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Cohen-Macaulay and Gorenstein Finitely Graded Rings.

CLAUDIA MENINI (*)

Introduction.

Let G be a group, with identity element e , $R = \bigoplus_{\sigma \in G} R_{\sigma}$ a graded ring of type G . R is called *finitely graded* if $R_{\sigma} = 0$ for almost every $\sigma \in G$. In particular R is finitely graded whenever G is finite.

The main purpose of this paper is to characterize Cohen-Macaulay and Gorenstein finitely graded rings. Our starting point was the following. Graded rings over $G = \mathbb{Z}/2\mathbb{Z}$ are the so called semi-trivial extensions and a particular case of semi-trivial extensions are the trivial ones (see Section 3 for details). Now a complete description of Gorenstein trivial extensions, essentially due to I. Reiten ([R]), can be found in [FGR]. In [F₂] R. Fossum investigates the general situation of commutative extensions. Part of his results is found also in [F₁] where it appears as the algebraic basis of the well known Ferrand's construction. Ferrand's construction itself has been extensively used in studying set theoretic complete intersections. In this setting trivial extensions still provide answers to specific questions (see, for example [BG]).

Our first idea was to study Gorenstein semi-trivial extensions. From the very beginning it looked more appropriate to regard this case as a particular case of finitely graded rings than as a generalization

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of trivial extensions. The main point is that, in general, a semitrivial extension, unlike a trivial one, of a local ring is not a local ring. On the other hand we already had found in [M₄] some technical tools which looked very useful for this general investigation. In fact, together with some basic commutative algebra, they led us to a complete solution of our problem. At the end this solution appears as a sophisticated generalization of Reiten's results, even if the employed techniques are quite different.

We now give a short description of the content of the paper.

In section 0 we recall some basic notions of graded ring and module theory. The reader, which is not too familiar with it, is suggested to refer to Năstăsescu and Van Oystaeyen's book [NV].

Always in section 0 we essentially quote from our paper [M₄] the above mentioned results.

In the following sections all rings are assumed to be commutative, but the group G of the gradation is not.

In section 1 we prove, first of all, the following basic result: a finitely graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$ is gr-local (i.e. it has a unique gr-maximal ideal, that is a unique graded ideal which is maximal among graded ideals of R) iff R_e is local (i.e. it has a unique maximal ideal).

After that, the characterization of Cohen-Macaulay and Gorenstein finitely graded rings is easily reduced (Lemma 1.10) to that of gr-local ones.

Another basic result proved in Section 1 is the following. If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a graded module over a gr-local and noetherian finitely graded ring, $R = \bigoplus_{\sigma \in G} R_\sigma$ and $T = R_e$, then the socle of ${}_R M$ is not zero iff ${}_R M$ contains a copy of every simple R -module iff ${}_T M$ contains a copy of the unique simple T -module. Using this result it is not difficult to prove that when R is Cohen-Macaulay, one can find elements $t_1, \dots, t_n \in T$, $n = \dim(R)$, which form a regular R -sequence. This fact leads us to the following characterization of Cohen-Macaulay gr-local finitely graded rings (Theorem 1.12): Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian gr-local finitely graded ring of type G , $T = R_e$. Then R is a Cohen-Macaulay ring iff R is a Cohen-Macaulay T -module iff, for every $\sigma \in G$, R_σ is a Cohen-Macaulay T -module and $\dim(R_\sigma) = \dim(T)$.

Section 2 is devoted to the study of Gorenstein gr-local finitely graded rings. Our main tools are the local cohomology functors

$H_{\Omega}^i(N)$'s on R -mod with respect to the maximal spectrum Ω of R , the local cohomology functors $H_{\mathfrak{m}}^i(N)$'s on T -mod with respect to the maximal ideal $\mathfrak{m}$ of T and a functor $X_R: T\text{-mod} \rightarrow R\text{-gr}$ which had already been very useful in [M₄]. After proving several technical results we give, in Theorem 2.13, a complete description of Gorenstein gr-local finitely graded rings.

Unfortunately, part of this theorem (perhaps the deepest one), involves too many details to be quoted in this introduction. Thus we state here only the following one. Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a noetherian finitely graded ring of type G , $T = R_e$. Assume that R is gr-local and let $n = \dim(R)$. Then R is a Gorenstein ring iff R is a Cohen-Macaulay ring and $H_{\Omega}^n(R)$ is an injective R -module iff R is equidimensional and there is a $\sigma \in G$ such that R_{σ} is a canonical module for T and $R \cong \text{Hom}_T(R, R_{\sigma})$ in R -mod iff R is equidimensional, T has a canonical module K and $R \cong \text{Hom}_T(R, K)$ in R -mod. The particular case when R is Gorenstein and artinian is further investigated in Proposition 2.15.

In section 3 we specialize our results to the case of semi-trivial extensions. In particular we get, as a Corollary, the above mentioned Reiten's result on trivial extensions.

0. Notations and preliminaries.

All rings are associative with identity $1 \neq 0$ and all modules are unital. Let R be a ring. R -mod will denote the category of left R -modules. The notation ${}_R M$ will be used to emphasize that M is a left R -module. Moreover if R and T are two rings we will write ${}_R M_T$ to mean that M is an R - T -bimodule (left R -module and right T -module). Maps between modules will be written on the opposite side to that of the scalars only in the non-necessarily commutative case. If $L, M \in R$ -mod, the group $\text{Hom}_R(L, M)$ will be also written as $\text{Hom}_R({}_R L, {}_R M)$ or $\text{Hom}({}_R L, {}_R M)$.

If ${}_R L_T$ is an R - T -bimodule and if ${}_R M \in R$ -mod, then we will often consider $\text{Hom}_R({}_R L, {}_R M)$ with its left T -module structure defined by setting

$$t\xi = \varrho_t \cdot \xi \quad t \in T, \xi \in \text{Hom}_R({}_R L, {}_R M)$$

where ϱ_t is the *right* multiplication by t on ${}_R L$. In this case we will also write $\text{Hom}_R({}_R L_T, {}_R M)$. If $M \in R$ -mod, we will denote by $\text{Soc}_R(M)$

or simply by $\text{Soc}(M)$ the socle of ${}_R M$ and we will say that M has *finite socle* if $\text{Soc}(M)$ has finite length.

$E_R(M)$ or simply $E(M)$ will denote the injective envelope of M in $R\text{-mod}$. If $Z \subset {}_R M$ we set

$$\text{Ann}_R(Z) = \{r \in R : rZ = 0\}.$$

If $I \subset R$ we set

$$\text{Ann}_M(I) = \{x \in M : Ix = 0\}.$$

Let G be a multiplicative group with identity element e . Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a *graded ring of type G* . Recall (see [NV]) that this means that $\{R_\sigma\}_{\sigma \in G}$ is a family of additive subgroups of the ring R such that R splits—as an abelian group—into the direct sum of the R_σ 's, $\sigma \in G$, and for every $\sigma, \tau \in G$, $R_\sigma R_\tau \subseteq R_{\sigma\tau}$. An $M \in R\text{-mod}$ is said to be a *graded left R -module* if there is a family $\{M_\sigma, \sigma \in G\}$ of additive subgroups of M such that $M = \bigoplus_{\sigma \in G} M_\sigma$ and $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. The notion of graded right R -module is analogous. Note that if G is not abelian one has to distinguish between graded left and right R -modules *even if R is commutative!*

Let M and N be graded left modules over the graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$. For every $\tau \in G$ we set

$$\text{HOM}_R(M, N)_\tau = \{f : M \rightarrow N : f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_{\sigma\tau} \ \forall \sigma \in G\}.$$

$\text{HOM}_R(M, N)_\tau$ is an additive subgroup of the group $\text{Hom}_R(M, N)$ of all R -linear maps from M into N .

An $f \in \text{HOM}_R(M, N)_\tau$ is called a *graded morphism of degree τ* .

$$\text{HOM}_R(M, N) = \bigoplus_{\tau \in G} \text{HOM}_R(M, N)_\tau$$

is a graded abelian group of type G .

We denote by $R\text{-gr}$ ($\text{gr-}R$) the category of left (right) R -modules where the morphisms are the graded morphisms of degree e , i.e.

$$\text{Hom}_{R\text{-gr}}(M, N) = \text{HOM}_R(M, N)_e$$

for every $M, N \in R\text{-gr}$.

The *forgetful functor* $R\text{-gr} \rightarrow R\text{-mod}$ will be denoted by F_R or by F .

A ring homomorphism $f: R \rightarrow S$ between two graded rings of type G is called a *graded ring homomorphism* if $f(R_\sigma) \subseteq S_\sigma$ for all $\sigma \in G$.

If $M = \bigoplus_{\lambda \in G} M_\lambda$ is a graded left R -module and $\sigma \in G$ then $M(\sigma)$ is the graded left module obtained from M by setting $M(\sigma)_\lambda = M_{\lambda\sigma}$; the graded left module $M(\sigma)$ is called the σ -*suspension* of M . If $M, N \in R\text{-gr}$, $f \in \text{Hom}_{R\text{-gr}}(M, N)$ and $\sigma \in G$, then we denote by $f(\sigma)$ the morphism f regarded as an element of $\text{Hom}_{R\text{-gr}}(M(\sigma), N(\sigma))$. If $M = \bigoplus_{\sigma \in G} M_\sigma \in R\text{-gr}$ we set $h(M) = \bigcup_{\sigma \in G} M_\sigma$.

