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Hopf Bifurcation from Infinity.

Marco Sabatini (*)

0. Introduction.

The aim of this paper is to give sufficient conditions for bifurcation from infinity of periodic solutions of second order autonomous O.D.E.'s.

Let us consider the parametrized equation

$$(E_\mu) \qquad x'' = F_\mu(x, x') \qquad x \in \mathbf{R}, \ \mu \in [0, \mu^\#)$$

where the map $(\mu, x, x') \to F_\mu(x, x')$ is continuous.

The generalized Liénard and Rayleigh equations

$$(L_\mu) \qquad x'' + f_\mu(x) x' + g_\mu(x) = 0$$

$$(R_\mu) \qquad x'' + f_\mu(x') + g_\mu(x) = 0$$

are special cases of (E_μ), that will be considered in detail in sections 3, 4. The main result is the following.

THEOREM A. *Let us suppose that* (E_μ) *defines a continuous family of dynamical systems* $\pi_\mu(t, x)$ *such that:*

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(i) $\sigma(t, x) := \pi_0(-t, x)$ *is ultimately bounded;*

(ii) $\pi_\mu(t, x)$ *is ultimately bounded for $\mu > 0$;*

(iii) *there are no equilibrium points of (E_μ) out of a fixed ball.*

Then there exists a family of asymptotically stable compact annuli, with closed orbits as boundaries, bifurcating from infinity.

We remark that the notion of ultimate boundedness for π_μ is equivalent to that of uniform ultimate boundedness for the solutions of (E_μ), that was studied by several authors (see [3], [4], [5], [7], [12]). Moreover, condition (i) is verified by the flow defined by (E_μ) if and only if the solutions of

$$(-E_\mu) \qquad\qquad x'' = F_\mu(x, -x')$$

are uniformly ultimately bounded (u.u.b.). Hence, if the solutions of $(-E_0)$ and (E_μ), for μ positive, are u.u.b., then there are periodic solutions of (E_μ) bifurcating from infinity.

Theorem A is obtained as a corollary of a more general one, concerning abstract dynamical systems in locally compact metric spaces.

THEOREM B. *Let π_μ be a one-parameter continuous family of dynamical systems on X, a locally compact metric space. Assume that:*

(i) $\sigma(t, x) = \pi_0(-t, x)$ *is ultimately bounded;*

(ii) $\pi_\mu(t, x)$ *is ultimately bounded for $\mu > 0$.*

Then a family of asymptotically stable, invariant, compact sets bifurcates from infinity.

The central idea of the proof is to extend the flows π_μ to the Alexandrov compactification $\tilde{X} := X \cup \{\omega\}$ of X by setting, for any μ

$$\tilde{\pi}_\mu(t, \omega) = \omega \qquad \forall t \in \mathbb{R}$$

$$\tilde{\pi}_\mu(t, x) = \pi_\mu(t, x) \qquad \forall t \in \mathbb{R}, \ \forall x \in X .$$

Auslander and Seibert showed in [1] that stability properties of $\{\omega\}$ with respect to $\tilde{\pi}$ are strictly related to global boundedness properties of π. Then, to prove Theorem B it is sufficient to apply a result by Marchetti-Negrini-Salvadori-Scalia, [11].

Part of the results contained in this work could be reached by a suitable use of the stereographic projection. Alexandrov' compactification has been chosen for it allows to study also dynamical systems on unbounded submanifolds of $\mathbf{R}^n$, where a stereographic projection is not defined in a natural way.

A result similar to theorem A has been obtained by Lins-De Melo-Pugh (see [9]) for the equation:

$$x'' + (ax^2 + bx + c)x' + x = 0 \,.$$

They studied it on Poincaré's sphere and proved that a closed orbit goes to infinity when (c/a) diverges negatively.

Oliva, too, used Poincaré's sphere to study delay-differential equations at infinity, with special regard to Hopf bifurcation (see [8], and its references). A shortcoming of Hopf bifurcation around a point at infinity on Poincaré's sphere is that closed orbits do not correspond to periodic solutions of the original system.

