

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

JAMES WIEGOLD

**A non-commutative free algebra of rank 0**

*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 77 (1987), p. 207-211

[http://www.numdam.org/item?id=RSMUP\\_1987\\_\\_77\\_\\_207\\_0](http://www.numdam.org/item?id=RSMUP_1987__77__207_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1987, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## A Non-Commutative Free Algebra of Rank 0.

JAMES WIEGOLD (\*)

### 1. Varieties of algebras.

Every algebra  $A$  has a set  $\Omega = \bigcup \Omega_n$  of operations, where each  $\alpha$  in  $\Omega_n$  is  $n$ -ary, that is, it gives rise to a mapping  $\alpha: A^n \rightarrow A$ . A variety  $\mathfrak{B}$  of algebras with domain  $\Omega$  is the class of all algebras  $A$  having  $\Omega$  as set of operations and satisfying some set of laws. Thus, the variety of semigroups is the class of algebras  $A$  where  $\Omega_n = \emptyset$  for  $n \neq 2$ ,  $\Omega_2 = \{\mu\}$ , and the associative law holds:

$$xy\mu z\mu = xyz\mu\mu$$

for all  $x, y, z \in A$ . Groups can be viewed in a number of different ways (see [4]). The way that we shall be concerned with here has  $\Omega_n = \emptyset$  for  $n > 2$ ,  $\Omega_2 = \{\mu\}$ ,  $\Omega_1 = \{\iota\}$ ,  $\Omega_0 = \{\nu\}$ . Here  $\mu$  is multiplication,  $\iota$  is inversion, and  $\nu$  is the nullary operator that acts on the empty set to produce the group identity:  $\{\} \nu = 1$ . The group laws read

$$(1.1) \quad xy\mu z\mu = xyz\mu\mu, \quad 1y\mu = y1\mu = y, \quad xxi\mu = xxi\mu = 1$$

with suitable quantifiers; as indicated, 1 stands for  $\{\} \nu$ .

An algebra  $A$  in a variety  $\mathfrak{B}$  is said to be  $\mathfrak{B}$ -free on a set  $X$  if  $A$  is generated by  $X$  *qua*  $\Omega$ -algebra and every map  $\varphi: X \rightarrow B$ ,  $B \in \mathfrak{B}$ ,

(\*) Indirizzo dell'A.: Department of Pure Mathematics, University College, Cardiff CF1 1XL, Wales, Gran Bretagna.

extends to an  $\Omega$ -homomorphism. We shall need the following fundamental result of Birkhoff [1]:

**THEOREM.** *For every set  $X$  and every variety  $\mathfrak{B}$  of algebras, there exists a  $\mathfrak{B}$ -free algebra on  $X$ .*

It is this simple but profound result that led to this note.

## 2. A special variety.

We consider the variety  $\mathfrak{D}$  of  $\Delta$ -algebras, where

$$\begin{aligned}\Delta &= \bigcup_{n=0}^{\infty} \Delta_n, \\ \Delta_n &= \emptyset \quad \text{if } n > 2, \\ \Delta_0 &= \{\nu\}, \\ \Delta_1 &= \{\iota, \varrho_2, \varrho_3, \dots, \varrho_n, \dots\}, \\ \Delta_2 &= \{\mu\},\end{aligned}$$

and the following laws are satisfied:

1) laws (1.1) for  $\nu, \iota, \mu$ , so that the algebras in  $\mathfrak{D}$  are groups with respect to these operations;

2) for each  $n$ , a law stating that  $x\varrho_n$  is an  $n$ -th root for  $x$  relative to  $\mu$ :

$$x\varrho_n x\varrho_n \dots x\varrho_n \mu \mu \dots \mu = x$$

for all  $x$ . We shall use juxtaposition for  $\mu, x^{-1}$  for  $x$  and  $1$  for  $\{\}$   $\nu$  in all cases, for simplicity. Thus law 2) becomes

$$(x\varrho_n)^n = x.$$

It is clear that  $\mathfrak{D}$  is simply the class of all divisible groups, in disguise. Every divisible group  $G$  becomes an algebra in  $\mathfrak{D}$ , in many different ways, of course. For each positive integer  $n$  and each  $x \in G$ , one simply has to specify which of the  $n$ -th roots of  $x$  we choose to

be  $x_{\rho_n}$ . In [3], J. D. Ledlie considered the class of divisible groups with *unique* roots as a variety of algebras, and he proved some representation theorems in the metabelian case.

