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A Theorem on Direct Products of Slender Modules.

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1. Introduction.

Let R be a ring. A class $\mathcal{C}$ of R-modules is called *transitive* if, for each, X, Y, Z in $\mathcal{C}$, $\mathrm{Hom}_R(X, Y) \neq 0 \neq \mathrm{Hom}_R(Y, Z)$ implies $\mathrm{Hom}_R(X, Z) \neq 0$. If $\mathrm{Hom}_R(X, Y) \neq 0 \neq \mathrm{Hom}_R(Y, X)$, then X and Y have the same *type*. Our main result is the following.

THEOREM 1. Let $\mathcal{C}$ be a transitive class of slender R-modules. If $\prod_I G_i = A \oplus B$ with G_i in $\mathcal{C}$ and I countable, then A is isomorphic to a direct product of members of $\mathcal{C}$ if this result is true whenever all G_i's have the same type.

In section 4, using a result from [4], we generalize Theorem 1 to the case where I is any set of non-measurable cardinality.

In section 5 we present some applications of the theorem. In particular Corollary 8 includes the case where R is the ring of integers and $\mathcal{C}$ is the class of rank one torsion-free reduced abelian groups. This case is Theorem 4.3 in [4]. The proof there was defective (Lemma 4.2 was false). Thus our proof here of Theorem 1 (hence of Corollary 8) supplants the proof of Theorem 4.3 in [4].

2. Preliminaries.

All rings are associative with unity and all modules (except in Cor. 9) are left unital. Let C be a transitive class of R-modules.

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« Having the same type » is an equivalence relation on the members of $\mathcal{C}$. If type $X = t$, type $Y = s$, and $\mathrm{Hom}_R(X, Y) \neq 0$, we write $t \leqslant s$. This relation is a partial order on the types of members of $\mathcal{C}$. By $t < s$ we mean $t \leqslant s$ and $s \not\leqslant t$. An R-module has *finite rank* if it is isomorphic to a submodule of a finite direct sum of members of $\mathcal{C}$. A submodule X is *fully invariant* in Y if: for any homomorphism $f: Y \to Y$, $f(X) \subseteq X$. In this case a decomposition of Y induces a decomposition of X.

The first infinite ordinal (a cardinal) is ω and it is identified with the set of finite ordinals. Let R^ω be the direct product of ω copies of R. An R-module X is *slender* if each R-homomorphism $R^\omega \to X$ sends all but a finite number of components of R^ω to 0.

We shall presume a basic knowledge of slender modules and direct products of modules such as is found in [1, 2 and 4] and in the papers referenced there. Lemmas 3.1 and 3.2 in [4] are basic and, being well-known, are often used without mention.

Observe that the class $\mathcal{C}$ in Theorem 1 satisfies the following *Clause*: If $\prod_I G_i = A \oplus B$ where $|I| \leqslant \omega$ and all G_i's have the same type, then A is isomorphic to a direct product of members of $\mathcal{C}$. It will be clear after Lemma 2 that the components of A have the same type as the G_i's.

3. Proof of Theorem 1.

Write $V = \prod_I G_i = A \oplus B$ as in Theorem 1. Let $\alpha: V \to A$, $\beta: V \to B$ and $\alpha_i: V \to A \to G_i$ be the obvious projections. Let each G_i have type t_i. For a fixed type s write $V_s = \prod_{t_i = s} G_i$ and $V^s = \prod_{t_i > s} G_i$. If $J \subseteq I$, then $V_J = \prod_{i \in J} G_i$. We will adhere strictly to this notation.

LEMMA 2. For each type s

(1) $V_s \oplus V^s$ and V^s are fully invariant in V,

(2) $\alpha(V_s \oplus V^s) = A \cap (V_s \oplus \beta(V^s)) \oplus \alpha(V^s)$,

(3) We may assume the decomposition $V = \prod_I G_i$ is such that $V_s = \prod_{X_s} G_i \oplus \prod_{Y_s} G_i$ for subsets X_s, Y_s of I so that α induces an isomorphism between V_{X_s} and $\alpha(V_{X_s}) = A \cap (V_s \oplus \beta(V^s))$, which is thus a direct product of members of $\mathcal{C}$ of type s.