Let N be a graded submodule of M . N is called *gr-essential* in M if N is essential in M as a subobject of M in $R\text{-gr}$. N is *gr-essential* in M iff $F(N)$ is essential in $F(M)$ (see [NV] Lemma I.2.8). M is called *gr-injective* if M is an injective object in $R\text{-gr}$. If $F(M)$ is injective in $R\text{-mod}$ then M is *gr-injective*. The converse is not true in general (see [NV] Corollary I.2.5 and Remark I.2.6.1).

A graded module $S \in R\text{-gr}$ is called *gr-simple* if 0 and S are its only graded submodules.

If $M \in R\text{-gr}$, the *gr-socle* of M is the sum of its *gr-simple* graded submodules.

A graded module $M \in R\text{-gr}$ is called *left gr-noetherian* (*left gr-artinian*) if M satisfies the ascending (descending) chain condition on graded left R -submodules of M .

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G . R is called *strongly graded* if $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$. $M = \bigoplus_{\sigma \in G} M_\sigma \in R\text{-gr}$ is called *finitely graded* if $M_\sigma = 0$ for almost every $\sigma \in G$. If ${}_R R$ is finitely graded, R is called a *finitely graded ring*. If R is commutative and I is a graded ideal of R , I is called *gr-maximal* if I is maximal among graded ideals of R and R is called *gr-local* if R has exactly one *gr-maximal* ideal.

Let R be a commutative ring. We denote by $\text{Spec}(R)$ and by $\text{Spec Max}(R)$ the prime spectrum and the maximal spectrum of R respectively. If I is an ideal of R , $\text{ht}(I)$ will denote the height of I . If $M \in R\text{-mod}$, $\dim(M)$ will denote the Krull dimension of M .

If R is a commutative local noetherian ring and M is a finitely generated R -module, $\text{depth}_R(M)$ or simply $\text{depth}(M)$ will denote the depth of M with respect to the maximal ideal of R .

$\mathbb{N}$ will denote the set of non negative integers, $\mathbb{Z}$ the ring of integers.

We end this section recalling some notations and results from

$[M_4]$ which we will use often in the paper. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G , $T = R_e$, $N \in T\text{-mod}$. If R is *finitely graded* then the left R -module ${}_R Y = \text{Hom}_T({}_T R, {}_T N)$ has a natural structure of graded R -module defined by

$$Y_\sigma = \text{Hom}_T(R_{\sigma^{-1}}, N) \quad \sigma \in G.$$

We denote this graded R -module by $X(N)$. It is easy to check that the assignment $N \rightarrow X(N)$ yields a functor $X: T\text{-mod} \rightarrow R\text{-gr}$. Let $\bar{X} = F \circ X: T\text{-mod} \rightarrow R\text{-mod}$. Moreover for every $N \in T\text{-mod}$ we set

$$\bar{N} = (X(N))_e \quad \text{and} \quad \overline{\bar{N}} = \bigoplus_{\sigma \in G} R_\sigma \bar{N}.$$

0.1. PROPOSITION. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G , $T = R_e$. Then:

a) If L is an essential T -submodule of $N \in T\text{-mod}$, then $\bar{L}$ is a gr-essential R -submodule of $X(N)$ and hence $F(L)$ is an essential R -submodule of $\bar{X}(N)$.

b) If S is a simple left T -module, then $\bar{S}$ is a gr-simple left R -module. Hence $\bigoplus_{\sigma \in G} R_\sigma \bar{S}$ is an artinian semisimple left T -module and the left R -module $F(\bar{S})$ has finite length.

c) If $E \in T\text{-mod}$ is injective in $T\text{-mod}$, then $\bar{X}(E)$ is injective in $R\text{-mod}$.

d) If $E \in T\text{-mod}$ is a cogenerator of $T\text{-mod}$, then $\bar{X}(E)$ is a cogenerator in $R\text{-mod}$.

e) If E has a finite essential socle, then $\bar{X}(E)$ has a finite essential socle.

PROOF. See $[M_4]$ Lemmata 1.2, 1.3 and 1.4.

0.2. Let $R = \bigoplus_{\tau \in G} R_\tau$ be a finitely graded ring of type G , $T = R_e$ and let $M = \bigoplus_{\tau \in G} M_\tau$ be a graded left R -module. Fix $\sigma \in G$. For every $\tau \in G$ and $x \in M_\tau$ let

$$(x)\mu_\sigma^M: R_{\sigma\tau^{-1}} \rightarrow M_\sigma$$

be the map defined by

$$(r)((x)\mu_\sigma^M) = rx \quad \text{for every } r \in R_{\sigma\tau^{-1}}.$$

Clearly $(x)\mu_\sigma \in \text{Hom}_T({}_T(R_{\sigma\tau^{-1}}), {}_T(M_\sigma))$.

Following proposition generalizes Proposition 4.3 of [M₄].

0.3. PROPOSITION. *Let $R = \bigoplus_{\tau \in G} R_\tau$ be a finitely graded ring of type G , $T = R$, and let $M = \bigoplus_{\tau \in G} M_\tau$ be a graded left R -module. Then, within the notations of 0.2, the mapping $x \mapsto (x)\mu_\sigma^M$, $x \in M_\tau$, defines a morphism of graded left R -modules $\mu_\sigma^M: M \rightarrow X_R(M_\sigma)(\sigma^{-1})$. Moreover $\text{Im}(\mu_\sigma^M)$ is a graded essential submodule of $X_R(M_\sigma)(\sigma^{-1})$ and $\bigcap_{\sigma \in G} \text{Ker}(\mu_\sigma^M) = 0$.*

PROOF. It is trivial to check that $\mu_\sigma^M: M \rightarrow X_R(M_\sigma)(\sigma^{-1})$ is a morphism of graded left R -modules.

Let $\tau \in G$ and let $0 \neq \xi \in (X_R(M_\sigma)(\sigma^{-1}))_\tau = \text{Hom}_\tau(R_{\sigma\tau^{-1}}, M_\sigma)$. Then there is an $r \in R_{\tau\sigma^{-1}}$ so that $0 \neq (r)\xi$. Let $y = (r)\xi \in M_\sigma$ and consider $(y)\mu_\sigma^M \in \text{Hom}_T(T, M_\sigma)$. Then $r \cdot \xi = (y)\mu_\sigma^M$. In fact also $r\xi \in \text{Hom}_T(T, M_\sigma)$ and for every $t \in T$ it is:

$$(t)(r\xi) = (tr)\xi = t \cdot (r)\xi = ty = t((y)\mu_\sigma^M).$$

Thus $\text{Im}(\mu_\sigma^M)$ is graded essential in $X_R(M_\sigma)(\sigma^{-1})$. Let $0 \neq x \in h(M) \cap \bigcap_{\sigma \in G} \text{Ker}(\mu_\sigma^M)$ and let $\tau \in G$ such that $x \in M_\tau$. Then $(x)\mu_\tau^M = 0$ and hence $x = 0$. Contradiction.

If $M = R$ we will simply write μ_σ instead of μ_σ^M , for every $\sigma \in G$.

1. Finitely graded Cohen-Macaulay rings.

1.1. LEMMA. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G , $T = R$, and let $M = \bigoplus_{\sigma \in G} M_\sigma$ be a graded left R -module. If M is left gr-noetherian (left gr-artinian) then, for every $\sigma \in G$, ${}_T(M_\sigma)$ is left noetherian (left artinian).*

PROOF. If $\sigma \in G$ and H is a left submodule of ${}_T(M_\sigma)$, then $L = \bigoplus_{\tau \in G} R_\tau H$ is a graded submodule of ${}_R M$ with gradation defined by $L_\tau = R_{\tau\sigma^{-1}}H$ for every $\tau \in G$. The proof follows straightforward from this remark.

1.2. COROLLARY. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G , $T = R_e$ and let $M = \bigoplus_{\sigma \in G} M_\sigma$ be a finitely graded left R -module. Then the following statements are equivalent:

- (a) ${}_R M$ is left gr-noetherian (left gr-artinian).
- (b) For every $\sigma \in G$, ${}_T(M_\sigma)$ is left noetherian (left artinian).
- (c) $F(M)$ is left noetherian (left artinian).

1.3. LEMMA. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G and assume that $T = R_e$ has a unique simple left T -submodule S . Let $V \in R\text{-gr}$ be a gr-simple left R -module. Then $F(V)$ is isomorphic to $F(\bar{S})$ in $R\text{-mod}$.

PROOF. Let E be the injective envelope of S in $T\text{-mod}$. Then E is the minimal injective cogenerator of $T\text{-mod}$ and, by Proposition 0.1, $\bar{X}(E)$ is an injective cogenerator of $R\text{-mod}$. As $V \in R\text{-gr}$ is gr-simple, $F(V)$ is finitely generated in $R\text{-mod}$ so that

$$\text{Hom}_R(F(V), \bar{X}(E)) = \text{HOM}_R(V, X(E))$$

(see [NV] Corollary I.2.11).

Thus, as $\bar{X}(E)$ is a cogenerator of $R\text{-mod}$, there is a $\tau \in G$ and an $f \in \text{Hom}_{R\text{-gr}}(V(\tau), X(E)) = \text{HOM}_R(V, X(E))_{\tau^{-1}}$, $f \neq 0$. As $V(\tau)$ is gr-simple too, f is injective.

By Proposition 0.1, $\bar{S}$ is gr-simple and gr-essential in $X(E)$. It follows that $V(\tau)$ is isomorphic to $\bar{S}$ in $R\text{-gr}$ and hence $F(V) = F(V(\tau))$ is isomorphic to $F(\bar{S})$ in $R\text{-mod}$.

From now on, if not otherwise expressly stated, we will consider only commutative rings.

