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Hölder-continuity of solutions for some Schrödinger equations


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Hölder-Continuity of Solutions
for Some Schrödinger Equations.

GIUSEPPE DI FAZIO (*)

0. Introduction.

Recently the local regularity properties for solutions of Schrödinger equations of the form

\[
Lu = - (a_{ij} u_{x_i})_{x_j} + Vu
\]

have been studied by many authors (see e.g. [A-S], [D-M], [C-F-G], [C-F-Z]) allowing \( V \) to be a very singular potential, precisely \( V \in S \), the Stummel-Kato class (see definition 1.1).

Under this assumption in [C-F-G] was established a Harnack inequality and proved a local continuity result for solutions of (\( \ast \)). It is easy to see that if \( \Omega \) is an open bounded set in \( \mathbb{R}^n \) then \( L^p(\Omega) \subset S \) for \( p > n/2 \); hence the result in [C-F-G] generalizes the well known Hölder estimates by Stampacchia [ST], Ladizhenskaya [L-U] etc.

We stress that high integrability of \( V \) does not play an essential role.

In fact also the Morrey space \( L^{1,\lambda}(\Omega) \) is contained in \( S \) for \( \lambda > n-2 \) and being in \( L^{1,\lambda}(\Omega) \), for any \( 0 < \lambda < n \), does not imply any extra integrability (see e.g. the examples in [P2]).

In this paper we assume \( V \) in \( L^{1,\lambda}(\Omega) (\lambda > n-2) \) and prove local hölder-continuity for solutions of (\( \ast \)) hence, in this special situation, we improve the continuity result in [C-F-G].

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Our technique is very close to the one in [C-F-G] heavily relying on the exploitation of well known estimates for the Green function of $L$.

There is however a technical difficulty.

It is impossible to use the usual $C^\infty$-approximation for $L$ and $V$ (as in [C-F-G]) because functions in Morrey spaces are not close, in general, to bounded functions in $L^{1,\lambda}(\Omega)$ (see [P1] p. 22 for an example of an $L^{1,\lambda}(\Omega)$ function with distance from $L^\infty(\Omega)$ equal to 1). We overcame this difficulty by developing a representation formula for solutions of $(\ast)$ that extends classical results on the Green function (see e.g. [ST]).

1. Some function spaces.

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ ($n > 3$).

We will need some mild regularity assumption to be satisfies by $\partial \Omega$ e.g.

$$\exists A \in ]0, 1[: |\Omega_r(x)| \leq A |B_r(x)| \quad \forall x \in \partial \Omega$$

where $r: 0 < r < \text{diam} \,(\Omega)$ \(^{(1)}\).

**Definition 1.1 (Stummel-Kato class).** We say that $V: \Omega \to \mathbb{R}$ belongs to the Stummel-Kato class $S$ iff there exists a non decreasing function $\eta(r) > 0$ with $\lim_{r \to 0} \eta(r) = 0$ such that

$$\left(1.1\right) \quad \sup_{x \in \Omega} \int_{\Omega_r(x)} |V(y)||x - y|^{2-n} dy \leq \eta(r)$$

Obviously $S \subseteq L^{1,\lambda}(\Omega)$.

**Definition 1.2 (Morrey spaces).** $L^{1,\lambda}(\Omega)$ ($0 < \lambda < n$) is the space of functions $f \in L^1(\Omega)$ such that

$$\|f\|_{L^{1,\lambda}(\Omega)} = \sup_{x \in \Omega} \int_{\Omega_r(x)} |f(y)| dy < +\infty.$$ 

\(^{(1)}\) $|E|$ denotes the Lebesgue measure of a measurable subset $E$ of $\mathbb{R}^n$:

$$B_r(x) =: \{ y \in \mathbb{R}^n : |x - y| < r \} ; \quad \Omega_r(x) =: \Omega \cap B_r(x).$$
LEMMA 1.1. If \( u \) belongs to \( L^{1,\lambda}(\Omega) \) \((n - 2 < \lambda < n)\) then \( u \) belongs to the Stummel-Kato class and

\[
\int_{\Omega} |u(y)| |x - y|^{2-n} \, dy \leq C r^{1-n+\lambda} \|u\|_{L^{1,\lambda}(\Omega)}
\]

where \( C \) depends only on \( \lambda \) and \( n \).

