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Oka-Analyticity of the Essential Spectrum.

ENRICO CASADIO TARABUSI (*)

SUMMARY - Let X be a complex Banach space, G an open set in $\mathbb{C}$, and $\lambda \mapsto T_\lambda$ a holomorphic family of closed operators on X . We show here that $\lambda \mapsto \sigma_e(T_\lambda)$ is an analytic multifunction, where σ_e denotes the essential spectrum in *any* one of its several definitions.

1. Introduction.

Let X, Y be (nonzero) complex Banach spaces, and $\mathfrak{B}(X, Y)$ the Banach space of bounded linear operators from X to Y . It is known (see [9, Corollary 3.3 p. 371]) that, if G is an open set of $\mathbb{C}$, and $\lambda \mapsto T_\lambda: G \rightarrow \mathfrak{B}(X) = \mathfrak{B}(X, X)$ is a holomorphic map, then the multifunction spectrum $\lambda \mapsto \sigma(T_\lambda): G \rightarrow \text{cl}(\mathbb{C}) = \{\text{closed subsets of } \mathbb{C}\}$ is *Oka-analytic*, i.e. it is upper semicontinuous (u.s.c.; viz. $\{\lambda \in G: \sigma(T_\lambda) \subset A\}$ is open in G if A is open in $\mathbb{C}$ and $\mathbb{C} \setminus A$ is compact) and the open set $\Omega = \{(\lambda, z) \in G \times \mathbb{C}: z \notin \sigma(T_\lambda)\}$ is pseudoconvex: by abuse of terminology we say that σ is Oka-analytic on $\mathfrak{B}(X)$. Denoting by $\mathcal{C}(X, Y)$ the set of closed linear operators from X to Y , the same result holds, more generally, (see [10, Theorem 1 p. 121]) if $G \ni \lambda \mapsto T_\lambda$ is a Kato-holomorphic family with values in $\mathcal{C}(X) = \mathcal{C}(X, X)$: that is (cf. [4, in Section VII.1.2 p. 366]) if, for every $\lambda_0 \in G$ there exist G_0 open neighborhood of λ_0 in G , a Banach space Y , and holomorphic families $\lambda \mapsto U_\lambda, V_\lambda: G_0 \rightarrow \mathfrak{B}(Y, X)$ such that U_λ is one-to-one and $T_\lambda = V_\lambda U_\lambda^{-1}$ for every $\lambda \in G_0$: we say that σ is Oka-analytic on $\mathcal{C}(X)$.

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We shall prove the same results for other multifunctions of spectral type, each one usually referred to as essential spectrum. The standard consequences of Oka-analyticity (we refer to [1], [11], [12] for precise statements and proofs) thus extend to them: for instance, several functions of each essential spectrum (such as the radius, the inverse of the distance from a fixed point, the k -th diameter, for any $k \in \mathbb{N}$, the capacity, etc.) are plurisubharmonic on $\mathcal{B}(X)$ (or $\mathcal{C}(X)$). Also, we have the analyticity of spectral sets, the finite scarcity and countable scarcity theorems; and many others.

2. Essential spectrum.

A linear operator $T \in \mathcal{C}(X, Y)$ will be said to be *Fredholm* if its range $R(T)$ is closed, and if the dimensions of its kernel $N(T)$ and of its co-kernel $Y/R(T)$ are finite: such dimensions will be called *nullity* and *deficiency* of T , resp., and denoted by $\text{nul}(T)$ and $\text{def}(T)$; while the *index* of T will be $\text{ind}(T) = \text{nul}(T) - \text{def}(T)$. If $\mathcal{C}(X, Y)$ is endowed with the «gap» (metrizable) topology (see [4, in Section IV.2.4 p. 201-202]), which induces the norm topology on the open set $\mathcal{B}(X, Y)$, then $\mathcal{F}(X, Y) = \{T \in \mathcal{C}(X, Y) \text{ that are Fredholm}\}$ is open in $\mathcal{C}(X, Y)$, and on each of its connected components the function ind is constant, while nul and def are, in general, just u.s.c. (cf. [4, Theorem IV.5.17 p. 235]). By $\mathcal{F}_0(X, Y)$ we shall denote the union of the components of $\mathcal{F}(X, Y)$ where $\text{ind} \equiv 0$; and by $\mathcal{K}(X, Y)$ the set of compact operators from X to Y .

