Centralizers and Lie ideals

Rendiconti del Seminario Matematico della Università di Padova, tome 78 (1987), p. 255-259

<http://www.numdam.org/item?id=RSMUP_1987__78__255_0>
Centralizers and Lie Ideals.

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SUMMARY. - Let \( R \) be an associative ring, \( Z(R) \) its center and \( T(U) = \{ a \in R | au^n = u^n a, n = n(u, a) > 1, \text{ all } u \in U \} \), where \( U \) is a non central Lie ideal of \( R \). We prove that if \( R \) is a prime ring of characteristic not 2 with no nil right ideals, then either \( T(U) = Z(R) \) or \( R \) is an order in a simple algebra of dimension at most 4 over its center.

Let \( R \) be an associative ring, \( Z(R) \) its center. The hypercenter theorem [4] asserts that in a ring with no nonzero nil ideals an element commuting with a suitable power of every element of the ring must be central.

In this note we want to extend this result to noncentral Lie ideals in case \( R \) is a prime ring of characteristic not 2 with no nil right ideals. Let \( T(U) = \{ a \in R : au^n = u^n a, \text{ all } u \in U \} \), where \( U \) is a noncentral Lie ideal of \( R \), then one cannot expect the same conclusion of [4], as the following example shows:

**Example.** Let \( R = F_2 \), the \( 2 \times 2 \) matrices over a field \( F \),

\[
U = [R, R] = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in F \right\}.
\]

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Research supported by a grant from M.P.I.
Then $U$ is a noncentral Lie ideal of $R$ and $u^2 \in Z(R)$ for every element $u \in U$, therefore $T(U) = R$, but $Z(R) \neq R$.

Then making use of a result of Felzenszwalb and Giambruno [2], we shall prove the following:

**Theorem.** Let $R$ be a prime ring of characteristic not 2 with no nil right ideals, $U$ a noncentral Lie ideal of $R$. Then either $T(U) = Z(R)$ or $R$ is an order in a simple algebra of dimension at most 4 over its center.

Notice that the conclusion of the theorem is false if one removes the assumption of primeness. In fact, let $F_k$ be the ring of $k \times k$ matrices over a field $F$. If $R = \prod_{k=2}^{\infty} F_k$, then $R$ is a semisimple ring. Take $U = \bigoplus_{k=2}^{\infty} U_k$, where $U_2 = [F_2, F_2]$ and $U_k = F_k$ for $k > 2$, then $U$ is a noncentral Lie ideal of $R$. Let $a = (c, 0, 0, \ldots)$ with $c \notin Z(F_2)$. Then $a \in T(U)$, but $a \notin Z(R)$ and moreover it is clear that $R$ does not satisfy any polynomial identity.

For $a, b \in R$ set $[a, b] = ab - ba$ and for subsets $U, V \subset R$, let $[U, V]$ be the additive subgroup generated by all $[u, v]$ for $u \in U$ and $v \in V$. We recall that a Lie ideal $U$ of $R$ is an additive subgroup of $R$ such that $[U, R] \subset U$.

In all that follows, unless otherwise stated, $R$ will be a 2-torsion free ring, $Z = Z(R)$ the center of $R$, $J(R)$ the Jacobson radical of $R$, $U$ a noncentral Lie ideal of $R$ (i.e. $U \notin Z$) and

$$T(U) = \{a \in R: aw^n = w^n a, n = n(u, a) \geq 1, \text{ all } u \in U\}.$$ 

We start with

**Lemma.** If $R$ is a primitive ring then either $T(U) = Z(R)$ or $R$ is a simple algebra of dimension at most 4 over its center.

