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M. D. PÉREZ-RAMOS

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A p -Stable Action of the Automorphism Group of a Group.

M. D. PÉREZ-RAMOS (*)

SUMMARY - The following Theorem is proved: « Let G be a group and $\text{Aut}(G)$ its automorphism group. If $\text{Aut}(G)$ is not p -stable over G , p an odd prime, then $\text{Aut}(G)$ involves $SL(2, p)$ ». This Theorem is a generalization of a classic result on stability ([4], Th. 3.8.3).

Introduction. Notation.

In this note all groups will be finite. p will denote a prime number, and $O_p(G \bmod N)$ the inverse image in G of the p -radical of G/N . If we assume that $A \leq \text{Aut}(G)$ and H is a subgroup of G which is fixed by A , then, whenever $a \in A$, we denote $[H, a] = \langle [x, a] = x^{-1}x^a / x \in H \rangle$, $[H, a, a] = [[H, a], a]$, $[H, A] = \langle [H, a] / a \in A \rangle$, and $[H, A, A] = [[H, A], A]$. The remainder of the notation is standard, and is taken mainly from [4]. In particular, $\Pi(G)$ is the set of all primes which divide the order of the group G .

The first concept concerning stability is the following:

DEFINITION A ([4], 3.8). Let G be a group with no nontrivial normal p -subgroups, p odd. A faithful representation ψ of G on a vector space V over $GF(p^n)$ will be called p -stable provided no p -element of $(G)\psi$ has a quadratic minimal polynomial on V . Moreover, G is said to be p -stable if all such faithful representations of G are p -stable. (We will say G is p -stable linear to distinguish it from later definitions of p -stability).

(*) Indirizzo dell'A.: Dept. de Algebra, Universidad de Valencia, Doctor Moliner 50, 46100 Burjassot (Spagna).

And the main result on p -stability linear:

THEOREM A ([4], 3.8.3). Let G be a group with no nontrivial normal p -subgroups, p odd. If G is not p -stable linear, then G involves $SL(2, p)$.

Later we can find other definitions of p -stable groups, ([3], [5]) π -stable groups, π a set of primes, ([1]), and in general, $\mathcal{F}$ -stable groups, where $\mathcal{F}$ is either a saturated Formation, ([2]), or a Fitting class, ([8]). As well, we find the close relationship between the stability of a group G , and the fact that G does not involve the special affin groups $SA(2, p)$, for adequated primes p , ([3], [5], [2], [7]).

The aim of this paper is to study the structure of the automorphisms group of a group by using these ideas.

We give the following Definition:

DEFINITION 1. Let G be a group. We say that the automorphisms group of G , $\text{Aut}(G)$, is p -estable over G if, whenever A is a p -subgroup of $\text{Aut}(G)$ and B is a p -subgroup of G , such that B is fixed by A , (that is, $A \leq N_{\text{Aut}(G)}(B)$), and $[B, A, A] = 1$, then

$$A \leq O_p(N_{\text{Aut}(G)}(B) \text{ mod } C_{\text{Aut}(G)}(B)).$$

We obtain the following result:

THEOREM 1. Let G be a group. If $\text{Aut}(G)$ is not p -stable over G , p odd, then $\text{Aut}(G)$ involves the special linear group $SL(2, p)$.

First, we give a preliminary result:

LEMMA. For a group G , the following are equivalent:

- i) $\text{Aut}(G)$ is p -stable over G .
- ii) Whenever B is a p -subgroup of G , and x is a p -automorphism of G which fixes B , and such that $[B, x, x] = 1$, then

$$x \in O_p(N_{\text{Aut}(G)}(B) \text{ mod } C_{\text{Aut}(G)}(B)).$$

- iii) Whenever B is a p -subgroup of G , and x is an automorphism of G which fixes B , and such that $[B, x, x] = 1$, then

$$x \in O_p(N_{\text{Aut}(G)}(B) \text{ mod } C_{\text{Aut}(G)}(B)).$$

PROOF. i) $\rightarrow$ ii) Let B and x be as in (ii). The conclusion follows easily by taking $A = \langle x \rangle$, and using (i).

ii) $\rightarrow$ iii) Let B and x be as in (iii). Let $\Pi(\langle x \rangle) = \{p_1, \dots, p_n\}$. Then $\langle x \rangle = \langle x_1 \rangle \times \dots \times \langle x_n \rangle$, where $\langle x_i \rangle$ is a p_i -group, for each $i = 1, \dots, n$. Obviously, each x_i , $i = 1, \dots, n$, fixes B and satisfies $[B, \langle x_i \rangle, \langle x_i \rangle] = 1$.