1.4. PROPOSITION. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G . Then R is gr-local iff $T = R_e$ is local.

PROOF. If $T = R_e$ is local, then R is gr-local by Lemma 1.3. Conversely if R is gr-local then all non invertible elements of T must be contained in the unique gr-maximal ideal M of R . Thus $M \cap T$ is the unique maximal ideal of T .

1.5. PROPOSITION. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G and assume that $\mathcal{K} = R_e$ is a field. Then for every $M \in \text{Spec Max}(R)$, R/M is a $\mathcal{K}$ -vector space of finite dimension and hence the R/M -vector space $\text{Hom}_{\mathcal{K}}(R/M, \mathcal{K})$ is one-dimensional. Moreover $\text{Spec Max}(R)$ is finite.

PROOF. By Proposition 0.1, $\bar{X}(\mathcal{K})$ is an injective cogenerator of $R\text{-mod}$ whose socle is contained in $F(\bar{\mathcal{K}})$ and moreover $F(\bar{\mathcal{K}})$ is an artinian semisimple $\mathcal{K}$ -module i.e. a finite dimensional $\mathcal{K}$ -vector space.

Let $M \in \text{Spec Max}(R)$. Then, as $\bar{X}(\mathcal{K})$ is a cogenerator of $R\text{-mod}$, $\text{Soc}(\bar{X}(\mathcal{K}))$ contains an R -module isomorphic to R/M and hence R/M is a finite dimensional $\mathcal{K}$ -vector space. In particular $\text{Spec Max}(R)$ is finite.

Note now that if L is a finite dimensional $\mathcal{K}$ -algebra then $\dim_{\mathcal{K}}(L) = \dim_{\mathcal{K}}(\text{Hom}_{\mathcal{K}}(L, \mathcal{K}))$ and hence, if L is a field, $\text{Hom}_{\mathcal{K}}(\mathcal{K}_{L_L}, \mathcal{K})$ is a one-dimensional L -vector space.

1.6. PROPOSITION. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G , $T = R_e$. Let $\mathfrak{m} \in \text{Spec Max}(T)$, $S = T/\mathfrak{m}$. Then $\mathfrak{m}$ is contained only in a finite number of distinct maximal ideals of R , say $M_1, \dots, M_n$, and

$$\text{Soc}_R(\bar{X}(S)) \cong R/M_1 \oplus \dots \oplus R/M_n.$$

PROOF. Let $M \in \text{Spec Max}(R)$. Then

$$\begin{aligned} \text{Hom}_R(R/M, \text{Hom}_T({}_T R_R, T/\mathfrak{m})) &\cong \\ &\cong \text{Hom}_R(R/M, \text{Hom}_T(R/\mathfrak{m}R, T/\mathfrak{m})) \cong \\ &\cong \text{Hom}_T(R/M \bigotimes_R R/\mathfrak{m}R, T/\mathfrak{m}) = \text{Hom}_T(R/(M + \mathfrak{m}R), T/\mathfrak{m}). \end{aligned}$$

Thus $\text{Hom}_R(R/M, \bar{X}(S)) \neq 0$ iff $\mathfrak{m} \subseteq M$ and in this case

$$\text{Hom}_R(R/M, \bar{X}(S)) = \text{Hom}_{\bar{R}}(\bar{R}/\bar{M}, \bar{X}_{\bar{R}}(\mathcal{K})) \cong \text{Hom}_{\mathcal{K}}(\bar{R}/\bar{M}, \mathcal{K})$$

where $\bar{R} = R/\mathfrak{m}R$, $\bar{M} = M/\mathfrak{m}R$ and $\mathcal{K} = T/\mathfrak{m}$ so that Proposition 1.5 applies.

1.7. COROLLARY. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring. If $T = R_e$ is semilocal and E is the minimal (injective) cogenerator of $T\text{-mod}$ then

$\bar{X}(E)$ is the minimal (injective) cogenerator of $R\text{-mod}$ and every maximal ideal of R contains a maximal ideal of T .

PROOF. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the distinct maximal ideals of T . Then $E = \bigoplus_{i=1}^n E(S_i)$ where, for every $i = 1, \dots, n$, $S_i = T/\mathfrak{m}_i$. Clearly $\bar{X}(E) \cong \bigoplus_{i=1}^n \bar{X}(E(S_i))$.

Let $M \in \text{Spec Max}(R)$. Then M cannot contain two distinct maximal ideals of T (as they are coprime!). Thus, by Proposition 1.6, $\text{Soc}(\bar{X}(E))$ splits into the direct sum of distinct simple R -modules. By Proposition 0.1, $\bar{X}(E)$ is a cogenerator of $R\text{-mod}$ with essential socle. It follows that $\bar{X}(E)$ is the minimal cogenerator of $R\text{-mod}$. Moreover as $\bar{X}(E)$ cogenerates $R\text{-mod}$, $\text{Soc}(\bar{X}(E))$ contains a copy of every R/M for every $M \in \text{Spec Max}(R)$. Thus, by Proposition 1.6, if $M \in \text{Spec Max}(R)$, then M must contain some $\mathfrak{m}_i$.

1.8. PROPOSITION. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G , $T = R_e$. Then R is a noetherian (resp. artinian) ring iff R is a noetherian (resp. artinian) T -module. If R is noetherian then

- a) $\dim(R) = \dim(T)$.
- b) For every $P \in \text{Spec}(R)$, $P \in \text{Spec Max}(R)$ iff

$$P \cap T \in \text{Spec Max}(T).$$

PROOF. Apply Corollary 1.2 and note that if R is a noetherian T -module, then R is an integral extension of T . Thus a) and b) follow from [M₃] Theorem 20 page 81 and [AM] Corollary 5.8 page 61.

1.9. LEMMA. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a finitely graded ring of type G . Assume that R is gr-local and noetherian. Let $M = \bigoplus_{\sigma \in G} M_\sigma$ be a graded (left) R -module. Then the following statements are equivalent:

- (a) $\text{Soc}_T(M) \neq 0$.
- (b) M contains a copy of every simple R -module.
- (c) $\text{Soc}_R(M) \neq 0$.

PROOF. (a) $\Rightarrow$ (b). Let $0 \neq x \in h(M)$ and such that $x \in \text{Soc}_T(M)$. Then Rx is a finite vector space over $T/\mathfrak{m}$, where $\mathfrak{m}$ is, by Proposition 1.4, the unique maximal ideal of T . Thus Rx is an R -module of finite length. Hence Rx can be also regarded as an object of finite length in $R\text{-gr}$. It follows that Rx contains a gr-simple R -module V . Let $S = T/\mathfrak{m}$. Then by Proposition 0.1 $F(\bar{S})$ contains $\text{Soc}_R(\bar{X}(S))$. By Lemma 1.3 $F(V)$ is isomorphic to $F(\bar{S})$ in $R\text{-mod}$. The conclusion now follows from Proposition 1.6 and Corollary 1.7.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) is trivial in view of Corollary 1.7 (or Proposition 1.8).

We will say that a commutative noetherian ring R is *Cohen-Macaulay* (resp. *Gorenstein*) iff, for every $M \in \text{Spec Max}(R)$, R_M is a local Cohen-Macaulay ring (resp. a local Gorenstein ring). Similarly, if N is a finitely generated R -module, we will say that N is a *Cohen-Macaulay* R -module iff N_M is a Cohen-Macaulay R_M -module for every $M \in \text{Spec Max}(R)$. For the definition of local Cohen-Macaulay ring and of Cohen-Macaulay module over a local noetherian ring see [S₁] or [M₃] or [HK]. For the definition of local Gorenstein ring see [B] or [HK].

1.10. LEMMA. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian finitely graded ring of type G , $T = R_e$. Then R is Cohen-Macaulay (Gorenstein) iff $R_{\mathfrak{m}}$ is Cohen-Macaulay (Gorenstein) for every $\mathfrak{m} \in \text{Spec Max}(T)$.

PROOF. Let $M \in \text{Spec Max}(R)$. Then $\mathfrak{m} = M \cap T \in \text{Spec Max}(T)$ by Proposition 1.8. Let $R' = R_{\mathfrak{m}}$ and $M' = MR_{\mathfrak{m}}$. Then $R'_{M'} \cong R_M$. Apply now theorem 1 in [B] and Theorem 30 page 107 in [M₃].

1.11. REMARK. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring, $T = R_e$, $\mathfrak{m} \in \text{Spec Max}(T)$. Then the ring $R_{\mathfrak{m}}$ is a graded ring of type G with gradation defined by

$$(R_{\mathfrak{m}})_\sigma = (R_\sigma)_{\mathfrak{m}} \quad \text{for every } \sigma \in G.$$

Thus, in view of Proposition 1.4, if R is finitely graded, $R_{\mathfrak{m}}$ is a gr-local ring and by Lemma 1.10 the characterization of Cohen-Macaulay and Gorenstein rings is reduced to that of gr-local ones.

1.12. THEOREM. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian gr-local finitely graded ring of type G , $T = R_e$. Then the following statements are equivalent:

- (a) R is a Cohen-Macaulay ring.
- (b) R is a Cohen-Macaulay T -module.
- (c) For every $\sigma \in G$, R_σ is a Cohen-Macaulay T -module and $\dim(R_\sigma) = \dim(T)$.

If these conditions are satisfied then, for every $M \in \text{Spec Max}(R)$, $\text{ht}(M) = \dim(R) = \dim(T) = n$ and there are elements $t_1, \dots, t_n \in T$ which form a regular R -sequence.