Each of the following sets is customarily referred to as essential spectrum of $T \in \mathcal{C}(X)$ (cf. [7, p. 365; 13, § 1 p. 142; 2, Definition 11 p. 107]):

a) the *Wolf spectrum*

$$\sigma_{ew}(T) = \{z \in \mathbb{C} : T - zI \notin \mathcal{F}(X) = \mathcal{F}(X, X)\};$$

b) the *Weyl spectrum* $\sigma_{em}(T) = \bigcap_{K \in \mathcal{K}(X) = \mathcal{K}(X, X)} \sigma(T + K)$;

c) the *Browder spectrum* $\sigma_{eb}(T) = \left\{z \in \mathbb{C} : z \text{ is an accumulation point of } \sigma(T), \text{ or } R(T - zI) \text{ is not closed, or } z \text{ is an eigenvalue of } T \text{ of infinite algebraic multiplicity (i.e. } \dim \left(\bigcup_{k=1, \dots, \infty} N(T - zI)^k \right) = \infty) \right\}$.

Thus the Weyl spectrum is the largest subset of the spectrum which is invariant under compact perturbations: furthermore (see [8, Theorem VII.5.4. p. 180])

$$\sigma_{em}(T) = \{z \in \mathbf{C}: T - zI \notin \mathcal{F}_0(X) = \mathcal{F}_0(X, X)\}.$$

If $\text{Cal}(X) = \mathcal{B}(X)/\mathcal{K}(X)$ is the Calkin algebra of X , then (see [8]) the Wolf spectrum of $T \in \mathcal{B}(X)$ coincides with the (Banach algebra) spectrum $\sigma_{\text{Cal}(X)}$ in $\text{Cal}(X)$ of the coset $[T]_{\text{Cal}(X)}$ of T .

Fixed $T \in \mathcal{C}(X)$, due to forementioned properties the function $z \mapsto \text{ind}(T - zI)$ is constant on each component W of the complement of $\sigma_{ew}(T)$: but we also have that $z \mapsto \text{nul}(T - zI)$, $\text{def}(T - zI)$ are constant on W , except for a discrete subset of W (see [4, Theorem IV.5.31 p. 41]). This implies that $\sigma_{eb}(T)$ is the complement of the union of those W 's which are not contained in $\sigma(T)$, viz. such that $W \ni z \mapsto \text{nul}(T - zI) \equiv \text{def}(T - zI) \equiv 0$ « a.e. » (see [2]).

Thanks to the various observations made so far, we have $\sigma_{ew}(T) \subset \sigma_{em}(T) \subset \sigma_{eb}(T) \subset \sigma(T)$ (all closed subsets), and each inclusion may be strict, even if $T \in \mathcal{B}(X)$. (If X is finite-dimensional, all the essential spectra are obviously empty; while if it is Hilbert and T is self-adjoint, then they coincide.)

THEOREM 1. *The Wolf spectrum is Oka-analytic on $\mathcal{C}(X)$.*

PROOF. The upper semicontinuity of σ_{ew} easily follows from the openness of $\mathcal{F}(X)$ in $\mathcal{C}(X)$.

Let $\text{Cal}(X, Y) = \mathcal{B}(X, Y)/\mathcal{K}(X, Y)$ as a Banach space: the product of composition $\mathcal{B}(X, Y) \times \mathcal{B}(Y, X) \rightarrow \mathcal{B}(Y)$ induces a continuous bilinear product $\text{Cal}(X, Y) \times \text{Cal}(Y, X) \rightarrow \text{Cal}(Y)$. As in the case $X = Y$, one shows that $T \in \mathcal{B}(X, Y)$ is Fredholm if and only if its coset $[T]_{\text{Cal}(X, Y)}$ has a two-sided inverse in $\text{Cal}(Y, X)$: if so, such inverse is unique by the associativity of the product, and (just like when $X = Y$) it depends continuously and holomorphically on $[T]_{\text{Cal}(X, Y)}$, so on T .