PROOF. (1) is clear and (2) follows (1) by standard arguments. From (1) the members of any new $\mathcal{C}$-decomposition of V_s have type s. Also $\beta(V_s \oplus V^s) = B \cap (V_s \oplus \alpha(V^s)) \oplus \beta(V^s)$ and $V_s \oplus V^s = [A \cap (V_s \oplus \oplus \beta(V^s)) \oplus B \cap (V_s \oplus \alpha(V^s))] \oplus V^s$. Now V_s is isomorphic to the summand in the bracket and, by the *Clause*, each summand in the bracket is isomorphic to a direct product of members of $\mathcal{C}$ of type s. It we project each of these summands to V_s we get the desired decomposition of V_s.

LEMMA 3. Let T be a finite set of types and let s be a minimal type in T. Assume Lemma 2. Set $V^T = \prod_J G_i$ where $J = \{i : t_i > \text{some } t \text{ in } T \text{ or } t_i \in T \setminus \{s\}\}$ and set ${}_TV = \prod_K G_i$ where $K = \{i : t_i \not\geq \text{any } t \text{ in } T\}$; so $V = {}_TV \oplus V_s \oplus V^T$. Then

(1) $V_s \oplus V^T$ and V^T are fully invariant in V,

(2) $A = A \cap ({}_TV \oplus \beta(V_s \oplus V^T)) \oplus \alpha(V_{X_s}) \oplus \alpha(V^T)$ for X_s as in Lemma 2.

PROOF. (1) is clear. Hence $A = A \cap ({}_TV \oplus \beta(V_s \oplus V^T)) \oplus A \cap \cap (V_s \oplus \beta(V^T)) \oplus \alpha(V^T)$. Now $A \cap (V_s \oplus \beta(V^s)) \subseteq A \cap (V_s \oplus \beta(V^T))$ which, being in $\alpha(V_s)$, is in $A \cap (V_s \oplus \beta(V^s))$. By Lemma 2 $\alpha(V_{X_s}) = A \cap (V_s \oplus \beta(V^T))$. Substitution yields (2).

LEMMA 4. If C is a finite rank direct summand of V, then C is isomorphic to a finite direct sum of members of $\mathcal{C}$.

PROOF. By slenderness C is a direct summand of a finite direct sum of G_i's. If all G_i's have the same type, the *Clause* applies. For the general case we may use Baer's classical proof for a direct summand of a finite direct sum of rank one torsionfree abelian groups (see Theorem 86.7 in [1]).

LEMMA 5. Suppose $m \in I$. Then $A = E \oplus F$ where E has finite rank and $\alpha_m(F) = 0$.

PROOF. Since G_m is slender, $\alpha_m(V_t) = 0$ for all types t except those in a finite set T. We induct on the order of T. If $T = \emptyset$, $\alpha_m(A) \subseteq \subseteq \alpha_m(V) = 0$; so $E = 0$ and $F = A$ satisfy the Lemma. Otherwise let s be a minimal type in T. From Lemma 3 we write $A = A \cap ({}_TV \oplus \oplus \beta(V_s \oplus V^T)) \oplus \alpha(V_{X_s}) \oplus \alpha(V^T)$. Note that the left summand is in $\alpha({}_TV)$ and $\alpha_m({}_TV) = 0$. Since $\alpha(V_{X_s})$ is a product of G_i's of type s, by slenderness $\alpha(V_{X_s}) = D \oplus E_1$ where $\alpha_m(D) = 0$ and E_1 is a finite direct sum of G_i's of type s. Let $F_1 = A \cap ({}_TV \oplus \beta(V_s \oplus V^T)) \oplus D$ Now $A = F_1 \oplus E_1 \oplus \alpha(V^T)$ where $\alpha_m(F_1) = 0$ and E_1 has finite rank. Next consider $V^T = \alpha(V^T) \oplus \beta(V^T)$. Let T_1 be the set of t's such that $V_t \subseteq V^T$ and $\alpha_m(V_t) \neq 0$. Then $T_1 = T \setminus \{s\}$ and $|T_1| < |T|$. By the induction hypothesis $\alpha(V^T) = E_2 \oplus F_2$ where E_2 has finite rank and $\alpha_m(F_2) = 0$. Therefore $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$ satisfy the lemma.