PROOF. (a) $\Rightarrow$ (b) By Proposition 1.8, $\dim(R) = \dim(T)$. Let $d = \dim(T)$ and let n be the depth of the T -module R . Let $\mathfrak{m}$ be the maximal ideal of T and let $t_1, \dots, t_n \in \mathfrak{m}$ be a maximal regular R -sequence. Set

$$\bar{T} = \frac{T}{(t_1, \dots, t_n)} \quad \text{and} \quad \bar{R} = \frac{R}{(t_1, \dots, t_n)R}.$$

Clearly $\bar{R}$ is a graded quotient ring of R and $(\bar{R})_e = \bar{T}$. In particular $\bar{R}$ is gr-local and, by the maximality of the sequence $t_1, \dots, t_n$, $\text{Soc}_{\bar{T}}(\bar{R}) \neq 0$. Thus, by Lemma 1.9, $\bar{R}$ contains a copy of every simple $\bar{R}$ -module and hence every maximal ideal of $\bar{R}$ is associated to 0 so that $\bar{R}$ is artinian. By Proposition 1.8 and 1.4 every maximal ideal M of R contains $\mathfrak{m}$ and hence it contains $t_1, \dots, t_n$. From the foregoing considerations it follows that every $M \in \text{Spec Max}(R)$ is associated to the ideal spanned by $t_1, \dots, t_n$ in R . It follows (see Theorem 155 page 133 in [K]) that

$$\text{ht}(M) = n \quad \text{for every } M \in \text{Spec Max}(R).$$

Thus $n = d$ and R is a Cohen-Macaulay R -module.

(b) $\Rightarrow$ (c) Let $t \in T$. Then t is regular on R iff t is regular on each R_σ . Thus we get

$$\dim(T) \geq \text{depth}({}_T R_\sigma) \geq \text{depth}({}_T R) = \dim({}_T R) = \dim(T).$$

(c) $\Rightarrow$ (b) Let $i \in \mathbb{N}$. Then $\text{Ext}_T^i(T/\mathfrak{m}, R) \cong \bigoplus_{\sigma \in G} \text{Ext}_T^i(T/\mathfrak{m}, R_\sigma)$.

By Proposition 6 page IV-14 in [S] we get $\text{depth}({}_T R) = \dim(T)$. As $R \geq T$, $\dim({}_T R) = \dim(T)$.

(b) $\Rightarrow$ (a) Let $t_1, \dots, t_n \in \mathfrak{m}$ be a regular R -sequence. Then $t_1, \dots, t_n$, regarded as elements of R , still form a regular R -sequence. Moreover, by Proposition 1.8 the elements $t_1, \dots, t_n$ are contained in the Jacobson radical of R and $\dim(R) = \dim({}_T R) = \dim(T)$.

1.13. THEOREM. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian finitely graded ring, $T = R_e$. Then the following statements are equivalent:

(a) R is a Cohen-Macaulay ring.

(b) R is a Cohen-Macaulay T -module.

(c) For every $\sigma \in G$, R_σ is a Cohen-Macaulay T -module and, for every $\mathfrak{m} \in \text{Spec Max}(T)$, $\dim((R_\sigma)_{\mathfrak{m}}) = \dim(T_{\mathfrak{m}})$.

PROOF. It follows by Proposition 1.4, Lemma 1.10, Remark 1.11 and Theorem 1.12.

1.14. COROLLARY. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian finitely graded ring, $T = R_e$ and assume that every R_σ is a projective T -module. Then R is a Cohen-Macaulay ring iff T is a Cohen-Macaulay ring.

1.15. COROLLARY. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a strongly graded noetherian ring of type G , G a finite group. Then R is a Cohen-Macaulay ring iff T is a Cohen-Macaulay ring.

PROOF. Apply Corollary I.3.3 page 15 in [NV] and Corollary 1.14 above.

2. Finitely graded Gorenstein rings.

In all this section, when $R = \bigoplus_{\tau \in G} R_\tau$ is a noetherian gr-local finitely graded ring of type G , T will denote R_e , $\mathfrak{m}$ will be the maximal ideal of T and $E = E(T/\mathfrak{m})$ the injective envelope of $T/\mathfrak{m}$ in $T\text{-mod}$.

2.1. Let R be a commutative noetherian ring, $\mathfrak{A}$ an ideal of R , $A = V(\mathfrak{A}) = \{\mathfrak{P} \in \operatorname{Spec}(R) : \mathfrak{P} \supseteq \mathfrak{A}\}$. For every $i \in \mathbb{N}$, let $H_A^i(N)$ denote the i -th local cohomology functor on $R\text{-mod}$ with respect to A i.e. the i -th right derived functor of the functor $\Gamma_A : R\text{-mod} \rightarrow R\text{-mod}$ defined by setting, for every $M \in R\text{-mod}$,

$$\Gamma_A(M) = \{x \in M : \exists n \in \mathbb{N} \setminus \{0\} \text{ such that } \mathfrak{A}^n x = 0\}$$

(see [HK]).

If $R = \bigoplus_{\sigma \in G} R_\sigma$ is a noetherian gr-local finitely graded ring of type G , for every $i \in \mathbb{N}$ we denote by $H_m^i(-)$ the i -th local cohomology functor on $T\text{-mod}$ with respect to $\{\mathfrak{m}\}$ and by $H_\Omega^i(-)$ the i -th local cohomology functor on $R\text{-mod}$ with respect to $\Omega = \operatorname{Spec} \operatorname{Max}(R) = V(\mathfrak{m}R)$ (see Proposition 1.8).

2.2. LEMMA. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian gr-local finitely graded ring of type G , $i \in \mathbb{N}$, $M = \bigoplus_{\sigma \in G} M_\sigma \in R\text{-gr}$. Then $H_\Omega^i(M)$ has a natural structure of graded left R -module and

$$(H_\Omega^i(M))_\sigma \cong H_m^i(M_\sigma) \quad \text{for every } \sigma \in G.$$

PROOF. It is well known that there is a natural isomorphism of T -modules

$$\varphi_M^i : H_\Omega^i(M) \rightarrow H_m^i(M)$$

(see e.g. [S₂] Theorem 4.3).

For every $\tau \in G$, let $\eta_\tau : M_\tau \rightarrow M$ be the canonical injection. Then

$$\bigoplus_{\tau \in G} H_m^i(\eta_\tau) : \bigoplus_{\tau \in G} H_m^i(M_\tau) \rightarrow H_m^i(M)$$

is a natural isomorphism (see [S₂] Theorem 3.2). For every $\tau \in G$ set

$$N_\tau = (\varphi_M^i)^{-1} \left(H_m^i(\eta_\tau) (H_m^i(M_\tau)) \right).$$

Then it is enough to show that for every $\sigma, \tau \in G$

$$R_\sigma N_\tau \subseteq N_{\sigma\tau}.$$

Let $r \in R_\sigma$ and let $\alpha: M \rightarrow M$ be the multiplication by r on M . Then $H_\Omega^i(\alpha): H_\Omega^i(M) \rightarrow H_\Omega^i(M)$ is the multiplication by r on $H_\Omega^i(M)$. Thus we have to show that

$$H_\Omega^i(\alpha)N_\tau \subseteq N_{\sigma\tau}.$$

As the φ_M^i 's are natural isomorphisms, this is equivalent to prove that $H_m^i(\alpha)(L_\tau) \subseteq L_{\sigma\tau}$, where $L_\tau = H_m^i(\eta_\tau)(H_m^i(M_\tau))$ for every $\tau \in G$.

Let $\pi_\tau: M \rightarrow M_\tau$ be the canonical projection. Then

$$(H_m^i(\pi_\xi) \circ H_m^i(\alpha))(L_\tau) = (H_m^i(\pi_\xi \circ \alpha \circ \eta_\tau))(H_m^i(M_\tau)) = 0$$

for every $\xi \in G \setminus \{\sigma\tau\}$.

Since $H_m^i(\pi_\xi) \circ H_m^i(\eta_\rho) = H_m^i(\pi_\xi \circ \eta_\rho)$ is equal to the identity on $H_m^i(M_\rho)$ iff $\xi = \rho$ and is equal to zero otherwise, we get $H_m^i(\alpha)(L_\tau) \subseteq L_{\sigma\tau}$.

2.3. Let R be a noetherian ring, $\mathfrak{A}$ an ideal of R , $r_1, \dots, r_n$ a system of generators of $\mathfrak{A}$, $A = V(\mathfrak{A})$, $M \in R\text{-mod}$. For every $i \in \mathbb{N}$, let $H_r^i(M)$ be the i -th cohomology functor of M with respect to $\mathbf{r} = \{r_1, \dots, r_n\}$ (see [H] page 19 or [HK] Def. 4.6 for the definition). Then, for every $i \in \mathbb{N}$ there is a natural isomorphism $\lambda_r^i(M)$ between $H_r^i(M)$ and $H_A^i(M)$ (see [H] Theorem 2.3). Moreover it is easy to check (see e.g. [M₂] Theorem 10) that $H_r^n(M)$ naturally identifies with $\varinjlim_{v \in \mathbb{N}} (M/\mathbf{r}^v M)$

where, for every $v \in \mathbb{N}$, $\mathbf{r}^v M = r_1^v M + \dots + r_n^v M$ and the transition morphisms

$$\psi_r^v(M): \frac{M}{\mathbf{r}^v M} \rightarrow \frac{M}{\mathbf{r}^{v+1} M}$$

in the direct limit are given by

$$(\psi_r^v(M))(x + \mathbf{r}^v M) = r_1 \dots r_n x + \mathbf{r}^{v+1} M \quad \text{for every } x \in M.$$

2.4. PROPOSITION. Let $R = \bigoplus R_\sigma$ be a noetherian gr-local finitely graded ring of type G . Assume that R is a Cohen-Macaulay ring, $n = \dim(R)$ and let $t_1, \dots, t_n \in T$ be a regular R -sequence (see Theorem 1.12); $\mathbf{t} = \{t_1, \dots, t_n\}$. Endow $H_\Omega^n(R)$ with its graded left R -module structure (see Lemma 2.2) and each $R/\mathbf{t}^v R$, $v \in \mathbb{N}$, with graded quotient ring structure. Then

$$H_\Omega^n(R) \cong \varinjlim_{v \in \mathbb{N}} R/\mathbf{t}^v R \quad \text{naturally in } R\text{-gr}$$

and for every $\nu \in \mathbb{N}$ the transition morphisms $\psi_i^\nu = \psi_i^\nu(R)$ in the direct limit are injective.