The notion of analytic multifunction being obviously local, we can assume the holomorphic families $\lambda \mapsto U_\lambda, V_\lambda$ of operators in $\mathcal{B}(Y, X)$ relative to $\lambda \mapsto T_\lambda$ (see introduction) to be defined on the whole of G (Y being a suitable Banach space). If $(\lambda, z) \in G \times \mathbf{C}$, then $T_\lambda - zI = (V_\lambda - zU_\lambda)U_\lambda^{-1}$, and the range, nullity and deficiency of $T_\lambda - zI$

are the same, resp., of $V_\lambda - zU_\lambda$; so

$$\begin{aligned}\Omega_{ew} &= \{(\lambda, z) \in G \times \mathbf{C} : z \notin \sigma_{ew}(T_\lambda)\} = \\ &= \{(\lambda, z) \in G \times \mathbf{C} : V_\lambda - zU_\lambda \in \mathcal{F}(Y, X)\} = \\ &= \{(\lambda, z) \in G \times \mathbf{C} : [V_\lambda - zU_\lambda]_{\text{Cal}(Y, X)} \text{ has a two-sided inverse in } \text{Cal}(X, Y)\}.\end{aligned}$$

In view of the remarks made above, the conclusion can be drawn exactly as in [3, Proof of the Theorem, p. 1] using the nonextendability to $(G \times \mathbf{C}) \cap \partial\Omega$ of any restriction of the holomorphic mapping

$$\lambda \mapsto ([V_\lambda - zU_\lambda]_{\text{Cal}(Y, X)})^{-1} : \Omega \rightarrow \text{Cal}(X, Y). \quad \square$$

(Examination of $V_\lambda - zU_\lambda$ instead of $(V_\lambda - zU_\lambda)U_\lambda^{-1}$, as made in the preceding proof, allows a quicker proof of the Oka-analyticity of the spectrum on $\mathcal{C}(X)$ than [10, Proof of Theorem 1 p. 123].)

THEOREM 2. *The Weyl spectrum is Oka-analytic on $\mathcal{C}(X)$.*

PROOF. Since Kato-holomorphic families are continuous (by [4, Theorem IV.2.29 p. 207]), if $\varphi: G \times \mathbf{C} \rightarrow \mathcal{B}(X)$ is given by $\varphi(\lambda, z) = T_\lambda - zI$, then the open set $\Omega_{em} = \{(\lambda, z) \in G \times \mathbf{C} : z \notin \sigma_{em}(T_\lambda)\} = \varphi^{-1}(\mathcal{F}_0(X))$ is a union of components of the open set $\Omega_{ew} = \varphi^{-1}(\mathcal{F}(X))$, which is pseudoconvex by Theorem 1. $\square$

As to the Browder spectrum, we cannot infer its Oka-analyticity directly from that of the Wolf or the Weyl spectrum as done for Theorem 2. In fact $\Omega_{eb} = \{(\lambda, z) \in G \times \mathbf{C} : z \notin \sigma_{eb}(T_\lambda)\}$ is not, in general, a union of components of Ω_{em} , because of the lack of lower semicontinuity of nul , def . Yet one could conjecture, in view of some of the properties recalled earlier, that the functions nul' , def' are locally constant on $\mathcal{F}_0(X)$, where

$$\text{nul}'(T) = \text{nul}(T - zI), \quad \text{def}'(T) = \text{def}(T - zI), \quad \text{for } 0 < |z| \ll 1.$$

This conjecture is true only when X is finite-dimensional (in which case nul' , def' vanish identically), as the following counter-example shows.

COUNTEREXAMPLE 3. Let: X be the Hilbert space $l^2 = L^2(\mathbf{Z}, \nu)$ (where ν is the counting measure); $P, A \in \mathcal{B}(X)$ the projection on the