If $p_i \neq p$, then $x_i \in C_{\text{Aut}(G)}(B)$. Otherwise, we have

$$x_i \in O_p(\text{Aut}(G)(B) \bmod C_{\text{Aut}(G)}(B))$$

by using (ii). So,

$$x \in O_p(N_{\text{Aut}(G)}(B) \bmod C_{\text{Aut}(G)}(B)).$$

iii) $\rightarrow$ i) Let A and B be as in Def. 1. For every $x \in A$, B and x satisfy the assumptions in (iii). Then, the conclusion is clear.

PROOF OF THEOREM 1. Because of the previous Lemma, if $\text{Aut}(G)$ is not p -stable over G , there exist a p -subgroup P of G , and a p -element x in $N_{\text{Aut}(G)}(P)$ such that $[P, x, x] = 1$, and

$$x \notin O_p(N_{\text{Aut}(G)}(P) \bmod C_{\text{Aut}(G)}(P)).$$

But, from ([5], Th. IX.7.8), this is equivalent to the existence of an element $g \in N_{\text{Aut}(G)}(P)$ such that $\langle x, x^g \rangle C_{\text{Aut}(G)}(P) / C_{\text{Aut}(G)}(P)$ is not a p -group.

Let $T = \langle x, x^g \rangle$. Let us refine the T -series $P \triangleright 1$ to a T -chief series $P = P_0 \triangleright P_1 \triangleright \dots \triangleright P_r = 1$ (*)

For each $i = 0, \dots, r-1$, let $\bar{P}_i = P_i / P_{i+1}$: So, it is clear that each $\bar{P}_i$, $i = 0, \dots, r-1$, is an abelian p -element group, that is, it is a vector space over $GF(p)$, and each $TC_{\text{Aut}(G)}(\bar{P}_i) / C_{\text{Aut}(G)}(\bar{P}_i) = \bar{T}_i$, $i = 0, \dots, r-1$, acts faithfully and irreducibly on $\bar{P}_i$: Therefore, since ([4], Th. 3.1.3), it follows that $O_p(\bar{T}_i) = 1$.

If $\bar{T}_i$ was a p -group for every $i = 0, \dots, r-1$, that is, $\bar{T}_i = 1$, then $T \leq \bigcap_{i=0}^{r-1} C_{\text{Aut}(G)}(\bar{P}_i)$. Whence T would stabilize the series (*) of P , and we would get a contradiction because $TC_{\text{Aut}(G)}(P) / C_{\text{Aut}(G)}(P)$ is not a p -group.

Therefore, there exists an $i \in \{0, \dots, r-1\}$ such that $\bar{T}_i \neq 1$, that is, $\bar{T}_i$ is not p -stable linear, (Def. A). Now, because of Th. A, $\bar{T}_i$ involves $SL(2, p)$, and clearly $\text{Aut}(G)$ involves $SL(2, p)$.

Using an argument similar to ([4], Th. 3.8.4), and Th. 1, we get the following:

COROLLARY. Let G be a group. If one of the following conditions is verified in $\text{Aut}(G)$:

- i) $\text{Aut}(G)$ is of odd order.
- ii) A Sylow 2-subgroup of $\text{Aut}(G)$ is abelian.
- iii) A Sylow 2-subgroup of $\text{Aut}(G)$ is dihedral.
- iv) $\text{Aut}(G)$ is isomorphic to $L_2(q)$.
- v) $\text{Aut}(G)$ is soluble and either $p \geq 5$ or $p = 3$ and $SL(2, 3)$ is not involved in $\text{Aut}(G)$.

Then, $\text{Aut}(G)$ is p -stable over G .

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