PROOF. First of all note that the ψ_i^ν 's, $\nu \in \mathbb{N}$, are morphisms in $R\text{-gr}$ so that $\{R/\mathfrak{t}^\nu R, \psi_i^\nu, \nu \in \mathbb{N}\}$ is an inductive system in $R\text{-gr}$ and hence we can consider the direct limit of this system in $R\text{-gr}$. This is nothing else than the usual direct limit in $R\text{-mod}$ endowed with the gradation defined by setting, for every $\tau \in G$,

$$(\lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} R/\mathfrak{t}^\nu R)_\tau = \lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} (R/\mathfrak{t}^\nu R)_\tau \cong \lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} (R_\tau/\mathfrak{t}^\nu R_\tau)$$

(see [NV] page 4).

Let $\tau \in G$ and let $\eta_\tau: R_\tau \rightarrow R$ be the canonical injection. Then (see 2.3) we have the commutative diagram

$$\begin{array}{ccc} H_m^n(R_\tau) & \xrightarrow{H_m^n(\eta_\tau)} & H_m^n(R) \\ \lambda_i^n(R_\tau) \uparrow \wr & & \uparrow \wr \lambda_i^n(R) \\ H_i^n(R_\tau) & \xrightarrow{H_i^n(\eta_\tau)} & H_i^n(R) \end{array}$$

This means that if we identify $H_m^n(R)$ with $\lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} R/\mathfrak{t}^\nu(R)$, then $H_m^n(R_\tau)$ identifies with $\lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} R_\tau/\mathfrak{t}^\nu R_\tau$.

Now it is easy to check that if we identify $H_\Omega^n(R)$ with $\lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} R/\mathfrak{t}^\nu R$ then the isomorphism $\varphi_R^n: H_\Omega^n(R) \rightarrow H_m^n(R)$ induces the identity on $\lim_{\substack{\longrightarrow \\ \nu \in \mathbb{N}}} R/\mathfrak{t}^\nu R$.

By [M₂] Theorem 8. (1) the ψ_i^ν 's are injective.

In the sequel we will identify, for every $\sigma \in G$, $(H_\Omega^n(R))_\sigma$ with $H_m^n(R_\sigma)$ and set

$$\chi_\sigma = \mu_\sigma^{H_\Omega^n(R)}: H_\Omega^n(R) \rightarrow X(H_m^n(R_\sigma))(\sigma^{-1}) \quad (\text{see 0.3}).$$

2.5. LEMMA. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian gr-local finitely graded ring of type G . Assume that R is a Gorenstein ring of Krull dimension n . Then there is a $\sigma \in G$ such that $\chi_\sigma: H_\Omega^n(R) \rightarrow X(H_m^n(R_\sigma))(\sigma^{-1})$ is injective. Moreover, for every $\sigma \in G$ such that χ_σ is injective, χ_σ is an isomorphism in $R\text{-gr}$, $H_m^n(R_\sigma) \cong E$ so that $H_\Omega^n(R)$ is injective in $R\text{-mod}$.*

PROOF. Our proof will be essentially a modification of that one of Satz 5.9 in [HK] to our case. Let $\mathfrak{t}$ be as in Prop. 2.4. First of all, for every $\nu \in \mathbb{N}$, $R/\mathfrak{t}^\nu R$ is an artinian Gorenstein ring and hence it is self-injective (see [B]). Then, as in the proof of Satz 5.9 in [HK], a standard argument using Artin-Rees Lemma shows that $H_\Omega^n(R)$ is injective in $R\text{-mod}$. By Proposition 2.4 we can identify $H_\Omega^n(R)$ with $\varinjlim_{\nu \in \mathbb{N}} R/\mathfrak{t}^\nu R$ in $R\text{-gr}$ and moreover, the transition morphisms in this direct limit are injective. Thus it is easy to check that

$$\text{Soc}_R(\varinjlim_{\nu \in \mathbb{N}} R/\mathfrak{t}^\nu R) = \text{Soc}_R(R/\mathfrak{t}R).$$

As $R/\mathfrak{t}R$ is gr-injective (Corollary I.2.5 in [NV]), gr-local and artinian, it has a gr-simple and essential socle. It follows that also $H_\Omega^n(R)$ has a gr-simple and essential socle. Now, by Proposition 0.3 $\bigcap_{\tau \in G} \text{Ker}(\chi_\tau) = 0$ and the χ_τ 's are morphisms in $R\text{-gr}$. Therefore there must be a $\sigma \in G$ such that $\text{Ker}(\chi_\sigma) = 0$. Let now σ be any element of G such that $\text{Ker}(\chi_\sigma) = 0$. Since, by Proposition 0.3, $\text{Im}(\chi_\sigma)$ is gr-essential in $X_R(H_m^n(R_\sigma))(\sigma^{-1})$ and since $H_\Omega^n(R)$ is injective we get that χ_σ is an isomorphism. Thus $\bar{X}_R(H_m^n(R_\sigma))$ is injective in $R\text{-mod}$ and hence, by Proposition 0.1, $X_R(H_m^n(R_\sigma)) = X_R(E(H_m^n(R_\sigma)))$ so that $H_m^n(R_\sigma)$ is injective in $T\text{-mod}$. Clearly $H_m^n(R_\sigma)$ is indecomposable in $T\text{-mod}$, otherwise $H_\Omega^n(R)$ would be decomposable in $R\text{-gr}$ while this is impossible as it has a gr-simple and essential socle. As $H_m^n(R_\sigma)$ is an artinian T -module (Lemma 1.1), by a classical Matlis' result (see $[M_1]$) we get $H_m^n(R_\sigma) = E$.

The last statement follows from Proposition 0.1.

Following [FGR] we shall say that a finitely generated module $K \neq 0$ over a noetherian ring R is a *canonical module for R* if for every $P \in \text{Spec}(R)$

$$\dim \mathcal{K}_{(P)} \text{Ext}_{R_P}^i(\mathcal{K}(P), K_P) = \delta_{i, \text{ht}(P)}$$

where $\mathcal{K}(P)$ is the residue field of the localization R_P of R at the prime ideal P . Thus K is a canonical module for R iff K is a Gorenstein module of rank 1 in the sense of Sharp $[S_3]$.

The results in the next theorem are essentially in $[S_3]$ and in $[HK]$, but see also [FGR] Theorem 5.6.

2.6. THEOREM. *Let R be a noetherian ring and let $K \neq 0$ be a finitely generated R -module. Then*

a) If K is a canonical module for R then R and K are Cohen-Macaulay.

b) If $\text{Spec}(R)$ is connected, K is a canonical module for R iff $K_{\mathfrak{m}}$ is a canonical module for $R_{\mathfrak{m}}$, for every $\mathfrak{m} \in \text{Spec Max}(R)$.

c) If R is local with maximal ideal $\mathfrak{m}$ and Krull dimension n then:

$c_1)$ K is a canonical module for R iff

$$\dim_{R/\mathfrak{m}}(\text{Ext}_R^i(R/\mathfrak{m}, K)) = \delta_{i,n}$$

$c_2)$ K is a canonical module for R iff R is Cohen-Macaulay and

$$K \otimes_R \hat{R} \cong \text{Hom}_{\hat{R}}(H_{\mathfrak{m}}^n(\hat{R}), E) = \text{Hom}_R(H_{\mathfrak{m}}^n(R), E)$$

as $\hat{R}$ -modules, where $\hat{R}$ is the $\mathfrak{m}$ -adic completion of R and $E = E(R/\mathfrak{m})$.

Not knowing any adequate reference, we give a proof of the following two results, even if we suspect it is already available in the literature.

2.7. LEMMA. *Let T be a local complete Cohen-Macaulay noetherian ring, $\mathfrak{m}$ its maximal ideal, $E = E(T/\mathfrak{m})$, $n = \dim(T)$. Let $H(-)$ be the n -th local cohomology functor of T with respect to $\{\mathfrak{m}\}$ and set $\bar{K} = \text{Hom}_T(H(T), E)$. Then*

1) *The canonical morphism $\psi: \text{Hom}_T(H(T), E) \otimes_T H(T) \rightarrow E$, defined by setting $\psi(f \otimes h) = f(h)$ for every $f \in \text{Hom}_T(H(T), E)$ and $h \in H(T)$, is an isomorphism. Thus $H(\bar{K}) \cong E$.*

2) *For every $M \in T\text{-mod}$ the assignment $f \mapsto H(f)$ yields an isomorphism between $\text{Hom}_T(M, \bar{K})$ and $\text{Hom}_T(H(M), H(\bar{K}))$.*

PROOF. 1) As it is well known, the canonical morphism $T \rightarrow \text{End}_T(H(T))$ is an isomorphism (see e.g. [M₂] Theorem 14) so that, by Matlis' duality ([M₁]), also $\text{End}_T(\bar{K})$ is canonically isomorphic to T . Now let

$$\begin{aligned} \varphi: \text{Hom}_T(\text{Hom}_T(H(T), E) \otimes_T H(T), E) &\rightarrow \\ &\rightarrow \text{Hom}_T(\text{Hom}_T(H(T), E), \text{Hom}_T(H(T), E)) \end{aligned}$$

be the obvious canonical isomorphism, $\omega: T \rightarrow \text{End}_T(E)$ the canonical morphism and set

$$\eta = \text{Hom}_T(\psi, 1_E): \text{Hom}_T(E, E) \rightarrow \text{Hom}_T\left(\text{Hom}_T(H(T), E) \underset{T}{\otimes} H(T), E\right).$$

Then it is easy to check that

$$\varrho \circ \eta \circ \omega: T \rightarrow \text{Hom}_T\left(\text{Hom}_T(H(T), E), \text{Hom}_T(H(T), E)\right) = \text{End}_T(\bar{K})$$

is exactly the canonical morphism. Thus, by above considerations, $\varrho \circ \eta \circ \omega$ is an isomorphism. On the other hand ω is an isomorphism too (see $[M_1]$) and hence we get that $\eta = \text{Hom}_T(\psi, 1_E)$ is also an isomorphism. By Matlis' duality we finally get that ψ is an isomorphism.