Indeed,

\[
\int_{\Omega} |u(y)| |x - y|^{2-n} \, dy = \sum_{k=0}^{+\infty} \int_{\Omega \cap \{ r/2^{k+1} \leq |x - y| < r/2^k \}} |u(y)| |x - y|^{2-n} \, dy \leq \sum_{k=0}^{+\infty} (r^{2-k-1})^{2-n} \int_{\Omega_{r/2^k}(x)} |u(y)| \, dy \leq r^{\lambda-n+2} C \|u\|_{L^{1,\lambda}(\Omega)}.
\]

REMARK 1.1:

\( L^{1,\lambda}(\Omega) \subseteq S \subseteq L^{1,\mu}(\Omega) \) where \( 0 < \mu < n - 2 < \lambda < n \).

Indeed the inclusion \( L^{1,\lambda}(\Omega) \subseteq S \) is an immediate consequence of Lemma 1.1 and the other inclusion is obvious.

We now recall the definitions of the Sobolev spaces \( H^{1,p}(\Omega) \), \( H^{1,p}_0(\Omega) \) and \( H^{-1,p}(\Omega) \).

DEFINITION 1.3. We say that \( u \) belongs to \( H^{1,p}(\Omega)[H^{1,p}_0(\Omega)](1 < p < +\infty) \) iff \( u \),

\[
\frac{\partial u}{\partial x_i} \in L^p(\Omega)[L^p_0(\Omega)] \quad (i = 1, 2, \ldots, n)
\]

\( H^{1,p}(\Omega) \) is a Banach space under the norm

\[
\|u\|_{H^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}
\]

\( H^{1,p}_0(\Omega) \) is the closure of \( \mathcal{D}(\Omega) \) with respect to the \( H^{1,p}(\Omega) \) norm; \( H^{-1,p}(\Omega) \) is the dual space of \( H^{1,p}_0(\Omega) \), where \( 1/p + 1/q = 1 \). We have \( T \in H^{-1,p}(\Omega) \) iff, \( \exists f_i \in L^p(\Omega) \) \((i = 1, 2, \ldots, n)\) such that \( T = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \).
2. Green’s function and a representation formula.

In the following sections we will consider the operator $L - V$ where $L$ is the divergence form elliptic operator

$$L = - \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)$$

satisfying

$$\begin{cases} a_{ij} \in L^\infty(\Omega), & a_{ij} = a_{ji} \quad (i, j = 1, 2, \ldots, n) \\ \exists \nu > 0: \nu|\xi|^2 < a_{ij} \xi_i \xi_j < \nu^{-1}|\xi|^2 & \forall \xi \in \mathbb{R}^n \end{cases}$$

and $V$ is a function

$$V \in L^{1, \lambda}(\Omega) \quad (\lambda > n - 2).$$

**Definition 2.1.** We say that $u \in H^{1,2}_{loc}(\Omega)$ is a local weak solution of the equation

$$Lu = Vu$$

iff

$$\int_{\Omega} a_{ij}(x) u_{x_i}(x) \psi_{x_j}(x) \, dx = \int_{\Omega} V(x) u(x) \psi(x) \, dx; \quad \forall \psi \in \mathcal{D}(\Omega).$$

Definition 2.1 is meaningful by the inclusion $L^{1,\lambda}(\Omega) \subseteq S$ and [S] p. 138-140.

We recall that under the weaker hypothesis $V \in S$ the following regularity result for weak solutions was proven in [C-F-G].

**Theorem 2.1.** There exist two positive constants $C = C(\nu, n)$, $r_0 = r_0(\nu, n, \eta)$ (\eta from definition 1.1) and a non decreasing function $\omega(r)$: $\lim_{r \to 0} \omega(r) = 0$ such that, for any local weak solution of $Lu + Vu = 0$ in $\Omega$ and for every ball $B_r(x_0) \subset \Omega$ ($0 < r < r_0$) we have:

$$\operatorname{osc} u \leq C \omega(r) \sup_{B_r(x_0)} |u|.$$
We now define a different class of solutions:

**Definition 2.2.** Let $L$ be such that (2.1) holds, let $\mu$ be a bounded variation measure in $\Omega$ and $T = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \in H^{-1,2}(\Omega)$.