By Birkhoff's theorem, free algebras exist in our variety  $\mathfrak{D}$ . Their structure is obscure, and even the free algebra  $F$  of rank 0 is more complicated and «non-commutative» than one might expect. We shall give some small insight into the group-theoretical structure of  $F$  to illustrate this point.

Much depends on the following lemma, which can be found in Kargapolov, Merzlyakov and Remeslennikov [2]:

**LEMMA 2.1.** *Every group  $G$  is embeddable in a divisible group  $D$  such that  $\text{var}(D') = \text{var}(G')$ . In particular, every soluble group  $G$  can be embedded in a divisible soluble group of the same solubility length as  $G$ .*

### 3. Properties of $F$ .

3.1. *The group  $F$  is insoluble, indeed  $\delta_n(F) > \delta_{n+1}(F)$  for all  $n$ .*

Here  $\delta_n(F)$  is the  $n$ -th term of the derived series of  $F$ : we remind the reader that we are referring to the canonical group structure of  $F$  relative to  $\{\nu, \iota, \mu\}$ .

For arbitrary  $n$ , let  $A$  be a soluble group of length  $n$  generated by an element  $a$  of order 2 and an element  $b$  of order 3. Use Lemma 2.1 to embed  $A$  in a divisible group  $D$  of solubility length  $n$ . Make  $D$  into a  $\Delta$ -algebra by assigning roots to the elements of  $D$ , ensuring that  $1_{\rho_2} = a$ ,  $1_{\rho_3} = b$ . Then the  $\Delta$ -subalgebra  $S$  generated by the empty set contains the group identity  $1 = \{\} \nu$  and therefore it contains  $1_{\rho_2} = a$ ,  $1_{\rho_3} = b$ , and thus it contains the original group  $A$ . But there is a  $\Delta$ -homomorphism, in particular a group homomorphism, of  $F$  onto  $S$ , and this is enough to establish 3.1.

3.2. *The subgroup  $V$  of  $F$  generated by the elements  $\{1_{\rho_i}: i = 2, 3, \dots\}$  is the free product of the cyclic groups  $\langle 1_{\rho_i} \rangle$ . Thus  $F$  has absolutely free subgroups.*

Let  $Y = Z_2 * Z_3 * \dots$  be the free product of cyclic groups  $Z_i = \langle a_i: a_i^i = 1 \rangle$ , and  $\bar{Y}$  a divisible group containing  $Y$ . Make  $\bar{Y}$  into a  $\mathfrak{D}$ -algebra, ensuring that  $1_{\rho_i} = a_i$  for each  $i$ . The subalgebra  $U$  of  $\bar{Y}$  generated by the empty set contains  $Y$ , and  $V$  maps to  $Y$  under

the  $\Delta$ -homomorphism from  $F$  to  $\bar{Y}$ . By the definition of free product,  $V$  and  $Y$  are isomorphic as groups.

3.3. Let  $F_c$  denote the free algebra of rank 0 of the variety of  $\mu$ -commutative algebras in  $\mathfrak{D}$ . As group,  $F_c$  is periodic, and for each prime  $p$ , the Sylow  $p$ -subgroup of  $F_c$  is a divisible  $p$ -group of rank  $\aleph_0$ . The commutator factor-group  $F/F'$  is isomorphic to  $F_c$  as group.

