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Proper Holomorphic Mappings between Reinhards Domains and Pseudoellipsoids.

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In recent years several results have been obtained on the conditions for the existence of proper holomorphic mappings between two domains D_1 and D_2 in $\mathbb{C}^n$ and particularly for mappings of polynomial type. It is a conjecture, due to Bell [2], that if R_1 and R_2 are Reinhardt domains related by a proper holomorphic mapping then there is such a map which is polynomial. We recall that a Reinhardt domain (respect to O) in $\mathbb{C}^n$ is an open connected set R such that if $z \in R$ for any $\theta \in \mathbb{R}^n$

$$T_\theta(z) = \left(\exp[i\theta_1] z_1, \ldots, \exp[i\theta_n] z_n\right) \in R .$$

If such a condition holds only when $\theta_1 = \theta_2 = \ldots = \theta_n$ then R is said to be a circular domain.

A Reinhardt domain R is complete if for any $z^0 = (z_1^0, \ldots, z_n^0) \in R$ the closed polydisc $\Delta_{z^0} = \{z \in \mathbb{C}^n\colon |z_i| \leqslant |z_i^0| \;\; i = 1, \ldots, n\}$ is contained in R.

For any $\alpha \in \mathbb{N}^n$ the pseudoellipsoid

$$\Sigma_n(\alpha) = \left\{z \in \mathbb{C}^n\colon \sum_{i=1}^{n} |z_i|^{2\alpha_i} < 1\right\}$$

is a complete bounded Reinhardt domain.

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If $a \in (\mathbb{C}^*)^n$, $T_a\colon \mathbb{C}^n \to \mathbb{C}^n$, defined as $T_a(z_1, \ldots, z_n) = (a_1 z_1, \ldots, a_n z_n)$ is a linear automorphism of $\mathbb{C}^n$ such that for any Reinhardt domain R, $T_a(R)$ is still a Reinhardt domain. So, we will say that two Reinhardt domains R_1 and R_2 in $\mathbb{C}^n$ are *T_a-equivalent*, $R_1 \cong R_2$, if there exists $a \in (\mathbb{C}^*)^n$ such that $R_1 = T_a(R_2)$. It is clear that any T_a does not affect the polynomial feature of a map $f\colon R_1 \to R_2$.

In this note we prove that Bell conjecture holds when $R_2 = \Sigma_n(\alpha)$. More exactly

THEOREM 1. Let R_1 be a Reinhardt domain in $\mathbb{C}^n$ with $0 \in R_1$. If there exists a proper holomorphic mapping $T\colon R_1 \to R_2 \cong \Sigma_n(\alpha)$ then there exists a proper polynomial holomorphic one.

In [4] the autors proved

THEOREM 2. Let R_1 be a Reinhardt domain in $\mathbb{C}^n$ and

$$f\colon R_1 \to R_2 \cong \Sigma_n(\alpha)\,,$$

a proper polynomial holomorphic mapping then $R_1 \cong \Sigma_n(\beta)$ where $\beta_i/\alpha_i \in N$ for $i = 1, \ldots, n$.

The previous theorems allow to characterize in the following corollary Reinhardt domains properly related to pseudoellipsoids.

COROLLARY 3. Let R_1 be a Reinhardt domain in $\mathbb{C}^n$ with $0 \in R_1$, $R_1 \cong \Sigma_n(\beta)$ if and only if there exists a proper holomorphic mapping $F\colon R_1 \to \Sigma_n(\alpha)$ on a pseudoellipsoid $\Sigma_n(\alpha)$.

PROOF OF THEOREM 1. First consider $R_2 = \mathbb{B}_n(0,1)$, the unit ball in $\mathbb{C}^n$. We require two key facts.

FACT 1 (Alexander [1]). Let N be a neighborhood of $p \in b\mathbb{B}_n$ and F a non-constant mapping holomorphic in $N \cap \mathbb{B}_n$ and C^∞ in $N \cap \overline{\mathbb{B}}_n$. If $F(N \cap b\mathbb{B}_n) \subseteq b\mathbb{B}_n$ then F extends holomorphically to an automorphism of $\mathbb{B}_n$.

FACT 2 (Bell [2]). A proper holomorphic mapping F between bounded complete Reinhardt domains extends holomorphically past the boundary and if $F^{-1}(0) = \{0\}$ then F is a polynomial mapping.

To apply Bell's results let us see that R_1 is complete and bounded. R_1 is complete: infact if $z^0 \in R_1$ T will extend to a holomorphic map $\hat{T}\colon \bar{\Delta}_{z^0} \to \mathbb{C}^n$ (see for example [5] theorem 2.4.6). The existence of $z \in \bar{\Delta}_{z^0} \cap (\mathbb{C}^n - R_1)$ would contradict the maximum principle for the function $\sum_{i=1}^{n} |\hat{T}_i(z)|^2$, where $\hat{T}_i$ are the components of $\hat{T}$.