Recently L. Malaguti [10] proved a theorem of bifurcation of cycles from infinity for the equation

$$x'' + [h(\mu) + f(x)]x' + x = 0 \,.$$

This paper is divided in four sections. In the first one the fundamental definitions and theorems about bifurcation in dynamical systems are given. In section 2 the relation between boundedness properties and stability at infinity is described and Theorems A and B are proved. In the remaining sections, the previous results are applied to the study of generalized Liénard and Rayleigh equations.

1. Preliminaries.

For basic definitions and notations we refer to [2]. If $M \subseteq X$, we denote by $A_\mu(M)$ the region of attraction of M with respect to the dynamical system π_μ. Furthermore, the negative dynamical system $\pi(-t, x)$ will be called $\sigma(t, x)$.

As already observed in the introduction, the appearance of bifurcation in dynamical systems is often related to a sudden change of the stability properties of suitable sets. In [11] the following definition has been given:

DEFINITION 1.1. *Let C be the set of all proper non empty compact subsets of X. Let us consider a map $K\colon [0, \mu^{\#}) \to C$, $\mu \mapsto K_{\mu}$, such that:*

(i) $\forall \mu \in [0, \mu^{\#})$, K_{μ} *is π_{μ}-invariant*

(ii) $\max \{\varrho(x, K_0)\colon x \in K_{\mu}\} \to 0$ *as $\mu \to 0$*

then $\mu = 0$ is said to be a bifurcation point *for the map K if there exists $\mu^* \in (0, \mu^{\#})$ and a second map $M\colon (0, \mu^*) \to C$, $\mu \mapsto M_{\mu}$ satisfying the conditions:*

(1) $\forall \mu \in (0, \mu^*)$, M_{μ} *is π_{μ}-invariant and $K_{\mu} \cap M_{\mu} = \emptyset$;*

(2) $\max \{\varrho(x, K_0)\colon x \in M_{\mu}\} \to 0$ *as $\mu \to 0$.*

In the same paper, the following basic theorem has been proved:

THEOREM 1.2. *Let X be connected and π_{μ} be a continuous family of flows on X. Let $\mu > 0$ and $K\colon [0, \mu^{\#}) \to C$ be a map as in Def. 1.1. If K_0 is π_0-asymptotically stable and K_{μ} is π_{μ}-completely unstable (i.e. negatively asymptotically stable) for $\mu \in (0, \mu^{\#})$, then $\mu = 0$ is a bifurcation point for K. Furthermore, the map M and μ^* can be chosen so that $\forall \mu \in (0, \mu^*)$, M_{μ} is π_{μ}-asymptotically stable.*

REMARK 1.3. The proof shows that the set M_{μ} can be identified as the largest π_{μ}-invariant compact set disjoint from K_0, contained in a suitable neighbourhood of K_0 independent of μ, for μ small.

When $X = \mathbb{R}^2$, theorem 1.2 can be used to prove the existence of periodic solutions of ordinary differential systems.
Let

$$x' = f_{\mu}(x) \qquad x \in \mathbb{R}^2, \ \mu \in [0, \mu^{\#})$$

be a family of ordinary differential systems defining a continuous family of dynamical systems on $\mathbb{R}^2$, and let us assume that, for small μ's, the origin is the unique critical point of our system contained in a fixed ball. Then:

THEOREM 1.4. *Let the origin be π_0-asymptotically stable and π_{μ}-completely unstable, $\forall \mu \in (0, \mu^{\#})$. Then $\mu = 0$ is a bifurcation point; μ^* and M can be determined so that, $\forall \mu \in (0, \mu^*)$:*

(i) M_{μ} *is π_{μ}-asymptotically stable;*

(ii) M_μ is the compact annular region enclosed between two cycles C_μ, C'_μ of π_μ; the inner one C_μ equal to $\partial A_\pi^-(\{0\})$.

For the proofs of theorems 1.2 and 1.4 see [11].