First of all, the group  $F_c$  is periodic. This is because the elements of finite order in  $F_c$  form a subgroup, since  $F_c$  is commutative. They form a  $\Delta$ -subalgebra since roots of elements of finite order are of finite order. Like  $F$ ,  $F_c$  has no proper subalgebras, so that  $F_c$  is equal to its periodic part. Finally, all we need do is to turn the periodic group  $A$  whose Sylow  $p$ -subgroup is a divisible group of rank  $\aleph_0$  into an algebra of rank 0 in  $\mathfrak{D}$ .

Present  $A$  on generators  $a_{ijp}$ , where  $i, j \in \mathbf{N}$  and  $p$  runs over the set  $\Pi$  of all primes, with defining relations:

$$\begin{aligned} [a_{ijp}, a_{klq}] &= 1 && \text{for } i, j, k, l \in \mathbf{N}, p, q \in \Pi; \\ a_{ijp}^p &= a_{i-1, j, p} && \text{for } i, j \in \mathbf{N}, i \geq 2, p \in \Pi; \\ a_{ijp}^p &= 1 && \text{for } j \in \mathbf{N}, p \in \Pi. \end{aligned}$$

Make  $A$  into an algebra in  $\mathfrak{D}$ , ensuring that

$$\begin{aligned} 1\varrho_p^n &= a_{nnp} && \text{for } n \in \mathbf{N}, p \in \Pi; \\ a_{ijp}\varrho_p &= a_{i+1, j, p} && \text{for } i, j \in \mathbf{N}, p \in \Pi. \end{aligned}$$

The  $\Delta$ -subalgebra generated by the empty set contains the group identity and thus all the elements  $a_{knp}$ . It therefore contains all  $a_{knp}$  with  $k \geq n$ . Since these elements generate  $A$  *qua* group, this means that  $A$  is generated by the empty set *qua*  $\Delta$ -algebra.

Since  $F_c$  maps to  $A$ ,  $F_c$  and  $A$  are isomorphic groups since  $F_c$  is countable. To prove that  $F/F'$  and  $F_c$  are isomorphic groups, it suffices to prove that  $F/F'$  is periodic. To see this, let  $a_i$  stand for the element  $1\varrho_i$  of  $F$ . Then elements of the form  $a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \dots a_{i_n}^{\varepsilon_n} = : x$  are of finite order mod  $F'$ , where the  $\varepsilon_i$  are integers. Thus, so are elements of the form  $x\varrho_{j_1} \varrho_{j_2} \dots \varrho_{j_k}$ . But  $F'$  is built up of expressions of this form using roots, inverses and products, and thus periodicity mod  $F'$  is ensured.

We end by posing the following problem, where the answer is probably yes.

**PROBLEM.** Is the group  $F$  residually soluble?

Clearly,  $F$  is not residually nilpotent since  $F/F'$  is periodic and divisible and thus every nilpotent image of  $F$  is abelian. A possible approach to solving the problem (and others like it) would be to establish a normal form for the elements of  $F$ , based on the elements  $1\varrho_i = a_i$ . But this is a task of very great notational and combinatorial complexity, which we do not tackle here.

I thank M. F. Newman for some useful conversations on this topic.

#### REFERENCES

- [1] G. D. BIRKHOFF, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc., **31** (1935), pp. 433-454.
- [2] M. I. KARGAPOLOV - YU. I. MERZLYAKOV - V. N. REMESLENNIKOV, *Completion of groups*, Dokl. Akad. Nauk SSSR, **134** (1960), pp. 518-520.
- [3] J. D. LEDLIE, *Representations of free metabelian  $D_\pi$ -groups*, Trans. Amer. Math. Soc., **153** (1971), pp. 307-446.
- [4] B. H. NEUMANN - E. C. WIEGOLD, *A semigroup representation of varieties of algebras*, Coll. Math., **14** (1966), pp. 111-114.

Manoscritto pervenuto in redazione il 4 marzo 1986.