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A Note on the Converse of the Clifford's Theorem and Some Consequences.

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Throughout this paper «group» means «finite group». All characters considered are **C**-characters. We use the standard notation from the books of B. Huppert [1] and I. M. Isaacs [2].

It is well known that the class of p -decomposable groups, where p is a prime, is a saturated Fitting formation. The purpose of this paper is to prove the following

THEOREM A. *Let G be a group. Let P be a Sylow p -subgroup of G . Then $G = P \times K$ if and only if $\forall \theta \in \text{Irr}_C(G) \theta_P = n\chi$, where $\chi \in \text{Irr}_C(P)$ and n is some integer.*

Before proving Theorem A, we show that the converse of the Clifford's Theorem is true. Exactly, we give the following result

THEOREM B. *Let G be a group and $N \leq G$ such that $\forall \theta \in \text{Irr}_C(G) \theta_N = n(\chi^{g_1} + \dots + \chi^{g_k})$, where $g_i \in N_G(N)$, $\chi \in \text{Irr}_C(N)$ and n is some integer. Then $N \trianglelefteq G$.*

PROOF OF THEOREM B. Let 1_N be the principal character of N . If $x \in G$, then

$$(1_N)^G(x) = 1/|N| \sum_{g \in G} (1_N)^g(g^{-1}xg) = 1/|N| \sum_{g^{-1} \times x \in N} 1_N(g^{-1}xg) = |T(x)|/|N|$$

where $T(x) = \{g \in G : x \in N^{g^{-1}}\}$.

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If χ is an irreducible character of G , then

$$\chi_N = n_x(\eta_x^{\sigma_1} + \dots + \eta_x^{\sigma_{k_x}}), \quad g_i \in N_G(N), \quad \eta_x \in \text{Irr}_C(N).$$

By Frobenius reciprocity:

$$((1_N)^G, \chi)_G = (1_N, \chi_N)_N = (1_N, n_x(\eta_x^{\sigma_1} + \dots + \eta_x^{\sigma_{k_x}}))_N.$$

Suppose that there is an integer s satisfying $\eta_x^{\sigma_s} = 1_N$.

Then $\eta_x^{\sigma_i} = 1_N$, $\forall i$, and so, we get

$$((1_N)^G, \chi)_G = n_x k_x \quad \text{and} \quad \chi_N = n_x k_x 1_N.$$

Otherwise, $((1_N)^G, \chi)_G = 0$. Then, we have,

$$(1_N)^G = \sum_{\chi_N = n_x k_x 1_N} n_x k_x \chi.$$

In particular:

$$|G|/|N| = (1_N)^G(1) = \sum_{\chi_N = n_x k_x 1_N} n_x k_x \chi(1) = \sum_{\chi_N = n_x k_x 1_N} (n_x k_x)^2.$$

Let $\tau = (1_N)^G$. Then

$$\tau_N = \sum_{\chi_N = n_x k_x 1_N} n_x k_x \chi_N = \left(\sum_{\chi_N = n_x k_x 1_N} (n_x k_x)^2 \right) 1_N = (|G|/|N|) 1_N.$$

Inducing on both sides:

$$(\tau_N)^G = (|G|/|N|) (1_N)^G.$$

If $x \in G$,

$$\begin{aligned} (\tau_N)^G(x) &= 1/|N| \sum_{g \in G} (\tau_N)^G(g^{-1}xg) = 1/|N| \sum_{g^{-1}xg \in N} \tau(g^{-1}xg) = \\ &= (\tau(x)/|N|) |T(x)| = |T(x)|^2/|N|^2. \end{aligned}$$

On the other hand, $(|G|/|N|) (1_N)^G(x) = (|G|/|N|^2) |T(x)|$. Then, we have $(|G|/|N|^2) |T(x)| = |T(x)|^2/|N|^2$.

If $x \in N$, since $|T(x)| \geq |N| > 0$, the last equality gives us $|G| = |T(x)|$. Hence, x belongs to all conjugates of N . Consequently, $N \trianglelefteq G$.

PROOF OF THEOREM A: By Theorem B, $P \trianglelefteq G$. It is enough to see that G is p -nilpotent. We use induction on $|G|$ and we can assume that $p \nmid |G|$. If N is a normal subgroup of G , it is not difficult to see that G/N satisfies the induction hypothesis.

Assume that there exists $\theta \in \text{Irr}_c(G)$ such that $\ker \theta = 1$.

By hypothesis, $\theta_p = n\chi$, $\chi \in \text{Irr}_c(P)$.

Since $P > 1$, it follows that $Z(P) > 1$. Let $x \in Z(P)$. χ is irreducible and faithful, then $|\chi(x)| = \chi(1)$. Thus,

$$|\theta(x)| = n|\chi(x)| = n\chi(1) = \theta(1),$$

which implies that $x \in Z(G)$.

Then, $Z(G) > 1$ and $G/Z(G)$ is p -nilpotent.

From this it follows that there exists $M \trianglelefteq G$ p -nilpotent such that $G = MZ(G)$. But then, $M \trianglelefteq G$ and G is p -nilpotent.

So, we can suppose that $\ker \theta > 1$, $\forall \theta \in \text{Irr}_c(G)$.

Now, by the induction hypothesis $G/\ker \theta$ is p -nilpotent.

Hence, $G \simeq G / \bigcap_{\theta \in \text{Irr}_c(G)} \ker \theta$ is p -nilpotent.

Clearly, the converse holds.

REFERENCES

- [1] B. HUPPERT, *Endliche Gruppen I*, Springer-Verlag, 1967.
- [2] I. MARTIN ISAACS, *Character Theory of Finite Groups*, Academic Press, 1976.

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