2. Alexandrov' compactification and bifurcation from infinity.

In this section we denote by $\tilde{X} = X \cup \{\omega\}$ the Alexandrov' compactification of a topological space X (see [6] for details). As already remarked in [1], a flow π defined on X can be extended in a unique way to a flow $\tilde{\pi}$ on $\tilde{X}$: since X is invariant with respect to π, its complement $\{\omega\}$ has to be invariant, hence a fixed point. Let us define a new flow:

$$(2.1) \qquad \tilde{\pi}_\mu(t, x) = \begin{cases} \pi_\mu(t, x) & \text{if } x \in X , \\ \omega & \text{if } x = \omega . \end{cases}$$

Group axioms are trivially verified by $\tilde{\pi}$, and its continuity comes from a standard compactness argument.

From our viewpoint, the relevant feature of $\tilde{\pi}$ consists of the strict relation existing between the stability properties of ω with respect to $\tilde{\pi}$ and the boundedness properties of π. In particular:

THEOREM 2.1. (Auslander-Seibert, [1]). π is ultimately bounded if and only if $\{\omega\}$ is negatively asymptotically stable with respect to $\tilde{\pi}$.

Now we are ready to state the main result of this section. Let X be a connected, locally compact, non compact metric space.

THEOREM 2.2 (B). Let $\pi_\mu(t, x)$ be a continuous family of flows on X s.t.

(1) $\pi_0(t, x)$ is negatively ultimately bounded;

(2) if $\mu > 0$, $\pi_\mu(t, x)$ is ultimately bounded.

Then $\exists\ \mu^* \in [0, \mu^\#)$ such that $\forall \mu \in (0, \mu^*)$:

(i) $\exists\ M_\mu$ compact, π_μ-invariant and asymptotically stable;

(ii) M_μ is the largest invariant set contained in the complement of a fixed compact;

(iii) $M_\mu \to \omega$ in the Hausdorff metric of $\tilde{X}$ as $\mu \to 0$.

PROOF. It is easy to prove, by the usual compactness argument, that if π_μ is a continuous family of flows on X, then $\tilde{\pi}_\mu$ has the same property.

Since X is separable, $\tilde{X}$ is metrizable, so it makes sense speaking of bifurcation. The hypotheses and theorem 2.1 imply that $\{\omega\}$ is $\tilde{\pi}_0$-asymptotically stable and $\tilde{\pi}_\mu$-completely unstable, for $\mu > 0$. By theorem 2.1, $\mu = 0$ is a bifurcation point. So there exists $\mu^* \in (0, \mu^\#)$ and a map $M: \mu \mapsto M_\mu$, defined on $(0, \mu^*)$, such that M_μ is not empty, compact in $\tilde{X}$, $\tilde{\pi}$-invariant and disjoint from $\{\omega\}$; moreover:

$$\sup \{\tilde{\varrho}(x, \omega) : x \in M_\mu\} \to 0 \quad \text{as } \mu \to 0 .$$

By remark 1.3, M_μ may be chosen as the largest asymptotically stable $\tilde{\pi}_\mu$-invariant set contained in a neighborhood of ω, such that $\omega \notin M_\mu$. This means that M_μ may be chosen as the largest asymptotically stable π_μ-invariant set contained in the complement of a compact of X. Since $\omega \notin M_\mu$, by a well known property of Alexandrov' compactification, M_μ is compact also in X. Point (iii) of the thesis comes directly from point (2) of definition 1.1. $\square$

As it has been made in [11] for Hopf bifurcation, the previous result may be used in the study of bifurcation from infinity for periodic solutions of differential systems in $\mathbb{R}^2$.

Let $f: [0, \mu^\#) \times \mathbb{R}^2 \to \mathbb{R}^2$ ($\mu^\# > 0$) be a continuous map satisfying the following conditions:

(i) the family of differential equations

$$(\Sigma_\mu) \qquad\qquad x' = f_\mu(x) \qquad \mu \in [0, \mu^\#)$$

(2.2)   defines a continuous family π_μ of dynamical systems on $\mathbb{R}^2$ (see remark 2.5);

(ii) there exists a compact set H out of which (Σ_μ) has no critical points.