REMARK. If $\mathcal{C}$ is the class of rank one torsion-free reduced abelian groups, Lemma 5 follows readily from the fact that V and any direct summand of V is coseparable (see Proposition 1.2 and Theorem 5.8 in [3]). Thus the kernel of the map $\alpha_m \colon A \to G_m$ must contain a direct summand F of A with finite rank complement E.

PROOF OF THEOREM 1. By Lemma 4 we may assume $I = \omega$. We may also assume V has the decomposition in Lemma 2. We wish to find submodules A_n, A^n in A for each n in ω such that:

(1) $A = A_n$, $A^n = A_n \oplus A^{n+1}$ for each n,

(2) Each A_n has finite rank,

(3) For fixed $mA_n(A_n)\alpha_m(A_n) = 0$ for almost all n,

(4) $\cap A^n = 0$.

Then, by Proposition 3.3 in [4], we will have $A \cong \prod_\omega A_n$ and Lemma 4 above will complete the proof.

Let $A = A^0$. If $m \geqslant 0$ and if A^m is a direct summand of V, then, by letting A^m be A in Lemma 5, we can find a decomposition $A^m = A_m \oplus A^{m+1}$ where A_m has finite rank and $\alpha_m(A^{m+1}) = 0$. By induction we can find A_n, A^n for each n in ω to satisfy (1) and (2) above and where $\alpha_n(A^{n+1}) = 0$ for each n. For fixed m $\alpha_m(A_n) \subseteq \alpha_m(A^n) = 0$ for all $n > m$. This yields (3) and (4) which completes the proof.

4. Generalization.

THEOREM 6. Let $\mathcal{C}$ be a transitive class of slender R-modules. If $\prod_I G_i = A \oplus B$ with $G_i \in \mathcal{C}$ and $|I|$ non-measurable, then A is isomorphic to a direct product of members of $\mathcal{C}$ if this statement is true whenever I is countable and all G_i's have the same type.

PROOF. The result follows from Theorem 1 and the following proposition.

PROPOSITION 7 (Theorem 3.7 in [4]). Suppose an R-module P has decompositions $P = \prod_I G_i = A \oplus B$ where $|I|$ is non-measurable and each G_i is slender. Then $A \cong \prod_J A_j$ where each A_j is isomorphic to a direct summand of a direct product of countably many G_i's.

As an aside we mention that the conclusion of Lemma 3.6 in [4] is misstated. It should be: Then $A \cong \prod_J A_j$ and $B \cong \prod_J B_j$ where $A_j = A \cap (P_j \oplus \beta(P^{j'}))$ and $B_j = B \cap (P_j \oplus \alpha(Pj'))$. The proof of the Lemma, with obvious modifications, remains the same.

5. Applications.

COROLLARY 8 (see Theorem 13 in [5]). Let R be a commutative Dedekind domain which is not a field or a complete discrete valuation ring. Let $V = \prod_I G_i = A \oplus B$ where $|I|$ is non-measurable and each G_i is a rank one torsion-free reduced R-module. Then A is a direct product of rank one R-modules.

PROOF. Let $\mathcal{C}$ be the class of rank one torsion-free reduced R-modules. Each module in $\mathcal{C}$ is slender by Proposition 3 in [5]. If X and Y are in $\mathcal{C}$ and $f: X \to Y$ is a non-zero homomorphism, it is a monomorphism. Hence $\mathrm{Hom}_R(X, Y) \neq 0$ if and only if X is isomorphic to a submodule of Y. It follows that $\mathcal{C}$ is a transitive class of slender R-modules and that, for this class, the definitions of « type » in this paper and in Definition 9 in [5] are equivalent. By Proposition 12 in [5] the Corollary is true if all G_i's have the same type. Theorem 6 above completes the proof.

COROLLARY 9. Let R be a ring and let $\mathcal{C}$ be a transitive class of slender left R-modules such that modules of the same type are isomorphic and, for each X in $\mathcal{C}$, projective right $\mathrm{End}_R X$-modules are free. If $\prod_I G_i = A \oplus B$ where $G_i \in \mathcal{C}$ and $|I|$ is non-measurable, then A is isomorphic to a direct product of G_i's.

PROOF. By Theorem 3.1 in [2] the result is true if all G_i's have the same type. Theorem 6 above completes the proof.

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