A theorem on direct products of Slender modules

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A Theorem on Direct Products of Slender Modules.

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1. Introduction.

Let $R$ be a ring. A class $C$ of $R$-modules is called transitive if, for each, $X, Y, Z$ in $C$, $\text{Hom}_R(X, Y) \neq 0 \neq \text{Hom}_R(Y, Z)$ implies $\text{Hom}_R(X, Z) \neq 0$. If $\text{Hom}_R(X, Y) \neq 0 \neq \text{Hom}_R(Y, X)$, then $X$ and $Y$ have the same type. Our main result is the following.

**Theorem 1.** Let $C$ be a transitive class of slender $R$-modules. If $\prod G_i = A \oplus B$ with $G_i$ in $C$ and $I$ countable, then $A$ is isomorphic to a direct product of members of $C$ if this result is true whenever all $G_i$'s have the same type.

In section 4, using a result from [4], we generalize Theorem 1 to the case where $I$ is any set of non-measurable cardinality.

In particular Corollary 8 includes the case where $R$ is the ring of integers and $C$ is the class of rank one torsion-free reduced abelian groups. This case is Theorem 4.3 in [4]. The proof there was defective (Lemma 4.2 was false). Thus our proof here of Theorem 1 (hence of Corollary 8) supplants the proof of Theorem 4.3 in [4].

2. Preliminaries.

All rings are associative with unity and all modules (except in Cor. 9) are left unital. Let $C$ be a transitive class of $R$-modules.

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« Having the same type » is an equivalence relation on the members of \( C \). If type \( X = t \), type \( Y = s \), and \( \text{Hom}_R(X, Y) \neq 0 \), we write \( t \leq s \). This relation is a partial order on the types of members of \( C \). By \( t < s \) we mean \( t \leq s \) and \( s \leq t \). An \( R \)-module has \textit{finite rank} if it is isomorphic to a submodule of a finite direct sum of members of \( C \). A submodule \( X \) is \textit{fully invariant} in \( Y \) if: for any homomorphism \( f: Y \to Y \), \( f(X) \subseteq X \). In this case a decomposition of \( Y \) induces a decomposition of \( X \).

The first infinite ordinal (a cardinal) is \( \omega \) and it is identified with the set of finite ordinals. Let \( R^\omega \) be the direct product of \( \omega \) copies of \( R \). An \( R \)-module \( X \) is \textit{slender} if each \( R \)-homomorphism \( R^\omega \to X \) sends all but a finite number of components of \( R^\omega \) to 0.

We shall presume a basic knowledge of slender modules and direct products of modules such as is found in [1, 2 and 4] and in the papers referenced there. Lemmas 3.1 and 3.2 in [4] are basic and, being well-known, are often used without mention.

Observe that the class \( C \) in Theorem 1 satisfies the following \textit{Clause}: If \( \prod_i G_i = A \oplus B \) where \( |I| < \omega \) and all \( G_i \)'s have the same type, then \( A \) is isomorphic to a direct product of members of \( C \). It will be clear after Lemma 2 that the components of \( A \) have the same type as the \( G_i \)'s.

3. \textbf{Proof of Theorem 1.}

Write \( V = \prod_i G_i = A \oplus B \) as in Theorem 1. Let \( \alpha: V \to A \), \( \beta: V \to B \) and \( \alpha_i: V \to A \to G_i \) be the obvious projections. Let each \( G_i \) have type \( t_i \). For a fixed type \( s \) write \( V_s = \prod_i G_i \) and \( V^s = \prod_{t_i > s} G_i \).

If \( J \subseteq I \), then \( V_J = \prod_{i \in J} G_i \). We will adhere strictly to this notation.