zeroth coordinate and the onestep shift to the right, resp.; $G = \mathbb{C}$; $\lambda \mapsto T_\lambda = A(I - \lambda P): G \rightarrow \mathcal{B}(X)$. If $\{e_j\}_{j \in \mathbb{Z}}$ is the canonical basis of l^2 , then $N(T_1) = [e_0]$, $R(T_1) = [e_1]^\perp$: so $T_1 \in \mathcal{F}_0(X)$, therefore $T_\lambda - zI \in \mathcal{F}_0(X)$ for $|\lambda - 1| \ll 1$, $|z| \ll 1$. But $N(T_1 - zI) = [\sum_{j=0, \dots, +\infty} z^j e_{-j}]$ for $|z| \ll 1$, so $\text{nul}'(T_1) = \text{def}'(T_1) = 1$; while T_λ is invertible for $0 < |\lambda - 1| \ll 1$, thus $\text{nul}'(T_\lambda) = \text{def}'(T_\lambda) = 0$. Hence $(1, 0) \in \Omega_{em} \setminus \Omega_{eb}$, whereas $(\lambda, 0) \in \Omega_{eb}$ if $0 < |\lambda - 1| \ll 1$.

The upper semicontinuity of the Browder spectrum is interesting in its own, so we give it separately.

LEMMA 4. *The Browder spectrum is u.s.c. on $\mathcal{C}(X)$.*

PROOF. Let $T \in \mathcal{C}(X)$. Then $\sigma(T) \setminus \sigma_{eb}(T)$ consists only of isolated points of $\sigma(T)$: if A is a compact-complemented open subset of $\mathbb{C}$ containing $\sigma_{eb}(T)$, then $\sigma(T) \setminus A = \{z_1, \dots, z_k\}$. Since σ is u.s.c., for $T' \in \mathcal{C}(X)$ near T we have $\sigma(T') \subset A \cup \left(\bigcup_{j=1, \dots, k} B(z_j, \varepsilon) \right)$ (where $B(z_j, \varepsilon) = \{z \in \mathbb{C}: |z - z_j| < \varepsilon\}$), with $\varepsilon > 0$ such that the above union is disjoint. Using the Dunford integral calculus, for every such T' [4, Theorem IV.3.16 p. 212] provides a splitting of X into a $(k+1)$ -ple direct sum $X = \bigoplus_{j=0, \dots, k} X_j(T')$ of the generalized eigenspaces associated to $\sigma(T') \cap A$, $\sigma(T') \cap B(z_j, \varepsilon)$, $j = 1, \dots, k$; furthermore $\dim X_j(T')$ is independent of T' . Fix $j = 1, \dots, k$. Since the restriction of $T - z_j I$ to $X_j(T)$ is a quasinilpotent operator, and its approximated nullity and deficiency (defined equal to the nullity and deficiency, resp., if the range is closed, or to $+\infty$ otherwise: cf. [4, Theorem IV.5.10 p. 233]) are finite (because $T - z_j I$ is Fredholm), [4, Theorem IV.5.30 p. 240] yields that $\dim X_j(T)$ is finite. Hence $\sigma(T') \cap B(z_j, \varepsilon)$ is a finite set $\{z_{j1}, \dots, z_{jk_j}\}$, and for $j' = 1, \dots, k_j$ the range of $T' - z_{jj'} I$ is closed and the algebraic multiplicity of $z_{jj'}$ in T' is finite; that is, $\sigma_{eb}(T') \cap B(z_j, \varepsilon)$ is empty. Therefore $\sigma_{eb}(T') \subset A$. $\square$

Let $\lambda \mapsto \Sigma_\lambda: G \rightarrow \text{cl}(\mathbb{C})$ be an Oka-analytic multifunction, and $\lambda \in G$. An isolated point z of Σ_λ is a *good isolated point*, or *g.i.p.*, (for Σ) at λ if there exists $\delta > 0$ such that $\sigma(T_{\lambda'}) \cap B(z, \delta)$ is finite for $|\lambda' - \lambda| < \delta$. The Oka-Nishino theorem (cf. [6, Corollary 5.5 p. 557]) asserts that the multifunction $\lambda \mapsto D\Sigma'_\lambda = \Sigma_\lambda \setminus \{\text{g.i.p.'s at } \lambda\}$ is itself Oka-analytic.