R_1 is bounded otherwise by Liouville theorem T would not be proper.

For any given proper mapping $T: R_1 \to \mathbf{B}_n$ and for any $g \in \mathrm{Aut}(R_1)$ we claim that there exists $\Phi_g \in \mathrm{Aut}(\mathbf{B}_n)$ such that $T \circ g = \Phi_g \circ T$ on R.

As T and g extend holomorphically past the boundary, one can find a point $P \in bR_1$ and a neighborhood U of P in $\mathbf{C}^n$ such that

i) $J_T(z) \neq 0 \;\; z \in U$,

ii) g is a biholomorphism on U,

iii) $J_T(\xi) \neq 0$, $\xi \in g(U)$, where

$$J_T(z) = \det\left(\partial T_i(z)/\partial z_j\right) \qquad j, i = 1, \ldots, n .$$

(By the way, one could show that J_T can vanish only on coordinate hyperplanes.)

Furthermore, for any $z \in U \cap bR$, $g(z) \in bR$ and $T(z) \in b\mathbf{B}_n$ and if $\zeta \in g(U) \cap bR$, $T(g(\zeta)) \in b\mathbf{B}_n$.

Hence one can define a biholomorphism $\varphi = T \circ g \circ T^{-1}: T(U) \to \to T(g(U))$ such that $\varphi(T(U) \cap b\mathbf{B}_n) \subseteq T(g(U)) \cap b\mathbf{B}_n$.

By fact 1 such a map extends to $\Phi_g \in \mathrm{Aut}(\mathbf{B}_n)$ and $\Phi_g \circ T$ and $T \circ g$ agree on U, hence on R_1.

As $\mathrm{Aut}(B_n)$ acts transitively one can find $\psi \in \mathrm{Aut}(\mathbf{B}_n)$ such that $\psi \circ T \equiv F: R_1 \to \mathbf{B}_n$ is a proper map and $F(0) = 0$.

For any $\theta \in \mathbf{R}^n$ let Φ_θ be the automorphism of $\mathbf{B}_n$ such that $\Phi_\theta \circ F = F \circ T_\theta$.

$$\Phi_\theta(0) = \Phi_\theta(F(0)) = F(T_\theta(0)) = F(0) = 0 .$$

This implies $F^{-1}(0) = \{0\}$ and by fact 2 F is polynomial. In fact if there exists $0 \neq a \in F^{-1}(0)$, for any $\theta \in \mathbf{R}^n$, $F(T_\theta(a)) = \Phi_\theta(F(a)) = 0$ hence F would not be proper.

In the general case $R_2 \simeq \Sigma_n(\alpha) \neq \mathbf{B}_n(0,1)$ consider

$$H_\alpha: \Sigma_n(\alpha) \to \mathbf{B}_n(0,1) ,$$

defined as $H_\alpha(w_1, \ldots, w_n) = (w_1^{\alpha_1}, \ldots, w_n^{\alpha_n})$.

$H_\alpha \circ T: R_1 \to \mathbf{B}_n$ is a proper holomorphic map hence R_1 can be properly mapped on $\mathbf{B}_n$ by a polynomial map and $R_1 \simeq \Sigma_n(\beta)$ for suitable β by theorem 2.

By results of Landucci [6] $\gamma_i = \beta_i/\alpha_i \in \mathbb{N}$, $i = 1, \dots, n$ and

$$(z_1, \dots, z_n) \to (z_1^{\gamma_1}, \dots, z_n^{\gamma_n})$$

s the required map from R_1 on R_2.

REMARK. One can obtain the same conclusion of theorem 1 for circular domains D under suitable conditions (see [3]) that imply the extendibility of $T: D \to R_2 \cong \Sigma_n(\alpha)$ applying results analogous to fact 2, for circular domains, due to Bell [3].

The following example shows anyhow that there are circular domains D in $\mathbb{C}^n$ such that there exists proper polynomial holomorphic mapping $P: D \to \mathbb{B}_n$ but which are not T_a-equivalent to pseudoelipsoids.

$$(z_1 + z_2, 2z_1 - 2z_2, z_3^2): \mathbb{C}^n \to \mathbb{C}^n \text{ maps } D =$$

$$= \{z \in \mathbb{C}^n: 5|z_1|^2 + 5|z_2|^2 - 6 \operatorname{Re} z_1 z_2 + |z_3|^4 < 1\}$$

on the ball, but D is not a Reinhardt domain.

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