2) Let $M \in T\text{-mod}$. Let

$$\varphi: \text{Hom}_T(M, \text{Hom}_T(H(T), E)) \rightarrow \text{Hom}_T(M \underset{T}{\otimes} H(T), E)$$

be the canonical isomorphism. Set $\chi = \text{Hom}_T(1_{M \underset{T}{\otimes} H(T)}, \psi)$,

$$\begin{aligned} \chi: \text{Hom}_T\left(M \underset{T}{\otimes} H(T), \text{Hom}_T(H(T), E) \underset{T}{\otimes} H(T)\right) \rightarrow \\ \rightarrow \text{Hom}_T(M \underset{T}{\otimes} H(T), E). \end{aligned}$$

Then it is easy to check that for every $f \in \text{Hom}_T(M, \bar{K})$ $(\chi^{-1} \circ \varphi)(f) = f \otimes 1_{H(T)}$. As the functor H is naturally equivalent to the functor $-\underset{T}{\otimes} H(T)$ (see e.g. $[M_2]$ Theorem 10), we get the conclusion.

2.8. PROPOSITION. *Let T be a local noetherian ring, n its Krull dimension, $\mathfrak{m}$ its maximal ideal, $E = E(T/\mathfrak{m})$, K a finitely generated T -module. Then K is a canonical module for T iff T is Cohen-Macaulay and $H_{\mathfrak{m}}^n(K) \cong E$.*

PROOF. If K is a canonical module for T , then, by Theorem 2.6 a), T is Cohen-Macaulay. Thus we can assume w. l. o. g. that T is Cohen-Macaulay. Let $\hat{T}$ be the $\mathfrak{m}$ -adic completion of T , $\mathfrak{m} = \mathfrak{m}\hat{T}$ and set $\hat{K} = K \otimes \hat{T}$ and $\bar{K} = \text{Hom}_T(H_{\mathfrak{m}}^n(T), E) = \text{Hom}_T(H_{\mathfrak{m}}^n(\hat{T}), E)$.

Then, in view of Theorem 2.6 c₂), it is enough to show that $\hat{K} \cong \bar{K}$ in $\hat{T}\text{-mod}$ iff $H_{\mathfrak{m}}^n(K) \cong E$.

Now, by Lemma 2.7, $\hat{K} \cong \bar{K}$ iff $H_{\hat{m}}^n(\hat{K}) \cong H_{\hat{m}}^n(\bar{K})$ in $\hat{T}$ -mod. As $H_{\hat{m}}^n(K) = H_{\hat{m}}^n(\hat{K})$ and, by Lemma 2.7, $H_{\hat{m}}^n(\bar{K}) \cong E$ we get the conclusion.

2.9. Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a commutative graded ring of type G and let $M \in R$ -mod. Assume that $M = \bigoplus_{\sigma \in G} M_\sigma$ where the M_σ 's are suitable subgroups of M . Then, if G is not commutative, we still have to distinguish between M being a left or right graded R -module, as we said in Section 0. In fact if $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for every $\sigma, \tau \in G$ we have that M is a graded left R -module while if $R_\sigma M_\tau \subseteq M_{\tau\sigma}$ for every $\sigma, \tau \in G$ then M will be a graded right R -module.

Now let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a commutative graded ring of type G , $T = R_e$, $M = \bigoplus_{\sigma \in G} M_\sigma$ a finitely graded left R -module, $N \in T$ -mod. Then, for every $\sigma \in G$, we can define a natural structure of graded right R -module, which depends on σ , on $\text{Hom}_T(M, N)$. Denoting by $\text{Hom}_T(M(\sigma), N)$ this graded right R -module we have that the gradation on it is defined by setting

$$\text{Hom}_T(M(\sigma), N)_\tau = \text{Hom}_T(M_{\tau^{-1}\sigma}, N) \quad \text{for every } \tau \in G.$$

In fact note that if $\xi, \tau \in G$, $r \in R_\xi$, $f \in \text{Hom}_T(M_{\tau^{-1}\sigma}, N)$ then $rf \in \text{Hom}_T(M_{\xi^{-1}\tau^{-1}\sigma}, N)$. If $\alpha: M_1 \rightarrow M_2$ is a graded morphism between finitely graded R -modules, then the transposed morphism

$$\text{Hom}_T(\alpha, 1_N): \text{Hom}_T(M_2, N) \rightarrow \text{Hom}_T(M_1, N)$$

can be regarded as a graded morphism, which we will denote by $\text{Hom}_T(\alpha(\sigma), 1_N)$, between the graded right modules $\text{Hom}_T(M_2(\sigma), N)$ and $\text{Hom}_T(M_1(\sigma), N)$.

Now, for every $\sigma \in G$, we denote by $\varrho_\sigma: R \rightarrow \text{Hom}_T(R(\sigma), R_\sigma)$ the morphism of graded (right) R -modules defined by setting, for every $\xi \in G$, $x \in R_\xi$

$$\varrho_\sigma(x): R_{\xi^{-1}\sigma} \rightarrow R$$

be the map defined by setting $(\varrho_\sigma(x))(a) = xa$ for every $a \in R_{\xi^{-1}\sigma}$. Clearly if G is commutative, $\text{Hom}_T(R(\sigma), N) = X(N)(\sigma^{-1})$ and $\varrho_\sigma = \mu_\sigma$ defined in 0.3.

Recall that a commutative noetherian ring R is called *equidimensional* if given any two maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ in R , $\dim(R_{\mathfrak{m}}) = \dim(R_{\mathfrak{n}})$.

2.10. LEMMA. Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a noetherian gr-local finitely graded ring of type G . Let $\hat{T}$ be the $\mathfrak{m}$ -adic completion of T , $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, $\hat{R} = R \otimes_T \hat{T}$. Then

- a) $\bar{R} = \bigoplus_{\sigma \in G} (R_{\sigma} \otimes_T \hat{T})$ is a noetherian gr-local finitely graded ring of type G and $\hat{R}_e = \hat{T}$.
- b) $E_{\hat{T}}(\hat{T}/\hat{\mathfrak{m}}) = E_T(T/\mathfrak{m})$.
- c) The maximal ideals of $\hat{R}$ are exactly those of the form $\hat{\mathfrak{n}} = \mathfrak{n} \otimes_T \hat{T} = \mathfrak{n}\hat{T}$ for $\mathfrak{n} \in \text{Spec Max}(R)$.
- d) If $\mathfrak{n} \in \text{Spec Max}(R)$, $\hat{R}_{\hat{\mathfrak{n}}} \cong R_{\mathfrak{n}} \otimes_T \hat{T}$ and $\dim(\hat{R}_{\hat{\mathfrak{n}}}) = \dim(R_{\mathfrak{n}})$.
- e) $\hat{R}$ is equidimensional iff R is equidimensional.
- f) $\hat{R}$ is a Cohen-Macaulay ring iff R is a Cohen-Macaulay ring.
- g) For every $\sigma \in G$, $H_{\mathfrak{m}}^n(R_{\sigma}) = H_{\hat{\mathfrak{m}}}^n(\hat{R}_{\sigma})$.

PROOF. a) Is clear. b) Is well known. Let $\hat{\mathfrak{n}}$ be a maximal ideal of $\hat{R}$ and set $\mathfrak{n} = \hat{\mathfrak{n}} \cap R$. By Proposition 1.8 b) $\hat{\mathfrak{n}}$ contains $\hat{\mathfrak{m}}$ and hence $\mathfrak{n}$ contains $\hat{\mathfrak{m}} \cap R = \hat{\mathfrak{m}} \cap T = \mathfrak{m}$. Clearly $\mathfrak{n}$ is a prime ideal of R thus, by Proposition 1.8 b) $\mathfrak{n}$ is a maximal ideal of R . Conversely let $\mathfrak{n}$ be a maximal ideal of R and set $\hat{\mathfrak{n}} = \mathfrak{n} \otimes_T \hat{T} = \mathfrak{n}\hat{T}$. Then, as $\hat{T}$ is a flat T -module, $\hat{R}/\hat{\mathfrak{n}} \cong (R/\mathfrak{n}) \otimes_T \hat{T}$. Now $(R/\mathfrak{n}) \otimes_T \hat{T} = R/\mathfrak{n}$ so that $\hat{\mathfrak{n}}$ is a maximal ideal of $\hat{R}$.

d) Follows from c) and e) follows from d).

f) Follows from Theorem 1.12 and g) is well known.

