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Proper Holomorphic Mappings
between Reinhardt Domains and Pseudoellipsoids.

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In recent years several results have been obtained on the conditions for the existence of proper holomorphic mappings between two domains \( D_1 \) and \( D_2 \) in \( \mathbb{C}^n \) and particularly for mappings of polynomial type. It is a conjecture, due to Bell [2], that if \( R_1 \) and \( R_2 \) are Reinhardt domains related by a proper holomorphic mapping then there is such a map which is polynomial. We recall that a Reinhardt domain (respect to \( O \)) in \( \mathbb{C}^n \) is an open connected set \( R \) such that if \( z \in R \) for any \( \theta \in \mathbb{R}^n \)

\[
T_\theta(z) = (\exp[i\theta_1]z_1, \ldots, \exp[i\theta_n]z_n) \in R.
\]

If such a condition holds only when \( \theta_1 = \theta_2 = \ldots = \theta_n \) then \( R \) is said to be a circular domain.

A Reinhardt domain \( R \) is complete if for any \( z^0 = (z_1^0, \ldots, z_n^0) \in R \) the closed polydisc \( A_{z^0} = \{z \in \mathbb{C}^n : |z_i^0| < |z_i|, i = 1, \ldots, n\} \) is contained in \( R \).

For any \( \alpha \in \mathbb{N}^n \) the pseudoellipsoid

\[
\Sigma_\alpha(z) = \left\{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i^{\alpha_i}| < 1\right\}
\]

is a complete bounded Reinhardt domain.

If \( a \in (\mathbb{C}^*)^n, T_a: \mathbb{C}^n \to \mathbb{C}^n \), defined as \( T_a(z_1, \ldots, z_n) = (a_1 z_1, \ldots, a_n z_n) \) is a linear automorphism of \( \mathbb{C}^n \) such that for any Reinhardt domain \( R \), \( T_a(R) \) is still a Reinhardt domain. So, we will say that two Reinhardt domains \( R_1 \) and \( R_2 \) in \( \mathbb{C}^n \) are \( T_a \)-equivalent, \( R_1 \cong R_2 \), if there exists \( a \in (\mathbb{C}^*)^n \) such that \( R_1 = T_a(R_2) \). It is clear that any \( T_a \) does not affect the polynomial feature of a map \( f: R_1 \to R_2 \).

In this note we prove that Bell conjecture holds when \( R_2 = \Sigma_n(x) \).

More exactly

**THEOREM 1.** Let \( R_1 \) be a Reinhardt domain in \( \mathbb{C}^n \) with \( 0 \in R_1 \). If there exists a proper holomorphic mapping \( T: R_1 \to R_2 \cong \Sigma_n(x) \) then there exists a proper polynomial holomorphic one.

In [4] the authors proved

**THEOREM 2.** Let \( R_1 \) be a Reinhardt domain in \( \mathbb{C}^n \) and

\[ f: R_1 \to R_2 \cong \Sigma_n(x), \]

a proper polynomial holomorphic mapping then \( R_1 \cong \Sigma_n(\beta) \) where \( \beta_i/\alpha_i \in \mathbb{N} \) for \( i = 1, \ldots, n \).

The previous theorems allow to characterize in the following corollary Reinhardt domains properly related to pseudoellipsoids.

**COROLLARY 3.** Let \( R_1 \) be a Reinhardt domain in \( \mathbb{C}^n \) with \( 0 \in R_1 \), \( R_1 \cong \Sigma_n(\beta) \) if and only if there exists a proper holomorphic mapping \( F: R_1 \to \Sigma_n(x) \) on a pseudoellipsoid \( \Sigma_n(x) \).

**PROOF OF THEOREM 1.** First consider \( R_2 = \mathbb{B}_n(0, 1) \), the unit ball in \( \mathbb{C}^n \). We require two key facts.

**FACT 1** (Alexander [1]). Let \( N \) be a neighborhood of \( p \in \partial \mathbb{B}_n \) and \( F \) a non-constant mapping holomorphic in \( N \cap \mathbb{B}_n \) and \( C^\infty \) in \( N \cap \partial \mathbb{B}_n \). If \( F(N \cap \partial \mathbb{B}_n) \subseteq \partial \mathbb{B}_n \) then \( F \) extends holomorphically to an automorphism of \( \mathbb{B}_n \).

**FACT 2** (Bell [2]). A proper holomorphic mapping \( F \) between bounded complete Reinhardt domains extends holomorphically past the boundary and if \( F^{-1}(0) = \{0\} \) then \( F \) is a polynomial mapping.