**Proof.** If $R$ is primitive, then $R$ is a dense ring of linear transformations on a vector space $V$ over a division ring $D$. If $\dim_D V \leq 2$, then $R$ is simple. Since $U$ is a noncentral Lie ideal of $R$, by Theorem 1.5 of [3], we may assume that $U = [R, R]$. Therefore

$$T(U) = \{a \in R: a(xy - yx)^n = (xy - yx)^n a, n = n(a, x, y) \geq 1, \text{ all } x, y \in R\}.$$
By a result of Felzenszwalb and Giambruno [2, Theorem 1], then we have the desired conclusion.

Suppose now that \( \dim_D V > 2 \). Since \( R \) is prime of characteristic different from 2, by [1, Lemma 1] there exists a nonzero ideal \( I \) such that \([I, R] \subseteq U \) and \([I, R] \not\subseteq Z \). It is also well known that \( I \) acts densely on \( V \) over \( D \) (see [5]).

Let \( a \neq 0 \) be an element of \( T(U) \) and suppose that for some \( v \in V \), the vectors \( v \) and \( va \) are linearly independent over \( D \). Since \( \dim_D V > 2 \), there exists a vector \( v_3 \) in \( V \) such that \( v_1 = v, v_2 = va, v_3 \) are linearly independent over \( D \).

The density of \( R \) and \( I \) on \( V \) gives \( r_2 \in R \) and \( i \in I \) with

\[
\begin{align*}
v_1 r_2 &= 0, & v_2 r_2 &= v_3, & v_3 r_2 &= 0, \\
v_1 i &= 0, & v_2 i &= 0, & v_3 i &= v_2.
\end{align*}
\]

Clearly \( a \) commutes with \( (ir_2 - r_2 i)^m \), for a suitable \( m \geq 1 \). Since \( 0 = v_1 (ir_2 - r_2 i) \) we get:

\[
0 = v_1 (ir_2 - r_2 i)^m = v_1 a (ir_2 - r_2 i)^m = v_2 (ir_2 - r_2 i)(ir_2 - r_2 i)^{m-1} = -v_2 (ir_2 - r_2 i)^{m-1} = \cdots = \pm v_2;
\]

a contradiction.

Thus given \( v \in V, v \) and \( va \) are linearly dependent over \( D \). As in [4, Lemma 2] it follows that \( a \) is central. In other words, if \( \dim_D V > 2 \), then \( T(U) = Z \). With this the lemma is proved.

We recall that a semisimple ring is a subdirect product of primitive rings \( R_\alpha \). For every \( \alpha \), let \( P_\alpha \) be a primitive ideal of \( R \) such that \( R_\alpha \cong R/P_\alpha \). Since \( J(R) = 0 \), then \( \bigcap_\alpha P_\alpha = 0 \). Remark that since \( R \) is 2-torsion free, we may assume that the homomorphic images \( R_\alpha \) are still of characteristic different from 2. In fact, put \( A = \bigcap_\alpha P_\alpha \) and \( B = \bigcap_\alpha P_\alpha \) and let \( x \in B \); then \( 2x \in B \) and also \( 2x \in 2R \subseteq \bigcap_\alpha P_\alpha = A \), therefore \( 2x \in A \cap B = 0 \). Since \( R \) is 2-torsion free \( x = 0 \) and so we have proved that \( B = 0 \). In this way \( 2R \not\subseteq P_\alpha \) (and therefore char \( R/P_\alpha \neq 2 \)) for every \( \alpha \). Now we are ready to prove the following:
THEOREM. Let \( R \) be a prime ring of characteristic not 2 with no nonzero nil right ideals, \( U \) a noncentral Lie ideal of \( R \). Then either \( T(U) = Z(R) \) or \( R \) is an order in a simple algebra of dimension at most 4 over its center.