2.11. LEMMA. Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a noetherian gr-local finitely graded ring of type G , $\hat{T}$ the $\mathfrak{m}$ -adic completion of T . Assume that R is equidimensional, that T has a canonical module K and that $R \cong \text{Hom}_T(R, K)$ in R -mod. Then $\hat{R} = R \otimes_T \hat{T}$ is a Gorenstein ring and hence it is also Cohen-Macaulay.

PROOF. Let $\mathfrak{n} \in \text{Spec Max}(R)$, $\hat{\mathfrak{n}} = \mathfrak{n}\bar{R}$. Then, by Lemma 2.10, $\hat{R}_{\hat{\mathfrak{n}}} \cong (\text{Hom}_{\hat{T}}(\hat{R}, K \otimes_{\hat{T}} \hat{T}))_{\hat{\mathfrak{n}}}$ and hence by Theorem 2.6 and by [HK] Definition 5.6 and Satz 5.12 and 5.9, $\hat{R}_{\hat{\mathfrak{n}}}$ is Gorenstein. Thus, by Lemma 2.10, $\hat{R}$ is Gorenstein.

2.12. PROPOSITION. Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a noetherian gr-local finitely graded ring of type G , $n = \dim(R)$, $\sigma \in G$. Then the following statements are equivalent:

(a) R is Cohen-Macaulay, $H_{\mathfrak{m}}^n(R_e) \cong E$ and

$$\chi_{\sigma}: H_{\Omega}^n(R) \rightarrow X_R(H_{\mathfrak{m}}^n(R_{\sigma}))(\sigma^{-1})$$

is an isomorphism (in R -gr).

(b) R is equidimensional, T is Cohen-Macaulay, $H_{\mathfrak{m}}^n(R_{\sigma}) \cong E$ and

$$\chi_{\sigma}: H_{\Omega}^n(R) \rightarrow X_R(H_{\mathfrak{m}}^n(R_{\sigma}))(\sigma^{-1})$$

is an isomorphism.

(c) R is equidimensional, R_{σ} is a canonical module for T and

$$\varrho_{\sigma}: R \rightarrow \text{Hom}_T(R(\sigma), R_{\sigma})$$

is an isomorphism (in gr - R).

PROOF. Let $\hat{T}$ be the $\mathfrak{m}$ -adic completion of T . Then, as $\hat{T}$ is a faithfully flat T -module, as the morphisms χ_{σ} and ϱ_{σ} are natural and by Lemma 2.10 it is easy to check that we can assume w. l. o. g. $T = \hat{T}$.

Set $L = H_{\Omega}^n(R)$, $L_{\sigma} = H_{\mathfrak{m}}^n(R_{\sigma})$ and

$$\lambda_{\sigma} = \text{Hom}_T(\chi_{\sigma}(\sigma), 1_{L_{\sigma}}): \text{Hom}_T(X_R(L_{\sigma}), L_{\sigma}) \rightarrow \text{Hom}_T(L(\sigma), L_{\sigma}).$$

Let $\omega: R \rightarrow \text{Hom}_T(X_R(L_{\sigma}), L_{\sigma})$ be the morphism defined by setting

$$(\omega(x))(\xi) = \xi(x)$$

for every $\xi \in \text{Hom}_T(R, L_{\sigma})$, $x \in R$.

Clearly ω is a morphism in $\text{gr-}R$ and hence also $\lambda_\sigma \circ \omega: R \rightarrow \text{Hom}_T(L(\sigma), L_\sigma)$ is a morphism in $\text{gr-}R$ and for every $\tau \in G$, $r \in R_\tau$, $x \in (L(\sigma))_\tau = L_{\tau^{-1}\sigma}$ we have

$$[(\lambda_\sigma \circ \omega)(r)](x) = rx.$$

Let $\alpha: \text{Hom}_T(R(\sigma), R_\sigma) \rightarrow \text{Hom}_T(L(\sigma), L_\sigma)$ be the morphism defined by setting

$$\alpha(f) = H_m^n(f), \quad f \in \text{Hom}_T(R(\sigma), R_\sigma).$$

Then α is a morphism in $\text{gr-}R$ and it is easy to check that the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\omega} & \text{Hom}_T(X_R(L_\sigma), L_\sigma) & \xrightarrow{\lambda_\sigma} & \text{Hom}_T(L(\sigma), L_\sigma) \\ & \searrow \varrho_\sigma & & \nearrow \alpha & \\ & & \text{Hom}_T(R(\sigma), R_\sigma) & & \end{array}$$

is commutative.

Assume now (b) is fulfilled. Then, as χ_σ is an isomorphism in $R\text{-gr}$, $\lambda_\sigma = \text{Hom}_T(\chi_\sigma(\sigma), 1_{L_\sigma})$ is an isomorphism in $\text{gr-}R$. Moreover as $L_\sigma \cong E$, R is a noetherian T -module and T is complete in the $\mathfrak{m}$ -adic topology, by Matlis' duality ω is an isomorphism. Now by Proposition 2.8, R_σ is a canonical module for T so that, by Lemma 2.7 α is an isomorphism (in fact recall that when regarded as a T -module, $H_\Omega^n(R)$ coincides with $H_m^n(R)$). Thus ϱ_σ is an isomorphism too and (c) holds.

(c) $\Rightarrow$ (a) By Theorem 2.6 and Lemma 2.7, α is an isomorphism. By Proposition 2.8, $H_m^n(R_\sigma) \cong E$ so that ω is an isomorphism too and by Matlis' duality $\chi_\sigma = \text{Hom}_T(\lambda_\sigma 1_{L_\sigma})(\sigma^{-1})$ is also an isomorphism. By Lemma 2.11, R is Cohen-Macaulay.

(a) $\Rightarrow$ (b) By Theorem 1.12.

2.13. THEOREM. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian finitely graded ring of type G , $T = R_e$. Assume that R is gr-local , i.e. that T is a local ring, and let $\mathfrak{m}$ be the maximal ideal of T , $E = E(T/\mathfrak{m})$, $n = \dim(T)$ ($= \dim(R)$). Then the following statements are equivalent:*

(a) *R is a Gorenstein ring.*

- (b) R is a Cohen-Macaulay ring and there is a $\sigma \in G$ such that $H_m^n(R_\sigma) \cong E$ and

$$\chi_\sigma: H_\Omega^n(R) \rightarrow X_R(H_m^n(R_\sigma))(\sigma^{-1})$$

is an isomorphism (in R -gr).

- (c) R is equidimensional, T is Cohen-Macaulay and there is a $\sigma \in G$ such that $H_m^n(R_\sigma) \cong E$ and

$$\chi_\sigma: H_\Omega^n(R) \rightarrow X_R(H_m^n(R_\sigma))(\sigma^{-1})$$

is an isomorphism (in R -gr).

- (d) R is a Cohen-Macaulay ring and $H_\Omega^n(R)$ is an injective R -module.

- (e) R is equidimensional and there is a $\sigma \in G$ such that R_σ is a canonical module for T and $\varrho_\sigma: R \rightarrow \text{Hom}_T(R(\sigma), R_\sigma)$ is an isomorphism (in gr - R).

- (f) R is equidimensional and there is a $\sigma \in G$ such that R_σ is a canonical module for T and $R \cong \text{Hom}_T(R, R_\sigma)$ in R -mod.

- (g) R is equidimensional, T has a canonical module K and $R \cong \text{Hom}_T(R, K)$, in R -mod.

PROOF.

(a) $\Rightarrow$ (b) by Lemma 2.5.

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e) by Proposition 2.12.

(e) $\Rightarrow$ (f) $\Rightarrow$ (g) is trivial.

(g) $\Rightarrow$ (e) Let $\hat{T}$ be the $\mathfrak{m}$ -adic completion of T . By Lemma 2.11, $\hat{R} = R \otimes_T \hat{T}$ is a Gorenstein ring and hence by Lemma 2.5 and Proposition 2.12 (e) holds for $\hat{R}$. Then as $\hat{T}$ is faithfully flat and χ_σ is natural, using Lemma 2.10 it is easy to see that (e) holds also for R .

(b) $\Rightarrow$ (d) by Proposition 0.1.

(d) $\Rightarrow$ (a) Let $t_1, \dots, t_n \in T$ be a regular R -sequence (see Theorem 1.12) and let $H = H_\Omega^n(R)$. Then, by Proposition 2.4,

$R/(t_1, \dots, t_n)R \cong \text{Ann}_H(t_1, \dots, t_n)$. As H is an injective R -module it follows that $\bar{R} = R/(t_1, \dots, t_n)R$ is an injective $\bar{R}$ -module. Hence $\bar{R}$ is a Gorenstein ring (see [B]) and thus R is a Gorenstein ring too.

2.14. LEMMA. *Let $R = \bigoplus_{\tau \in G} R_\tau$ be a noetherian gr-local finitely graded ring of type G . Let $\sigma \in G$ and assume that $R_\sigma \cong E$. Then $\mu_\sigma: R \rightarrow X_R(R_\sigma)(\sigma^{-1})$ is an isomorphism when it is injective.*

PROOF. By Theorem 207 page 157 in [K], T is artinian. For every finitely generated T -module M , let $l(M)$ denote the length of M and recall (see [HK] Korollar 1.36) that the T -module $\text{Hom}_T(M, E)$ has finite length equal to $l(M)$. As μ_σ is injective, for every $\tau \in G$ we have

$$l(R_\tau) = l(\mu_\sigma(R_\tau)) \leq l(\text{Hom}_T(R_{\sigma\tau^{-1}}, R_\sigma)) = l(R_{\sigma\tau^{-1}}).$$

As $l(R) = \sum_{\tau \in G} l(R_\tau) = \sum_{\tau \in G} l(R_{\sigma\tau^{-1}})$ we get that μ_σ must be surjective.