To apply Bell's results let us see that \( R_1 \) is complete and bounded. \( R_1 \) is complete: infact if \( z^p \in R_1 \) \( T \) will extend to a holomorphic map \( \bar{T}: \mathbb{A}_n \to \mathbb{C}^n \) (see for example [5] theorem 2.4.6). The existence of \( z \in \mathbb{A}_n \cap (\mathbb{C}^n - R_1) \) would contradict the maximum principle for the function \( \sum_{i=1}^n |\bar{T}_i(z)|^2 \), where \( \bar{T}_i \) are the components of \( \bar{T} \).
$R_1$ is bounded otherwise by Liouville theorem $T$ would not be proper.

For any given proper mapping $T: R_1 \to B_n$ and for any $g \in \text{Aut}(R_1)$ we claim that there exists $\Phi_g \in \text{Aut}(B_n)$ such that $T \circ g = \Phi_g \circ T$ on $R$.

As $T$ and $g$ extend holomorphically past the boundary, one can find a point $P \in \partial R_1$ and a neighborhood $U$ of $P$ in $\mathbb{C}^n$ such that

1. $J_T(z) \neq 0$ for $z \in U$,
2. $g$ is a biholomorphism on $U$,
3. $J_T(\xi) \neq 0$, $\xi \in g(U)$, where

$$J_T(z) = \det \left( \frac{\partial T_i(z)}{\partial z_j} \right), \quad j, i = 1, \ldots, n.$$

(By the way, one could show that $J_T$ can vanish only on coordinate hyperplanes.)

Furthermore, for any $z \in U \cap \partial R$, $g(z) \in \partial R$ and $T(z) \in bB_n$ and if $\zeta \in g(U) \cap \partial R$, $T(g(\zeta)) \in bB_n$.

Hence one can define a biholomorphism $\varphi = T \circ g \circ T^{-1}: T(U) \to T(g(U))$ such that $\varphi(T(U) \cap bB_n) \subseteq T(g(U)) \cap bB_n$.

By fact 1 such a map extends to $\Phi_g \in \text{Aut}(B_n)$ and $\Phi_g \circ T$ and $T \circ g$ agree on $U$, hence on $R_1$.

As $\text{Aut}(B_n)$ acts transitively one can find $\psi \in \text{Aut}(B_n)$ such that $\psi \circ T \equiv F: R_1 \to B_n$ is a proper map and $F(0) = 0$.

For any $\theta \in \mathbb{R}^n$ let $\Phi_\theta$ be the automorphism of $B_n$ such that $\Phi_\theta \circ F = F \circ T_\theta$.

$$\Phi_\theta(0) = \Phi_\theta(F(0)) = F(T_\theta(0)) = F(0) = 0.$$

This implies $F^{-1}(0) = \{0\}$ and by fact 2 $F$ is polynomial. In fact if there exists $0 \neq a \in F^{-1}(0)$, for any $\theta \in \mathbb{R}^n$, $F(T_\theta(a)) = \Phi_\theta(F(a)) = 0$ hence $F$ would not be proper.

In the general case $R_n \cong \Sigma_n(\alpha) \neq B_n(0, 1)$ consider

$$H_\alpha: \Sigma_n(\alpha) \to B_n(0, 1),$$

defined as $H_\alpha(w_1, \ldots, w_n) = (w_1^{\alpha}, \ldots, w_n^{\alpha})$.

$H_\alpha \circ T: R_1 \to B_n$ is a proper holomorphic map hence $R_1$ can be properly mapped on $B_n$ by a polynomial map and $R_1 \cong \Sigma_n(\beta)$ for suitable $\beta$ by theorem 2.
By results of Landucci [6] \( \gamma_i = \beta_i/\alpha_i \in \mathbb{N}, \ i = 1, \ldots, n \) and
\[
(z_1, \ldots, z_n) \rightarrow (z_1^{\gamma_1}, \ldots, z_n^{\gamma_n})
\]
s the required map from \( R_1 \) on \( R_2 \).

**Remark.** One can obtain the same conclusion of theorem 1 for circular domains \( D \) under suitable conditions (see [3]) that imply the extendibility of \( T: D \rightarrow R_2 \cong \Sigma_n(x) \) applying results analogous to fact 2, for circular domains, due to Bell [3].

The following example shows anyhow that there are circular domains \( D \) in \( \mathbb{C}^n \) such that there exists proper polynomial holomorphic mapping \( P: D \rightarrow B_n \) but which are not \( T_n \)-equivalent to pseudoellipsoids.

\[
(z_1 + z_2, 2z_1 - 2z_2, z_3^2): \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ maps } D = \{z \in \mathbb{C}^n: 5|z_1|^2 + 5|z_2|^2 - 6 \text{ Re } z_1 z_2 + |z_3|^4 < 1\}
\]
on the ball, but \( D \) is not a Reinhardt domain.

**References**


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