PROOF. Suppose \( R \) is semisimple. If \( U_\alpha \) is the image of \( U \) in \( R_\alpha \), then \( U_\alpha \) is a Lie ideal of \( R_\alpha \). Let \( \mathcal{F} = \{ P_\alpha : U_\alpha \subset Z(R_\alpha) \} \). Set \( A = \bigcap_{P_\alpha \in \mathcal{F}} P_\alpha \) and \( B = \bigcap_{P_\alpha \in \mathcal{F}} P_\alpha \). Since \( R \) is prime and \( AB \subset A \cap B = 0 \), we must have either \( A = 0 \) or \( B = 0 \). If \( A = 0 \), then \( U \subset Z \), a contradiction. Thus \( B = 0 \) and so for every \( \alpha \), \( U_\alpha \) is a noncentral Lie ideal of \( R_\alpha \).

For each \( \alpha \) let \( T_\alpha \) be the image of \( T(U) \) in \( R_\alpha \). Since \( U_\alpha \notin Z(R_\alpha) \), \( T_\alpha \subset T(U_\alpha) \) for each \( \alpha \) and by the previous Lemma we get either \( T_\alpha \subset Z(R_\alpha) \) or \( R_\alpha \) satisfies \( S_4 \), the standard identity in four variables.

Put \( I = \{ \cap P_\alpha : T_\alpha \subset Z(R_\alpha) \} \) and \( J = \{ \cap P_\alpha : T_\alpha \notin Z(R_\alpha) \} \). Since \( R \) is prime and \( IJ = 0 \) we must have either \( I = 0 \) or \( J = 0 \).

If \( I = 0 \), we conclude that \( T(U) = Z(R) \), the desired conclusion. If \( J = 0 \) then, for every \( \alpha \), \( R_\alpha \) satisfies \( S_4 \) and so \( R \) satisfies \( S_4 \); even in this case we are done.

Therefore we may assume that \( J(R) \neq 0 \). As we remarked before, there exists a nonzero ideal \( I \) of \( R \) such that \( [I, R] \subset U \). Since \( R \) is prime, \( I \cap J(R) \) is a nonzero ideal of \( R \).

Let \( T = T([I, R]) \). If \( T \) centralizes \( J(R) \cap I \), then, since the centralizer of a nonzero ideal in a prime ring is equal to the centre of the ring, \( T \subset C_R(J(R) \cap I) = Z(R) \).

Suppose then that \( a \in T \), \( x \in J \cap I \) and \( ax - xa \neq 0 \). Now

\[
0 \neq (ax - xa)(1 + x)^{-1} = a - (1 + x)a(1 + x)^{-1} \in T.
\]

Therefore \( 0 \neq (ax - xa)(1 + x)^{-1} \) is in \( T \cap I \cap J \) and so \( T \cap I \cap J \neq 0 \).

Consider the following subset of \( R \):

\[
T(I) = \{ x \in I : a[x, y]^n = [x, y]^n a, \ n = n(a, x, y) \geq 1, \ \text{all} \ x, y \in I \}.
\]

Since \( I \) as a ring satisfies the same hypotheses placed on \( R \), by Theorem 1 of [2] either \( T(I) \subset Z(I) \subset Z(R) \) or \( I \) satisfies \( S_4 \).

If the first possibility occurs, since \( 0 \neq T \cap J \cap I \subset T(I) \subset Z(R) \) we have \( (ax - xa)(1 + x)^{-1} \in Z \). Also, if \( b \in T \), then

\[
b(ax - xa)(1 + x)^{-1} \in T \cap J \cap I \subset Z.
\]
since both $0 \neq (ax - xa)(1 + x)^{-1} \in Z$ and $b(ax - xa)(1 + x)^{-1} \in Z$
and since elements in $Z$ are not zero divisors in $R$, these relations would imply that $b \in Z$ and we would get $T = T([I, R]) = Z(R)$ and so $T(U) \subseteq Z(R)$.

Suppose now $T(U) \neq Z(R)$. By the above $T(I) \neq Z(I)$, then $I$
and so $R$ is an order in a simple algebra of dimension at most 4 over
its center, the desired conclusion.

REFERENCES

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Pervenuto in redazione il 9 maggio 1986 e in forma revisionata il 28 ot-
tobre 1986.