We say that $u \in L^1(\Omega)$ is a very weak solution of the equation

$$Lu = \mu + T$$

if and only if

$$\int_{\Omega} u(x) L\varphi(x) \, dx = \int_{\Omega} \varphi(x) \, d\mu - \sum_{i=1}^{n} \int_{\Omega} f_i(x) \frac{\partial \varphi}{\partial x_i} \, dx$$

for every $\varphi \in H^{1,2}_0(\Omega) \cap C^0(\overline{\Omega})$ such that $L\varphi \in C^0(\overline{\Omega})$. In much the same way as in [ST] it is possible to show

**Lemma 2.1.** Assume $\mu$ is a bounded variation measure and $T = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \in H^{-1,2}(\Omega)$. If $u \in H^{1,2}_0(\Omega)$ is a weak solution of the equation

$$Lu = \mu + T$$

i.e.

$$\int_{\Omega} a_{ij}(x) u(x) \varphi_{x_i}(x) \, dx = \int_{\Omega} \varphi(x) \, d\mu - \sum_{i=1}^{n} \int_{\Omega} f_i(x) \frac{\partial \varphi}{\partial x_i} \, dx; \quad \forall \varphi \in H^{1,2}_0(\Omega)$$

then $u$ is the very weak solution of the same equation.

The proof is an easy consequence of the definitions above. We now recall the definition of fundamental solution.

Let $y \in \Omega$ and $\delta_y$ the Dirac mass at $y$.

Consider the equation

$$Lu = \delta_y.$$ 

We call its (very weak) solution the Green’s function relative to the operator $L$ with pole at $y$ and we denote it by $g(x, y)$.

By the definition above the solution $\varphi \in H^{1,2}_0(\Omega) \cap C^0(\overline{\Omega})$ of $L\varphi = \varphi$, 

where \( \varphi \in C^0(\Omega) \) is given by the formula

\[
\varphi(y) = \int_\Omega g(x, y) \psi(x) \, dx = \langle \psi(x), g(x, y) \rangle.
\]

Consider:

(2.8) \quad Lu = \mu + T

where \( \mu \) is a bounded variation measure, \( T \in H^{-1,p}(\Omega) \) \((p > n)\). We have the following

**Theorem 2.2:**

\[
u(x) = \langle \mu(y), g(x, y) \rangle + \langle T(y), g(x, y) \rangle
\]

is the very weak solution of (2.8).

**Proof.** We consider only the case \( \mu = 0 \) (for the case \( T = 0 \) see [ST] Th. 8.3 p. 227).

We will show that

\[
u(x) = \langle T(y), g(x, y) \rangle
\]

satisfies:

\[
\langle L\varphi(x), \langle T(y), g(x, y) \rangle \rangle = \langle T(y), \varphi(y) \rangle; \quad \forall \varphi \in H^1_0(\Omega) \cap C^0(\overline{\Omega})
\]

such that \( L\varphi \in C^0(\overline{\Omega}) \).

Let

\[
T = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}, \quad \text{where } f_i \in L^p(\Omega), \quad i = 1, 2, \ldots, n.
\]

Then

\[
\langle L\varphi(x), \langle T(y), g(x, y) \rangle \rangle = \int_\Omega L\varphi(x) \left( -\int_\Omega \frac{\partial g}{\partial y_i} f_i(y) \, dy \right) \, dx.
\]

We observe that

\[
|L\varphi(x) \frac{\partial g}{\partial y_i} f_i(y)| \in L^1(\Omega \times \Omega).
\]
Indeed we have:

\[
\int \left( \int_{\Omega} |L^p(x)| \left| \frac{\partial g}{\partial y_i} \right| f_i(y) \, dy \right) \, dx = \int \left( \int_{\Omega} \frac{\partial g}{\partial y_i} \left| f_i(y) \right| \, dy \right) \, dx \leq \\
\leq \int_{\Omega} \left| L^p(x) \right| \left\| \frac{\partial g}{\partial y_i} \right\|_{L^p(\Omega)} \left\| f_i \right\|_{L^p(\Omega)} \, dx \leq \max_{\Omega} \left| L^p(x) \right| \left\| f_i \right\|_{L^p(\Omega)} \int_{\Omega} \left\| \frac{\partial g}{\partial x_i} \right\|_{L^p(\Omega)} \, dx.
\]

Then (see [ST] p. 220 (8.6))

\[
\int \left( \int_{\Omega} |L^p(x)| \left| \frac{\partial g}{\partial y_i} \right| f_i(y) \, dy \right) \leq C \max_{\Omega} \left| L^p(x) \right| \left\| f_i \right\|_{L^p(\Omega)}.
\]