**Lemma 2.** For each type \( s \)

1. \( V_s \oplus V^s \) and \( V^s \) are fully invariant in \( V \),
2. \( \alpha(V_s \oplus V^s) = A \cap (V_s \oplus \beta(V^s)) \oplus \alpha(V^s) \),
3. We may assume the decomposition \( V = \prod_i G_i \) is such that \( V_s = \prod_{X_s} G_i \oplus \prod_{Y_s} G_i \) for subsets \( X_s, Y_s \) of \( I \) so that \( \alpha \) induces an isomorphism between \( V_{X_s} \) and \( \alpha(V_{X_s}) = A \cap (V_s \oplus \beta(V^s)) \), which is thus a direct product of members of \( C \) of type \( s \).
PROOF. (1) is clear and (2) follows (1) by standard arguments. From (1) the members of any new C-decomposition of \( V_s \) have type \( s \). Also \( \beta(V_s \oplus V^s) = B \cap (V_s \oplus \alpha(V^s)) \oplus \beta(V^s) \) and \( V_s \oplus V^s = [A \cap (V_s \oplus \alpha(V^s))] \oplus \beta(V^s) \). Now \( V_s \) is isomorphic to the summand in the bracket and, by the Clause, each summand in the bracket is isomorphic to a direct product of members of \( C \) of type \( s \). It we project each of these summands to \( V_s \) we get the desired decomposition of \( V_s \).

**Lemma 3.** Let \( T \) be a finite set of types and let \( s \) be a minimal type in \( T \). Assume Lemma 2. Set \( V_T = \prod_j G_j \) where \( J = \{i : t_i > \text{some } t \} \) in \( T \) or \( t_i \in T \setminus \{s\} \) and set \( \tau V = \prod_k G_k \) where \( K = \{i : t_i \neq \text{any } t \} \) in \( T \); so \( V = \tau V \oplus V_s \oplus \tau V^s \). Then

1. \( V_s \oplus V^s \) and \( V^s \) are fully invariant in \( V \),
2. \( A = A \cap (\tau V \oplus \beta(V_s \oplus V^s)) \oplus \alpha(V^s) \oplus \alpha(V^s) \) for \( X_s \) as in Lemma 2.

**Proof.** (1) is clear. Hence \( A = A \cap (\tau V \oplus \beta(V_s \oplus V^s)) \oplus \alpha(V^s) \) which, being in \( \alpha(V_s) \), is in \( \alpha(V^s) \). By Lemma 2 \( \alpha(V^s) = A \cap (V_s \oplus \beta(V^s)) \). Substitution yields (2).

**Lemma 4.** If \( C \) is a finite rank direct summand of \( V \), then \( C \) is isomorphic to a finite direct sum of members of \( C \).

**Proof.** By slenderness \( C \) is a direct summand of a finite direct sum of \( G_i \)'s. If all \( G_i \)'s have the same type, the Clause applies. For the general case we may use Baer's classical proof for a direct summand of a finite direct sum of rank one torsionfree abelian groups (see Theorem 86.7 in [1]).

**Lemma 5.** Suppose \( m \in I \). Then \( A = E \oplus F \) where \( E \) has finite rank and \( \alpha_m(F) = 0 \).

**Proof.** Since \( G_m \) is slender, \( \alpha_m(V_i) = 0 \) for all types \( t \) except those in a finite set \( T \). We induct on the order of \( T \). If \( T = \emptyset \), \( \alpha_m(A) \subseteq \alpha_m(V) = 0 \); so \( E = 0 \) and \( F = A \) satisfy the Lemma. Otherwise let \( s \) be a minimal type in \( T \). From Lemma 3 we write \( A = A \cap (\tau V \oplus \beta(V_s \oplus V^s)) \oplus \alpha(V^s) \oplus \alpha(V^s) \). Note that the left summand is in \( \alpha(\tau V) \) and \( \alpha_m(\tau V) = 0 \). Since \( \alpha(V^s) \) is a product of \( G_i \)'s of type \( s \),
by slenderness $\alpha(V) = D \oplus E_1$ where $\alpha_m(D) = 0$ and $E_1$ is a finite direct sum of $G_i$'s of type s. Let $F_1 = A \cap (\alpha(V) \oplus \beta(V)) \oplus D$ Now $A = F_1 \oplus E_1 \oplus \alpha(V)$ where $\alpha_m(F_1) = 0$ and $E_1$ has finite rank. Next consider $V^\beta = \alpha(V^\beta) \oplus \beta(V^\beta)$. Let $T_1$ be the set of $t$'s such that $V_t \subseteq V^\beta$ and $\alpha_m(V_t) \neq 0$. Then $T_1 = T \setminus \{s\}$ and $|T_1| < |T|$. By the induction hypothesis $\alpha(V^\beta) = E_2 \oplus F_2$ where $E_2$ has finite rank and $\alpha_m(F_2) = 0$. Therefore $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$ satisfy the lemma.