We may now state the following:

THEOREM 2.3 (A). *Let π_0 be negatively ultimately bounded and π_μ be ultimately bounded, for $\mu \in (0, \mu^\#)$.*

Then $\mu = 0$ is a bifurcation point from infinity. Furthermore, $\mu^* > 0$ and M_μ may be determined in such a way that the conclusions of theorem 2.2 hold and M_μ is the annulus enclosed between two cycles C_μ, C'_μ, with $C_\mu = \partial A_\pi^-(\omega)$.

PROOF. Let us consider the homeomorphism:

$$\alpha: \widetilde{\mathbb{R}}^2 = S^2 \to S^2 \qquad \alpha(z) = \begin{cases} z/|z|^2 & z \neq 0, \omega \\ 0 & z = \omega \\ \omega & z = 0 . \end{cases}$$

The dynamical system:

$$\eta_\mu(t, z) := \alpha\big(\pi_\mu(t, \alpha^{-1}(z))\big)$$

has a fixed point at the origin whose stability properties coincide with ω's ones. We may apply theorem 1.4 to η_μ, in order to obtain the existence of an annulus N_μ enclosed between two cycles Γ_μ, Γ'_μ. where $\Gamma_\mu = \partial A_\pi^-(0)$. Now, setting $M_\mu = \alpha^{-1}(N_\mu)$ and $C_\mu = \alpha^{-1}(\Gamma_\mu)$, $C'_\mu = \alpha^{-1}(\Gamma'_\mu)$, we get the thesis. $\square$

REMARK 2.4. If $C_\mu = C'_\mu$, then $M_\mu = C_\mu = C'_\mu$ is an asymptotically stable closed orbit.

REMARK 2.5. If $(\mu, x) \mapsto f_\mu(x)$ is continuous and existence and uniqueness of solutions of (S_μ) are guaranteed, we always may define a continuos family of dynamical systems having the same orbits as (S_μ), possibly reparametrizing the time. For example, the differential system

$$x' = f_\mu(x)$$

may be susbstituted by the following one:

$$x' = f_\mu(x)/[1 + |f_\mu(x)|^2],$$

to obtain the desired result.

REMARK 2.6. In general the map M is not continuous with respect to Hausdorff metric, for $\mu \neq 0$. In the following example the origin is a cluster point of discontinuities.

Let us consider the family of dynamical systems defined by the

following differential systems:

$$(2.3)_\mu \qquad \begin{cases} x' = -y + \varphi_\mu(x^2 + y^2)x \\ y' = x + \varphi_\mu(x^2 + y^2)y \end{cases} \qquad \mu \in [0, 1]$$

where

$$\varphi_\mu(s) = s(2\mu - s)(s^2 - 2\mu s + \mu^2 - \mu^r \sin(1/\mu)) \qquad r > 2 .$$

To study the stability properties of the unique equilibrium point of $(2.3)_\mu$, the origin, we use the Liapunov function

$$V(x, y) = x^2 + y^2$$

whose derivative, along the solutions of $(2.3)_\mu$, is

$$V'_\mu(x, y) = 2(x^2 + y^2)\varphi_\mu(x^2 + y^2) .$$

For $\mu = 0$, the origin is asymptotically stable, because

$$V'_0(x, y) = -2(x^2 + y^2)^5 .$$

For $\mu > 0$, the origin is completely unstable, because

$$V'_\mu(x, y) = 4(x^2 + y^2)^2 (\mu^3 - \mu^{r+1} \sin(1/\mu)) + o[(x^2 + y^2)^2]$$

is positive definite in a neighbourhood of $(0, 0)$.

Hence, a family of asymptotically stable sets M_μ bifurcates from the origin. M_μ consists of a single cycle when the quadratic polynomial

$$s^2 - 2\mu s + (\mu^2 - \mu^r \sin(1/\mu))$$

has no real roots. This happens when

$$\Delta(\mu) = 4\mu^r \sin(1/\mu)$$

is negative. In this case the cycle is the circumference of equation

$$x^2 + y^2 = 2\mu .$$

When $\varDelta(\mu)$ changes sign, a new cycle appears and suddenly bifurcates in two cycles of equations

$$x^2 + y^2 = \mu \pm [\mu^r \sin (1/\mu)]^{\frac{1}{2}} .$$

If $\varDelta(\mu) \geqslant 0$, M_μ can be characterized as follows:

$$M_\mu = \{(x, y) \in \mathbb{R}^2 : \mu - [\mu^r \sin (1/\mu)]^{\frac{1}{2}} \leqslant x^2 + y^2 \leqslant 2\mu\} .$$

It is immediate to verify that M is Hausdorff-discontinuous where $\varDelta(\mu)$ changes sign, and that $\mu = 0$ is an accumulation point of such discontinuities.