By Tonelli and Fubini’s theorems we have:

\[
\int_{\Omega} L^p(x) \left( - \int_{\Omega} \frac{\partial g}{\partial y_i} f_i(y) \, dy \right) \, dx = \int_{\Omega} f_i(y) \left( - \int_{\Omega} \frac{\partial g}{\partial y_i} L^p(x) \, dx \right) \, dy = \\
= \int_{\Omega} f_i(y) \left( - \frac{\partial}{\partial y_i} \int_{\Omega} g(x, y) L^p(x) \, dx \right) \, dy = \\
= \int_{\Omega} f_i(y) \left( - \frac{\partial}{\partial y_i} \left\langle g(x, y), L^p(x) \right\rangle \right) = \left\langle \frac{\partial f_i}{\partial y_i}, \left\langle g(x, y), L^p(x) \right\rangle \right\rangle = \\
= \left\langle T(x), \psi(x) \right\rangle.
\]

**Remark 2.1.** In the proof above we may differentiate under the integral; i.e.

\[-\int_{\Omega} \frac{\partial g}{\partial y_i} L^p(x) \, dx = - \frac{\partial}{\partial y_i} \int_{\Omega} g(x, y) L^p(x) \, dx.
\]

In fact, for every \( \varphi \in \mathcal{D}(\Omega) \) we have, using Fubini’s theorem:

\[- \left\langle \frac{\partial}{\partial y_i} \int_{\Omega} g(x, y) L^p(x) \, dx, \varphi(y) \right\rangle = \left\langle \int_{\Omega} g(x, y) L^p(x) \, dx, \frac{\partial \varphi}{\partial y_i} \right\rangle = \]
3. Hölder-continuity of local solutions.

We now state the main result of this paper

**Theorem 3.1.** There exist positive numbers \( r_0 = r_0(v, \|V\|_{1,1}, \lambda, n) \)
\( \alpha = \alpha(v, n) \), \( C = C(v, n, \|V\|_{1,1}, \lambda) \) such that for any local solution \( u \)
of \( Lu = Vu \) in \( \Omega \) and for any ball \( B_r(x_0) \), with \( B_r(x_0) \subset \Omega \), \( 0 < r < r_0 \)
we have

\[
|u(x) - u(x_0)| \leq C \sup_{B_{2r}(x_0)} |u|^{\lambda - n + 2} \cdot \left( |x - x_0|^{\alpha/2} r^{-\alpha/2} + |x - x_0|^{(\lambda - n + 2)/2} r^{-(\lambda - n + 2)/2} + \left( \frac{|x - x_0|}{r} \right)^{\alpha} \right).
\]

**Proof.** Let \( V \in L^{1,\lambda}(\Omega) \) and \( u \) a local weak solution of \( Lu = Vu \)
i.e. \( u \in H^{1,q}_{loc}(\Omega) \) such that:

\[
\int \alpha_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = \int V(x) \varphi(x) dx \quad \forall \varphi \in \mathcal{D}(\Omega).
\]

Let \( \varphi \in \mathcal{D}(\Omega) \). It is easy to see that \( u \varphi \) is such that

\[
\int \alpha_{ij}(x) \frac{\partial (u \varphi)}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = \int V(x) u(x) \varphi(x) dx +
\]

\[
+ \int \alpha_{ij}(x) u(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx - \int \alpha_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \varphi(x) dx
\]

holds.

Therefore, by Lemma 2.1, \( u \varphi \) is a very weak solution of

\[
L(u \varphi) = V(x) u(x) \varphi(x) - \frac{\partial}{\partial x_j} \left( \alpha_{ij}(x) u(x) \frac{\partial \varphi}{\partial x_i} \right) - \alpha_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}.
\]
By Theorem 2.2 we have

$$u(x)\varphi(x) = \int_{\Omega} V(y)u(y)\varphi(y)g(x, y)\,dy + \int_{\Omega} \frac{\partial g}{\partial y_i}(y)a_{i}(y)u(y)\frac{\partial \varphi}{\partial y_j}\,dy - \int_{\Omega} g(x, y)a_{i}(y)\frac{\partial u}{\partial y_i}\frac{\partial \varphi}{\partial y_j}\,dy.$$  

Now we choose $\varphi \in \mathcal{D}(\Omega)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ in $B_{1}\varepsilon(x_0)$, $\text{supp}(\varphi) \subseteq B_{2r}(x_0)$, $|\nabla \varphi| < C/r$ where $0 < r < r_0$ and $r_0$ is determined by the local boundedness theorem 1.4 in [C-F-G].

