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Groups with Subnormal Subgroups of Bounded Defect.

CARLO CASOLO (*)

1. The principal object of this note are soluble p -groups with a bound on the defects of their subnormal subgroups. We denote by $\mathfrak{B}_n$ the class of groups in which every subnormal subgroup has defect at most n , and put $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$.

It is well known that soluble groups in $\mathfrak{B}_1$ (more often called T -groups) are metabelian [3], [9]; while there is no bound on the derived length of soluble groups in $\mathfrak{B}_2$, as shown by Robinson in [11]. However in [2] we proved that periodic soluble groups in $\mathfrak{B}_2$ have bounded derived length and that a soluble p -group in $\mathfrak{B}_2$ has derived length at most 4. Here we prove:

THEOREM 1. *For every prime p and every positive integer d , there exists a soluble p -group in $\mathfrak{B}_4$ with derived length exactly d .*

We leave open the question whether soluble p -groups in $\mathfrak{B}_3$ have bounded derived length.

Soluble p -groups in the class $\mathfrak{B}$ are the object of a paper by McDougall [8]. He denotes with $\mathbf{Q}$ the class of periodic abelian divisible groups, and with $P\mathbf{Q}$ and $\hat{P}\mathbf{Q}$ the classes of groups admitting respectively a finite or an ascending series whose factors are $\mathbf{Q}$ -groups. As a consequence of Theorem 1, we give an answer to a question posed by McDougall in his paper.

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THEOREM 2. *For every prime p , there exists a non soluble p -group which is a $\dot{P}Q$ -group (hence hyperabelian) and has a bound on the defects of its subnormal subgroups.*

We will show that the groups constructed in order to prove Theorem 2 actually are in the class $\mathfrak{B}_5$.

Theorem 1 yields also a similar result for SD -groups, that is groups admitting a descending series with abelian factors.

THEOREM 3. *For every prime p , there exists a locally finite p -group in $\mathfrak{B}_4$, which is a SD -group and whose Baer radical is trivial.*

We wish to observe here that Leinen has recently shown in his doctoral dissertation [6], that locally finite p -groups in $\mathfrak{B}_1$ need not to be soluble, thus answering a question which had been open for a certain time (see Robinson [13], Vol. I, pag. 174).

Finite soluble groups in $\mathfrak{B}_2$ have derived length at most 5 ([1], [7]). By contrast Hawkes [5] proves that every finite soluble group is isomorphic to a subgroup of a finite soluble group in $\mathfrak{B}_3$. The groups in $\mathfrak{B}_3$ obtained by Hawkes's procedure have, at least for one prime p , Sylow p -subgroups of « high » derived length; therefore *T. Hawkes* asked whether there are A -groups in $\mathfrak{B}_n$ of arbitrary derived length, where an A -group is a finite group all of whose Sylow subgroups are abelian. Using the same technique used to prove Theorem 1, we give an answer to this question for $n = 4$.

THEOREM 4. *For every positive integer d , there exists a soluble A -group in $\mathfrak{B}_4$, with derived length exactly d .*

Throughout the paper, $H \text{ wr } K$ will always denote the restricted standard wreath product of the group H by the group K , and $\gamma_n(G)$ will denote the n -th term of the lower central series of the group G .

2. The proofs of Theorems 1 and 4 are similar and based upon a construction for which is basic the following extension of a lemma of P. Hall's.

LEMMA 1. *Let H and K be groups, and B denote the base group of $H \text{ wr } K$. If $S \leq H \text{ wr } K$ is subnormal of defect n in BS and the exponent ⁽¹⁾ of BS/B is greater than n , then $S \geq \gamma_{n+1}(B)$.*

⁽¹⁾ By « exponent » of group G , we mean the supremum among the orders of the elements of G .

PROOF. We consider $H \text{ wr } K$ as the semidirect product $[B]K$, and proceed by induction on n .

Let $S \triangleleft BS$ (P. Hall [4], Lemma 4); there exists $1 \neq x \in K$ such that $xB \in SB$, and let $b \in B$ such that $bx \in S$. Let H_z be the coordinate subgroup of B related to $z \in K$, and let $h, g \in H_z$. We have $[h, bx] \in S$ and therefore:

$$S \ni [h, bx, g] = [h^{-1}h^{bx}, g] = [h^{-1}, g]^{h^{bx}}[h^{bx}, g] = [h^{-1}, g]$$

since $h^{bx} \in H_z^{bx} = H_{zx}$ and $[H_{zx}, H_z] = 1$ as $x \neq 1$. Hence $S \geq H'_z$. The same is true for any $z \in K$, therefore $S \geq B' = \gamma_2(B)$.