**Remark.** If $C$ is the class of rank one torsion-free reduced abelian groups, Lemma 5 follows readily from the fact that $V$ and any direct summand of $V$ is coseparable (see Proposition 1.2 and Theorem 5.8 in [3]). Thus the kernel of the map $\alpha_m: A \to G_m$ must contain a direct summand $F$ of $A$ with finite rank complement $E$.

**Proof of Theorem 1.** By Lemma 4 we may assume $I = \omega$. We may also assume $V$ has the decomposition in Lemma 2. We wish to find submodules $A_n, A^n$ in $A$ for each $n$ in $\omega$ such that:

1. $A = A_n$, $A^n = A_n \oplus A^{n+1}$ for each $n$,
2. Each $A_n$ has finite rank,
3. For fixed $m A_n(A_n) \alpha_m(A_n) = 0$ for almost all $n$,
4. $\cap A^n = 0$.

Then, by Proposition 3.3 in [4], we will have $A \cong \prod A_n$ and Lemma 4 above will complete the proof.

Let $A = A^0$. If $m > 0$ and if $A^m$ is a direct summand of $V$, then, by letting $A^m$ be $A$ in Lemma 5, we can find a decomposition $A^m = A_m \oplus A^{m+1}$ where $A_m$ has finite rank and $\alpha_m(A^{m+1}) = 0$. By induction we can find $A_n, A^n$ for each $n$ in $\omega$ to satisfy (1) and (2) above and where $\alpha_n(A^n) = 0$ for each $n$. For fixed $m \alpha_m(A_n) \subseteq \alpha_m(A^n) = 0$ for all $n > m$. This yields (3) and (4) which completes the proof.

4. Generalization.

**Theorem 6.** Let $C$ be a transitive class of slender $R$-modules. If $\prod G_i = A \oplus B$ with $G_i \in C$ and $|I|$ non-measurable, then $A$ is isomorphic to a direct product of members of $C$ if this statement is true whenever $I$ is countable and all $G_i$'s have the same type.
PROOF. The result follows from Theorem 1 and the following proposition.

PROPOSITION 7 (Theorem 3.7 in [4]). Suppose an $R$-module $P$ has decompositions $P = \prod G_i = A \oplus B$ where $|I|$ is non-measurable and each $G_i$ is slender. Then $A \cong \prod A_i$ where each $A_i$ is isomorphic to a direct summand of a direct product of countably many $G_i$'s.

As an aside we mention that the conclusion of Lemma 3.6 in [4] is misstated. It should be: Then $A \cong \prod A_j$ and $B \cong \prod B_j$ where $A_j = A \cap (P_j \oplus \beta(P_j'))$ and $B_j = B \cap (P_j \oplus \alpha(P_j'))$. The proof of the Lemma, with obvious modifications, remains the same.

5. Applications.

COROLLARY 8 (see Theorem 13 in [5]). Let $R$ be a commutative Dedekind domain which is not a field or a complete discrete valuation ring. Let $= A \oplus B$ where $|I|$ is non-measurable and each $G_i$ is a rank one torsion-free reduced $R$-module. Then $A$ is a direct product of rank one $R$-modules.

PROOF. Let $C$ be the class of rank one torsion-free reduced $R$-modules. Each module in $C$ is slender by Proposition 3 in [5]. If $X$ and $Y$ are in $C$ and $f:X \to Y$ is a non-zero homomorphism, it is a monomorphism. Hence $\text{Hom}_R(X, Y) \neq 0$ if and only if $X$ is isomorphic to a submodule of $Y$. It follows that $C$ is a transitive class of slender $R$-modules and that, for this class, the definitions of « type » in this paper and in Definition 9 in [5] are equivalent. By Proposition 12 in [5] the Corollary is true if all $G_i$'s have the same type. Theorem 6 above completes the proof.

COROLLARY 9. Let $R$ be a ring and let $C$ be a transitive class of slender left $R$-modules such that modules of the same type are isomorphic and, for each $X$ in $C$, projective right $\text{End}_R X$-modules are free. If $\prod G_i = A \oplus B$ where $G_i \in C$ and $|I|$ is non-measurable, then $A$ is isomorphic to a direct product of $G_i$'s.

PROOF. By Theorem 3.1 in [2] the result is true if all $G_i$'s have the same type. Theorem 6 above completes the proof.
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REFERENCES


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