THEOREM 5. *The Browder spectrum is Oka-analytic on $\mathcal{C}(X)$.*

PROOF. Let $\lambda \in G$. From the proof of Lemma 4 we get that all the points in $\sigma(T_\lambda) \setminus \sigma_{eb}(T_\lambda)$ are g.i.p.'s at λ , but in general not conversely: for instance, if $T_\lambda \equiv 0$ on G , then 0 is a g.i.p. at any $\lambda \in G$, but belongs to $\sigma_{eb}(T_\lambda)$ if X is infinite-dimensional. Thus we have $D\sigma(T_\lambda) \subset \sigma_{eb}(T_\lambda) \subset \sigma(T_\lambda)$ for each $\lambda \in G$, the first and third multifunction being Oka-analytic. In order to prove that Ω_{eb} is pseudoconvex it will suffice to show that for each $\lambda_0 \in G$ and $z_0 \in \sigma_{eb}(T_{\lambda_0}) \setminus D\sigma(T_{\lambda_0})$ there exists a neighborhood U of (z_0, λ_0) in $D\Omega = \{(\lambda, z) \in G \times \mathbb{C} : z \notin D\sigma(T_\lambda)\}$ such that $U \cap \Omega_{eb}$ is pseudoconvex. Because $D\Omega$ is open, U can be taken to be a bidisk; also, we will assume $\lambda_0 = z_0 = 0$.

Since 0 is isolated in $\sigma(T_0)$ and Ω is open we can choose $\varepsilon > \delta > 0$ so that $\sigma(T_\lambda) \cap B(0, \varepsilon) = \sigma(T_\lambda) \cap B(0, \varepsilon - \delta)$ for $|\lambda| < \delta$: so $\lambda \mapsto \Sigma_\lambda = \sigma(T_\lambda) \cap B(0, \varepsilon) : B(0, \delta) \rightarrow \text{cl}(\mathbb{C})$ is still Oka-analytic. Because 0 is a g.i.p. at 0, by [1, Theorem 3.8] δ may be taken small enough that the cardinality of Σ_λ be finite and independent of $\lambda \in B(0, \delta) \setminus \{0\}$, say k . Furthermore for any such λ there exists $\delta_\lambda > 0$ and k holomorphic functions $h_1, \dots, h_k : B(\lambda, \delta_\lambda) \rightarrow \mathbb{C}$ such that $\Sigma_{\lambda'} = \{h_1(\lambda'), \dots, h_k(\lambda')\}$ for each $\lambda' \in B(\lambda, \delta_\lambda)$. As in the proof of Lemma 4 we have that $h_j(\lambda') \in \sigma_{eb}(T_{\lambda'})$ if and only if the dimension of the generalized eigenspace of $T_{\lambda'}$ associated to $h_j(\lambda')$ is finite; such dimension being stable, either $h_j(\lambda') \in \sigma_{eb}(T_{\lambda'})$ for all $\lambda' \in B(\lambda, \delta_\lambda)$, or for none of them. Therefore k_0 exists, with $1 \leq k_0 \leq k$, such that for each $\lambda \in B(0, \delta) \setminus \{0\}$ the function h_j can be rearranged in such a way that the former alternative holds for exactly $h_1, \dots, h_{k_0}$. If $f : U = B(0, \delta) \times B(0, \varepsilon) \rightarrow \mathbb{C}$ is defined through $f(\lambda, z) = \prod_{j=1, \dots, k_0} (z - h_j(\lambda))$ for $\lambda \neq 0$, and $f(0, z) = z^{k_0}$, such f is well-defined, holomorphic where $\lambda \neq 0$, and (by the upper semicontinuity of $\lambda \mapsto \Sigma_\lambda$, which is implied by that of $\lambda \mapsto \sigma(T_\lambda)$) continuous where $\lambda = 0$. By Raddò's theorem the function f is holomorphic on U : since $\{f = 0\} = U \setminus \Omega_{eb}$, no restriction of $1/f : U \cap \Omega_{eb} \rightarrow \mathbb{C}$ can be extended to any point of $U \setminus \Omega_{eb}$. Thus $U \cap \Omega_{eb}$ is pseudoconvex, because U is. $\square$

The following corollary of the three preceding theorems appeared in [5, Theorem 13 p. 320] for the Weyl case, while it can be easily proven directly in the Wolf case using the Oka-analyticity of the spectrum on a Banach algebra (Cal(X) here: see [1, Theorem 3.2 p. 46]).

COROLLARY 6. *The Wolf, Weyl, and Browder spectrum are all Oka-analytic on $\mathfrak{B}(X)$.* $\square$

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