Let us introduce a new topology τ in C:

DEFINITION 2.7. *Let us set* $U(M, \varepsilon) := \{K \in C : K \subset S[M, \varepsilon]\}$. *We call* τ *the topology having, for any* $M \in C$, *the family* $\{U(M, \varepsilon) : \varepsilon > 0\}$ *as a fundamental system of neighborhoods.*

The topology induced on C by Hausdorff metric is strictly finer than τ. Moreover, τ is T_0 but not T_1 (see [6] for separation axioms). The restriction of τ to the subspace J of C consisting of all Jordan closed curves is T_2, that ensures the uniqueness of the limit in J.

We say that a map $M : [0, \mu^\#) \to C$, continuous with respect to τ, defines a « continuous family of compact sets ». In some cases , we may prove that the bifurcating families of theorems 1.4 and 2.3 are τ-continuous.

COROLLARY 2.8. *In the hypotheses of Theorem* (2.3), *if* π_μ *has at most one cycle for* $\mu \in (0, \mu^\#)$, *then* M *defines a continuous family of asymptotically stable cycles.*

PROOF. By theorem 3.1 in [11], the asymptotic stability of $M_{\bar{\mu}}$ entails its total stability. So, for any positive ε, there exists $\delta > 0$ such that $|\mu - \bar{\mu}| + \varrho(x, M_{\bar{\mu}}) < \delta$ implies that $\varrho(\pi_\mu(t, x), M_{\mu^\bullet}) < \varepsilon$ for any positive t. So the ω-limit set of x is compact for each $x \in S(M_\mu, \delta)$. By Poincaré-Bendixson theory, it is a cycle Γ_μ. For the uniqueness of cycles of π_μ, we have $\Gamma_\mu = M_\mu$ and the thesis.

3. Generalized Liénard equation.

Let us consider a one-parameter family of second order differential equations:

$$(E_\mu) \qquad\qquad x'' = F_\mu(x, x') \qquad x \in \mathbb{R}^n, \ \mu \in [0, \mu^\#)$$

where $(\mu, x, x') \mapsto F_\mu(x, x')$ is a continuous real function. (E_μ) is equivalent to the system

$$(S_\mu) \qquad\qquad x' = y , \qquad y' = F_\mu(x, y) .$$

If $\forall \mu \in [0, \mu^\#)$, $F_\mu(x, x')$ is locally lipschitzian, then (S_μ) defines a dynamical system $\pi_\mu(t, x)$, possibly by a suitable reparametrization of its orbits (see remark 2.5).

The following differential system

$$(-S_\mu) \qquad\qquad x' = -y , \qquad y' = -F_\mu(x, y)$$

defines the (negative) dynamical system $\sigma_\mu(t, x) = \pi_\mu(-t, x)$. $(-S_\mu)$ is equivalent to

$$(-E_\mu) \qquad\qquad x'' = F_\mu(x, -x') .$$

Hence, (S_μ) is negatively ultimately bounded if and only if the solutions of $(-E_\mu)$ are uniformly ultimately bounded. This allows us to use boundedness theorems in order to obtain unboundedness of solutions of the given differential equations. Then, by virtue of theorems 2.2 and 2.3, to any boundedness theorem we may associate a result of bifurcation from infinity. An example of this procedure will be given in this section for the generalized Liénard equation

$$(L) \qquad\qquad x'' + f(x)x' + g(x) = 0 .$$

In next section, we will give other examples concerning Rayleigh equation.

We recall below a theorem by J. R. Graef [7], relative to equation (L).