2.15. PROPOSITION. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian gr-local finitely graded ring of type G , $T = R_e$, $\mathfrak{m}$ the maximal ideal of T . Then the following statements are equivalent:*

- (a) R is Gorenstein and artinian.
- (b) R is self-injective.
- (c) There is a $\sigma \in G$ such that μ_σ is injective and $R_\sigma \cong E$.
- (d) There is a $\sigma \in G$ such that μ_σ is an isomorphism and $R_\sigma \cong E$.
- (e) There is a $\sigma \in G$ such that $R(\sigma) \cong X_R(E)$ in R -gr.
- (f) $R \cong \bar{X}_R(E)$ in R -mod.

PROOF. (a) $\Leftrightarrow$ (b) is well known (see [B]).

(a) $\Leftrightarrow$ (d) by Theorem 2.13.

(d) $\Rightarrow$ (e) $\Rightarrow$ (f) is trivial.

(f) $\Rightarrow$ (b) by Proposition 0.1.

(d) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) by Lemma 2.14.

2.16. PROPOSITION. *Let $R = \bigoplus_{\sigma \in B} R_\sigma$ be a noetherian finitely graded ring of type G , $T = R_e$ and assume that every R_σ is a projective T -module. Then:*

- a) *If R is Gorenstein then T is Gorenstein.*
- b) *If T is Gorenstein and there is a $\sigma \in G$ such that $R_\sigma \cong T$ and ϱ_σ is an isomorphism, then R is Gorenstein.*

PROOF. Apply Lemma 1.10, Theorem 2.13, Corollary 1.14 and Theorem 1.12 and recall that if a free module is a canonical module then it must be of rank 1.

Part of the following result is due to C. Năstăsescu (see [N] Corollary 2.9).

2.17. COROLLARY. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a noetherian strongly graded ring of type G , G a finite group, $T = R_e$: Then R is Gorenstein iff T is Gorenstein.*

PROOF. $\varrho_e: R \rightarrow \text{Hom}_T(R, T)$ is an isomorphism as $(\varrho_e)_e: T \rightarrow \text{Hom}_T(T, T)$ is an isomorphism (see [NV] Corollary I.3.5).

3. Semi-trivial extensions.

Let T be a not necessarily commutative ring and let H be a T - T -bimodule. Assume that

$$\lambda = [-, -]: H \underset{T}{\otimes} H \rightarrow T$$

is a T - T -bilinear map satisfying

$$(*) \quad [h_1, h_2]h_3 = h_1[h_2, h_3] \quad \text{for all } h_1, h_2, h_3 \in H.$$

Define a multiplication on the abelian group $T \times H$ by setting

$$(t, h)(t', h') = (tt' + [h, h'], th' + ht')$$

for all $t, t' \in T$, $h, h' \in H$.

In this way $T \times H$ becomes a ring which will be denoted by $T \times_{\lambda} H$ and called the *semi-trivial extension of T by H and λ* . The ring $R = T \times_{\lambda} H$ can be considered as a graded ring of type $G = \{-1, 1\}$ by setting $R_1 = T$ and $R_{-1} = H$. Moreover every ring of type $G = \{-1, 1\}$ is of this form.

If $\lambda = 0$ then $T \times_{\lambda} H$ is usually denoted by $T \bowtie H$ and called the *trivial extension of T by H* .

It is easy to see that $T \times_{\lambda} H$ is a commutative ring iff T is a commutative ring, the left and the right T -module structures on H coincide and $[h_1, h_2] = [h_2, h_1]$ for every $h_1, h_2 \in H$ i.e. if the form λ is symmetric.

$[\cdot, \cdot]$ is said to be *non-degenerate* iff $\chi \in H$ and $[h, \chi] = 0$ for every $h \in H$ implies $\chi = 0$.

$[\cdot, \cdot]$ is said to be *strongly non-degenerate* iff the assignment $h \mapsto [h, -]$ yields an isomorphism between H and $\text{Hom}_T(H, T)$.

For the definition of μ_1 and μ_{-1} see 0.3 and for those of ϱ_1 and ϱ_{-1} see 2.9.

3.1. LEMMA. Assume that $R = T \times_{\lambda} H$ is commutative. Then:

- 1) $\mu_1 = \varrho_1$ is injective iff $[\cdot, \cdot]$ is non-degenerate.
- 2) $\mu_1 = \varrho_1$ is an isomorphism iff $[\cdot, \cdot]$ is strongly non degenerate
- 3) $\mu_{-1} = \varrho_{-1}$ is injective iff ${}_T H$ is faithful.
- 4) $\mu_{-1} = \varrho_{-1}$ is an isomorphism iff $\text{End}({}_T H)$ is canonically isomorphic to T .

PROOF. Straightforward.

3.2. PROPOSITION. Assume that $R = T \times_{\lambda} H$ is commutative. Then:

- a) R is gr-local iff T is local.
- b) R is noetherian iff T and ${}_T H$ are noetherian. In this case $\dim(R) = \dim(T)$.
- c) R is Cohen-Macaulay iff T is a Cohen-Macaulay ring, ${}_T H$ is a Cohen-Macaulay T -module and for every $\mathfrak{m} \in \text{Spec Max}(T)$, $\dim(T_{\mathfrak{m}}) = \dim(H_{\mathfrak{m}})$.

PROOF. a) Follows from Proposition 1.4, b) follows from Proposition 1.8, c) follows from Theorem 1.13.

3.3. PROPOSITION. *Assume that $R = T \times_{\lambda} H$ is commutative, gr-local and noetherian. Then the following statement are equivalent:*

- (a) *R is Gorenstein.*
- (b) *R is equidimensional and either H is a canonical module for T or $[\ , \]$ is strongly non-degenerate and T is Gorenstein.*

PROOF. See Theorem 2.13 and Lemma 3.1 and recall that if H is a canonical module for T then $\text{Hom}_T(H, H)$ is canonically isomorphic to T (see Theorem 2.6 and Lemma 2.7 or [HK] Satz 6.1).

3.4. PROPOSITION. *In the hypothesis of Proposition 3.3 the following statements are equivalent:*

- (a) *R is self-injective.*
- (b) *Either $H = E(T/\mathfrak{m})$ or T is self-injective and $[\ , \]$ is non-degenerate.*

PROOF. See Proposition 2.15 and Lemma 3.1.

Following result is essentially due to I. Reiten (see [R]) but see also [FGR] Theorem 5.6.

3.5. COROLLARY. *Let T be a commutative noetherian ring, H a finitely generated T -module, $R = T \ltimes H$. Let*

$$\Omega_1 = \{\mathfrak{m} \in \text{SpecMax}(T) : H_{\mathfrak{m}} \neq 0\}, \quad \Omega_2 = \text{SpecMax}(T) \setminus \Omega_1.$$

Then:

- (a) *R is a Gorenstein ring iff, for every $\mathfrak{m} \in \Omega_1$, $H_{\mathfrak{m}}$ is a canonical module for $T_{\mathfrak{m}}$ and, for every $\mathfrak{m} \in \Omega_2$, $T_{\mathfrak{m}}$ is Gorenstein.*
- (b) *R is self-injective iff, for every $\mathfrak{m} \in \Omega_1$, $H_{\mathfrak{m}}$ is the injective envelope of $T_{\mathfrak{u}}/\mathfrak{m}T_{\mathfrak{m}}$ in $T_{\mathfrak{m}}\text{-mod}$ and for every $\mathfrak{m} \in \Omega_2$, $T_{\mathfrak{m}}$ is self-injective.*

PROOF. FIRST of all note that when $\lambda = 0$ then λ is non-degenerate iff $H = 0$. Let $\mathfrak{m} \in \text{SpecMax}(T)$. Then $R_{\mathfrak{m}} = T_{\mathfrak{m}} \ltimes H_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{m}T_{\mathfrak{m}} \times H_{\mathfrak{m}}$. The conclusion now follows by Lemma 1.10 and Propositions 3.3 and 3.4.

3.6. AN EXAMPLE. We give now an example of a gr-local Gorenstein ring $R = T \times_{\lambda} H$ where ${}_{\tau}H$ is not faithful (and hence it is not a canonical module for T !). Let $\mathcal{K}$ be a Gorenstein local ring, $T = \mathcal{K}[x]/(x^2)$, $x = x + (x^2) \in T$. Set $L = T[z]/(z^2 - x, xz)$ $z = z + (z^2 - x, xz) \in L$ and let H be the cyclic T -submodule spanned by z in L .

Let $\lambda = [\ , \]: H \otimes_T H \rightarrow T$ be the T -bilinear map defined by

$$[t_1 z, t_2 z] = t_1 t_2 x \quad t_1, t_2 \in T.$$

Then $[\ , \]$ satisfies condition $(*)$ and it is easy to check that

$$T \times_{\lambda} H \cong \frac{\mathcal{K}[x, z]}{(x^2, z^2 - x, xz)} = R.$$

Clearly ${}_{\tau}H$ is not faithful as $xH = 0$. Anyway $[\ , \]$ is strongly non-degenerate. In fact if $0 \neq \chi \in H$ then $\chi = kx$ where $0 \neq k \in \mathcal{K}$ and then $[\chi, z] = kx \neq 0$. Clearly $\text{Ann}_T(x) = Tx$ and hence every morphism $f \in \text{Hom}_T({}_{\tau}H, T)$ is of the form $[h, -]$, $h \in H$.

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