and W_α is a maximal $\mathfrak{X}$-subgroup of H_α. Having obtained W_α, we can obtain $W_{\alpha+1}$ since $H_{\alpha+1}/H_\alpha$ is finite, or as in Corollary 2.3. If β is a limit ordinal and the previous W_α have been obtained, we put $W_\beta = \bigcup_{\alpha<\beta} W_\alpha$, which is the join of ascendant $\mathfrak{X}$-subgroups and so belongs to $\mathfrak{X}$, and is clearly a maximal $\mathfrak{X}$-subgroup of H_β. Now we show by induction that $W_\alpha \cap M$ is an $\mathfrak{X}$-injector of $H_\alpha \cap M$, for each $\alpha \leqslant \lambda$. Since $W_0 \cap M = U \cap H_0 \cap M$, the case $\alpha = 0$ is clear. Lemma 3.1. deals with the passage to limit ordinals. If $W_\alpha \cap M$ is known to be an $\mathfrak{X}$-injector of $H_\alpha \cap M$, then Corollary 2.3 shows that $W_{\alpha+1} \cap M$ is an $\mathfrak{X}$-injector of $H_{\alpha+1} \cap M$. Finally, we find that $W_\lambda \cap M$ is an $\mathfrak{X}$-injector of $H_\lambda \cap M$. Therefore $W_\lambda \cap M = (U \cap H_\lambda)^x =$ $= (V \cap H_\lambda)^x \cap M$ for some $x \in H_\lambda \cap M$, since this group has conjugate $\mathfrak{X}$-injectors. By Lemma 2.2, W_λ and $(V \cap H_\lambda)^x$ are conjugate in H_λ. But W_λ contains W_0, so $W_\lambda \not\leqslant M$, while $V \cap H_\lambda = \bigcup_{\alpha<\lambda} (V \cap H_\alpha)$. Since $V \cap H_\alpha$ is not a maximal $\mathfrak{X}$-subgroup of H_α if $\alpha < \lambda$, and $|V \cap H_\alpha \colon U \cap H_\alpha|$ is either 1 or p, we have $V \cap H_\alpha = U \cap H_\alpha \leqslant M$. Therefore $V \cap H_\lambda \leqslant M$ and $(V \cap H_\lambda)^x \leqslant M$, a contradiction.

This completes the proof that V is an $\mathfrak{X}$-injector of G.

Proof of Main Theorem. The conjugacy of $\mathfrak{X}$-injectors is given in Corollary 2.4. For the existence, we first note that $G_{\mathfrak{X}}$ is the unique $\mathfrak{X}$-injector of $G_{\mathfrak{X}}R$. For Lemma 3.2 shows that if H ser $G_{\mathfrak{X}}R$, then $H \cap G_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of H. Now by the remarks in the introduction, we have an ascending series $(G_\alpha \colon \alpha \leqslant \varrho)$ of G with finite abelian factors and such that $G_0 = G_{\mathfrak{X}}R$. We show by induction on α that G_α (and hence all its serial subgroups) has an $\mathfrak{X}$-injector. For $\alpha = 0$ this has been remarked. The step from α to $\alpha + 1$ follows from Lemma 3.3. The limit ordinal step is made by forming a tower of injectors and using Lemma 3.1.

The following results can be deduced exactly as in the finite case [3].

Theorem 3.4 (i). *Let* $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ *be a series of* G *with locally nilpotent factors and* $V \leqslant G$. *Then* V *is an* $\mathfrak{X}$-*injector of* G *if and only if* $V \cap G_i$ *is a maximal* $\mathfrak{X}$-*subgroup of* G_i *for* $i = 0, 1, \ldots, n$.

(ii) *If* V *is an* $\mathfrak{X}$-*injector of* G *and* $V \leqslant L \leqslant G$, *then* V *is an* $\mathfrak{X}$-*injector of* L.

(iii) *The* $\mathfrak{X}$-*injectors of* G *are pronormal in* G.

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