THEOREM 3.1. *Let f be continuous and g be locally lipschitzian. Let us suppose that there exist positive constants k, c such that:*

(i) $xF(x) > 0$,   *if* $|x| > k$;

(ii) $xg(x) > 0$,   *if* $|x| > k$;

(iii) $F(x) \geqslant c > 0$,   *if* $x \geqslant k > 0$ *or*

(iii)' $F(x) \leqslant -c < 0$,   *if* $x \leqslant -k < 0$.

where $F'(x) = f(x)$, $F(0) = 0$.
Further, if

(iv) $\displaystyle\int_0^{\pm\infty} [f(x) + |g(x)|]\, dx = \pm\infty$,

then the solutions of (L) are uniformly ultimately bounded.

Now let us consider the family of equations

(L_μ) $x'' + f_\mu(x)\, x' + g_\mu(x) = 0$

where, $\forall \mu \in [0, \mu^\#)$, f_μ is continuous and g_μ is locally lipschitzian. The results of section 2 may be applied to the study of bifurcation of periodic solutions of (L_μ) from infinity, as it is shown by the following:

THEOREM 3.2. *Let us assume that*

(i) $(-L_0)$ *and* (L_μ), *for* $\mu > 0$, *verify the hypotheses of theorem* 3.1;

(ii) *there exists a fixed compact H that contains all fixed points (L_μ), for μ positive.*

Then $\mu = 0$ is a point of bifurcation from infinity for a family of compact, invariant, asymptotically stable annuli.

PROOF. By remark 2.5, we see that for $\forall \mu \in [0, \mu^\#)$, (L_μ) defines a continuous family of dynamical systems $\pi_\mu(t, x)$. Moreover, the family π_μ verifies the hypotheses of theorem 2.3. This yields the thesis. $\square$

An existence result of a continuous family of asymptotically stable cycles bifurcating from infinity may be derived from theorem 3.2. For that, we need the following:

THEOREM 3.3 (Zhang Zhifen [14], [15]). *Let us consider the differential system in* $\mathbb{R}^2$:

(3.1) $$x' = -\Phi(y) - F(x) , \quad y' = g(x) .$$

If

    (i) *g locally lipschitzian,* $xg(x) > 0$ *for* $x \neq 0$, $G(\pm \infty) = +\infty$ *(where* $G'(x) = g(x)$, $G(0) = 0$*);*

    (ii) $F(x) \in C^1(\mathbb{R})$, $F(0) = 0$, $F'(x)/g(x)$ *is increasing for* $x \neq 0$;

    (iii) $y\Phi(y) > 0$ *for* $y \neq 0$, $\Phi(\pm \infty) = \pm \infty$, Φ *is locally lipschitzian and non decreasing,* $\exists \phi'(0) \neq 0$;

then (3.1) has at most one closed orbit.

Now we are in position to prove the following corollary of theorem 3.2.

COROLLARY 3.4. *Suppose that* (L_μ) *verifies the hypotheses of 3.2 for* $\mu \in [0, \mu^i)$. *Moreover, assume that for* $\mu > 0$:

    (iv) $xg_\mu(x) > 0 \quad \forall x \neq 0$;

    (v) $G_\mu(\pm \infty) = +\infty$;

    (vi) $f_\mu(x)/g_\mu(x)$ *is increasing for* $x \neq 0$.

Then, a continuous family of asymptotically stable cycles bifurcates from infinity when μ *becomes positive.*

PROOF. (L_μ) is equivalent to the following system:

$$x' = -y - F(x) ; \quad y' = g(x)$$

where $F'(x) = f(x)$, $F(0) = 0$. The above assumptions ensure that Zhang Zhifen theorem holds. By remark 2.4, the bifurcating annuli reduce to single orbits Γ_μ, and we may apply Corollary 2.8. $\quad\square$

When f_μ and g_μ are polynomials, bifurcation from infinity may happen when the leading coefficient of f becomes positive.