Obviously, for every $x \in B_{2r}(x_0)$ we have:

$$u(x) - u(x_0) = \int_{\Omega} V(y)u(y)\varphi(y)(g(x, y) - g(x_0, y))\,dy -$$

$$\int_{\Omega} (g(x, y) - g(x_0, y))a_{i}(y)\frac{\partial u}{\partial y_i}\frac{\partial \varphi}{\partial y_j}\,dy +$$

$$+ \int_{\Omega} \left( \frac{\partial g}{\partial y_i}(x, y) - \frac{\partial g}{\partial y_i}(x_0, y) \right) a_{i}(y)u(y)\frac{\partial \varphi}{\partial y_j}\,dy = I - II + III.$$  

We begin estimating $I$.

$$I = \int_{|x - y| > N|x - x_0|} (g(x, y) - g(x_0, y))V(y)u(y)\varphi(y)\,dy +$$

$$+ \int_{|x - y| \leq N|x - x_0|} (g(x, y) - g(x_0, y))V(y)u(y)\varphi(y)\,dy = A + B.$$  

Where $N$ is a positive number to be fixed later.

To estimate $A$ we use the inequality (see [G-T] p. 200 Th. 8.22 and Harnack’s Theorem)

$$|g(x, y) - g(x_0, y)| \leq C(v, n)\left( \frac{|x - x_0|}{r} \right)^{\alpha}g(x_0, y) \leq$$

$$\leq \frac{C(v, n)}{N^\alpha} g(x_0, y) \leq \frac{C(v, n)}{N^\alpha|x_0 - y|^{n-2}}.$$
hence

\[ A \leq \frac{C(v, n)}{N^\alpha} \int_{B_{4r}(x_0)} \frac{|V(y)|}{|x_0 - y|^{\alpha - 2}} \, dy \, \text{sup} \left\{ u \right\}_{B_{4r}(x_0)} \]

and by Lemma 1.1

\[ A \leq \frac{C(\|V\|_{L^{1,\infty}(\Omega)}, v, n, \lambda)}{N^\alpha} \, r^{\lambda - n + 2} \, \text{sup} \left\{ u \right\}_{B_{4r}(x_0)} . \]

To estimate \( B \) we use Lemma 1.1 and the following bound

\[ g(x, y) \leq \frac{C(v, n)}{|x - y|^{\alpha - 2}} \]

proven in [L-S-W].

We obtain:

\[ |g(x, y) - g(x_0, y)| \leq \frac{C(v, n)}{|x - y|^{\alpha - 2}} + \frac{C(v, n)}{|x_0 - y|^{\alpha - 2}} \]

and therefore

\[ B \leq C(v, n) \int_{|x_0 - y| \leq N|x - x_0|} \frac{|V(y)|}{|x - y|^{\alpha - 2}} \, dy \, \sup \left\{ u \right\}_{B_{4r}(x_0)} \leq \]

\[ \leq C(v, n) \int_{|x_0 - y| \leq (N + 1)|x - x_0|} \frac{|V(y)|}{|x - y|^{\alpha - 2}} \, dy \leq \]

\[ \leq C(v, n) \|V\|_{L^{1,\infty}(\Omega), \lambda} \, \sup \left\{ u \right\}_{B_{4r}(x_0)} \| (N + 1)|x - x_0| \|^\lambda_n - n + 2 . \]

Now, if we choose \( N = (r/|x - x_0|)^{\frac{1}{\alpha}} > 1 \) we obtain

\[ |I| \leq C(\|V\|_{L^{1,\infty}(\Omega), \lambda}, v, n) \, \sup \left\{ u \right\}_{B_{4r}(x_0)} \|x - x_0\|^{\alpha / 2 \, r^{\lambda - n + 2 - \alpha / 2}} + \]

\[ + C(\|V\|_{L^{1,\infty}(\Omega), \lambda}, v, n) \, \sup \left\{ u \right\}_{B_{4r}(x_0)} \|x - x_0\|^{(\lambda - n + 2) / 2 \, r^{(\lambda - n + 2) / 2}} . \]

Estimating II and III as in [C-F-G] we obtain

\[ |II| \leq C(v, n) \left( \frac{|x - x_0|}{r} \right)^{\alpha} \left( \int_{B_{4r}(x_0)} u(y)^2 \, dy \right)^{\frac{1}{2}} \]
and

\[ |III| \leq C(n, \kappa) \left( \frac{|x - x_0|}{r} \right)^\alpha \left( \int_{B_r(x_0)} u(y)^2 \, dy \right)^{\frac{1}{2}}. \]

The theorem now follows.

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