COROLLARY 3.5. *Let* f_μ *and* g_μ *be*:

$$f_\mu(x) = a_n(\mu)x^n + \ldots + a_0(\mu) , \quad g_\mu(x) = b_m(\mu)x^m + \ldots + b_0(\mu)$$

where the coefficients of f_μ and g_μ are real continuous functions defined on $[0, \mu^\#)$. Assume that n is even, m is odd, $a_n(\mu)$, $b_m(\mu)$ are positive for $\mu > 0$. If

$$a_n(0) = 0 , \quad a_{n-1}(0) = 0 , \quad a_{n-2}(0) < 0 , \quad b_m(0) > 0$$

then $\mu = 0$ is a point of bifurcation from infinity, and conclusions of theorem 3.2 hold.

PROOF. Since, for $\mu > 0$, $F_\mu(x)$ and $g_\mu(x)$ are odd-degree polynomials with positive leading coefficients, Graef' theorem holds. For $\mu = 0$ the same theorem holds for equation $(-L_0)$, because

$$- F_0(\pm \infty) = g_0(\pm \infty) = \pm \infty .$$

To prove the thesis it is sufficient to show that all critical points of (L_μ), for $\mu \in [0, \mu^\#/2]$, are contained in a fixed compact. That is equivalent to prove that the roots of $g_\mu(x)$ are contained in a fixed ball. If x_μ is a root of $g_\mu(x)$, we have:

$$x_\mu^m = - [b_{m-1}(\mu)/b_m(\mu)] x_\mu^{m-1} - \ldots - b_0(\mu)/b_m(\mu) .$$

Let us set

$$B_j = \sup \{|b_j(\mu)/b_m(\mu)|, \ \mu \in [0, \mu^\#/2]\} .$$

Now

$$|x_\mu^m| = | - [b_{m-1}(\mu)/b_m(\mu)] x_\mu^{m-1} - \ldots - b_0(\mu)/b_m(\mu)| \leqslant B_{m-1}|x_\mu|^{m-1} + \ldots + B_0$$

hence

(3.2) $$|x_\mu|^m - B_{m-1}|x_\mu|^{m-1} - \ldots - B_0 \leqslant 0 .$$

Since

$$\lim_{y \to +\infty} (y^m - B_{m-1} y^{m-1} - \ldots - B_0) = + \infty$$

inequality (3.2) holds in a compact H, indepent of μ. So all roots of $g_\mu(x)$ are contained in H, for any μ in $[0, \mu^\#/2]$. $\square$

What follows is the analogue of 3.5 for polynomial Liénard equation:

COROLLARY 3.6. *Under the hypotheses of* Cor. 3.5, *if* $n > m$, $x g_\mu(x) > 0$ *for* $x \neq 0$ *and the polynomial* $f'_\mu g_\mu - g'_\mu f_\mu$ *has no real roots, then* $\mu = 0$ *is a point of bifurcation from infinity for a continuous family of asymptotically stable cycles.*

PROOF. Zhang Zhifen theorem holds if $(f_\mu/g_\mu)' = (f'_\mu g_\mu - f_\mu g'_\mu)/g^2_\mu > 0$. Since the leading coefficient of $f'_\mu g_\mu - f_\mu g'_\mu$ is $(n - m) a_n(\mu) b_m(\mu) > 0$, if $f'_\mu g_\mu - f_\mu g'_\mu$ has no real roots, the function (f_μ/g_μ) is increasing in $\mathbf{R}/\{0\}$, and Cor. 2.8 applies. □

4. Rayleigh equation.

In this section we deal with the same problem of section 3 for a continuous family of generalized Rayleigh equations:

$$(R_\mu) \qquad\qquad x'' + f_\mu(x') + g_\mu(x) = 0$$

In the whole section it is assumed that all the functions f_μ, g_μ are locally lipschitzian. Let us recall a simplified version of a theorem by Cartwright and Swinnerton Dyer [4], about the equation

$$(R) \qquad\qquad x'' + f(x') + g(x) = 0 \,.$$

THEOREM 4.1. *If there are constants* $\eta, k, b, d > 0$ *such that:*

(i) $f(x)x \geqslant 0$ *for* $|x| \geqslant k$;

(ii) $f(x) \operatorname{sgn} x \geqslant b$ *for* $x > k$ (*or for* $x < - k$) ;

(iii) $g(x) \operatorname{sgn} x \geqslant 0$ *for* $|x| \geqslant k$;

(iv) $g(x) \operatorname{sgn} x \geqslant d + \sup \{|f(x)| : |x| \leqslant k\}$ *for* $|x| \geqslant k + \eta$,

then the solutions of (R) *are uniformly ultimately bounded.*

Let us call $(- R_0)$ the equation obtained from (R_0) by applying the procedure described at the beginning of section 3. As it has been done in the previous section, we may write the bifurcation theorem associated to Cartwright-Swinnertone Dyer's one.

THEOREM 4.2. *Let us suppose that* $(- R_0)$ *and* (R_μ), *for* $\mu > 0$, *satisfy the hypotheses of theorem 4.1. If all equilibrium points of* (R_μ) *are*

contained in a fixed ball, then $\mu = 0$ is a point of bifurcation from infinity for a family of compact, invariant, asymptotically stable annuli.

PROOF. Like in theorem 3.2. □

Zhang Zhifen' theorem allows us to state the following:

THEOREM 4.3. *Under the assumption of the above theorem, if,* $\forall \mu \in (0, \mu^\#)$:

(i) $f_\mu \in C^1(\mathbb{R})$, $f_\mu(0) = 0$;

(ii) $xg_\mu(x) > 0$, $\forall x \neq 0$;

(iii) g_μ *is non decreasing,* $g_\mu(\pm \infty) = \pm \infty$;

(iv) $\exists g'_\mu(0) \neq 0$;

(v) $f'_\mu(x)/x$ *is increasing for* $x \neq 0$;

then $\mu = 0$ is a bifurcation point from infinity for a continuous family of asymptotically stable cycles.

PROOF. (R_μ) is equivalent to the system:

(4.1) $x' = y$, $y' = - g_\mu(x) - f_\mu(y)$.

After the substitution:

$$X = y , \quad Y = x$$

the system becomes:

$$X' = - f_\mu(X) - g_\mu(Y) , \quad Y' = X .$$

Then, like in corollary 3.4, the conclusion comes from theorem 3.3 and corollary 2.8. □

The following corollaries are the analogue of 3.5 and 3.6 for Rayleigh equation.

COROLLARY 4.4. *Let f_μ and g_μ be polynomials. If they have odd degree and all other hypotheses of corollary 3.5 hold, then $\mu = 0$ is a point of bifurcation from infinity and the thesis of theorem 4.2 holds.*

PROOF. On the same line of corollary 3.5. It is immediate to verify that the boundedness properties requested by theorem 4.2 hold. It remains to prove that all equilibrium points of (R_μ) are contained in a fixed compact. To see this, we consider the equivalent system (4.1), whose critical points have coordinates $(x, 0)$, where x is a root of the polynomial $g_\mu(x) + a_0(\mu)$. Now it is sufficient to repeat the second part of the proof of corollary 3.5 to have the thesis. $\square$

COROLLARY 4.5. *Under the hypotheses of corollary 4.4, if $n > 2$ and:*

   (i) $a_0(\mu) = b_0(\mu) = 0$,   *for* $\mu \in (0, \mu^\sharp)$;

   (ii) $g'_\mu(x) \geqslant 0$, $\forall x \in \mathbb{R}$, $g'(0) > 0$;

   (iii) $x f''_\mu - f'_\mu$ *has no real roots;*

then $\mu = 0$ is a bifurcation point from infinity for a continuous family of asymptotically stable cycles.

PROOF. If the polynomial $x f''_\mu - f'_\mu$ has no real roots, then it is always positive, since its leading coefficient is $n(n-2)a_n(\mu) > 0$. Hence the function (f'_μ/x), which has $[(x f''_\mu - f'_\mu)/x^2]$ as derivative, is increasing, and the system

$$x' = -f_\mu(x) - g_\mu(y) , \qquad y' = x ,$$

has at most one closed orbit, by Zhang Zhifen' theorem. By exchanging x and y, we obtain system (4.1), that is equivalent to (R_μ). By corollary 2.8 we get the thesis. $\square$

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