Assume now $n > 1$, and let $y \in S$ such that $(yB)^i = y^i B \neq B$ for $i = 1, \dots, n$, and write $y = bx$ with $b \in B$, $x \in K$ and $x^i \neq 1$ for $i = 1, \dots, n$. Let $S \triangleleft S_1 \triangleleft \triangleleft BS$, with S_1 of defect $n-1$ in $BS_1 = BS$. Since $\exp(S_1 B/B) > n$, we have, by inductive hypothesis, $S_1 \geq \gamma_n(B)$. Let H_z be the coordinate subgroup of B corresponding to $z \in K$, and let $f \in H_z$ and $g \in \gamma_n(H_z) \leq S_1$. Since $x^i \neq 1$ for $i = 1, \dots, n$, the groups H_{zx^i} and H_{zx^k} permute if $1 \leq j, k \leq n$ and $j \neq k$. Therefore we have:

$$[f, {}_n x] = f^{(x^{-1})^n} = f^{(-1)^n} t, \quad \text{where } t \in H_{zx} \times \dots \times H_{zx^n}$$

and, by induction on n , it is easy to check that:

$$[f, {}_n y] = [f, {}_n bx] = f^{(-1)^n} u, \quad \text{where } u \in H_{zx} \dots H_{zx^n}.$$

Since $g \in S_1$, $S \triangleleft S_1$ and S has defect n in BS , we now have:

$$S \ni [g, [f, {}_n y]] = [g, f^{(-1)^n} u] = [g, f^{(-1)^n}]^u [g, u] = [g, f^{(-1)^n}]$$

as H_z commutes with $H_{zx} \dots H_{zx^n}$.

Therefore $S \geq [H_z, \gamma_n(H_z)] = \gamma_{n+1}(H_z)$. The same is true for any $z \in K$, hence $S \geq \gamma_{n+1}(B)$. $\square$

REMARK. The hypothesis $\exp(SB/B) > n$ in Lemma 1 cannot be relaxed to $|SB/B| > n$. Let, in fact, H be the symmetric group on three objects and K be an elementary abelian group of order 4. It is easy to check that $W = H \text{ wr } K$ has a subnormal subgroup S of defect 3, such that $SB = W$ but S does not contain B' , where B is again the base group of W .

3. In his paper on soluble p -groups in the class $\mathfrak{B}$ [8], McDougall proves that every soluble p -group in $\mathfrak{B}$ is the extension of a PQ -group (that is a group admitting a finite series whose factors are periodic abelian divisible groups) by a nilpotent group. Since, by a Theorem of Roseblade's [14], nilpotent groups in $\mathfrak{B}_n$ have class bounded by a function of n , in order to prove Theorem 1 we need to construct PQ -groups in $\mathfrak{B}_4$ with arbitrary derived length.

For the properties of PQ -groups that we will need, we refer to the quoted paper by McDougall.

LEMMA 2. *Let H be a p -group in PQ , K a periodic group, B the base group of H wr K , and let S be a subnormal subgroup of BS . Then either $S \leq B$ or $S \geq B'$.*

PROOF. Assume $S \not\leq B$. We prove that $S \geq B'$ by induction on the defect n of S in BS .

In $n = 1$, then the result is a particular case of Lemma 1. Assume $n > 1$ and let $S \triangleleft S_1 \triangleleft \triangleleft SB$ with S_1 of defect $n - 1$ in SB . Clearly $S_1 \not\leq B$, hence, by inductive hypothesis, $S_1 \geq B'$. It is clear that B is a PQ -group and, therefore, B/B' is a normal periodic abelian divisible subgroup of SB/B' ; since K and, in particular, SB/B' is a periodic group, we have that B/B' normalizes $S_1 B'/B' = S_1/B'$ (see Robinson [10], Lemma 2.2). Hence S_1 is normal in SB and so $n = 2$. If SB/B has exponent greater than 2, then by Lemma 1, $S \geq \gamma_3(B)$ and, since B is a PQ -group, we have $\gamma_3(B) = B'$ (McDougall [8], Th. 4.8), yielding $S \geq B'$. Assume now $\exp(SB/B) = 2$. Let $y \in S \setminus B$, then $y = bx$ with $b \in B$, $x \in K$ and $x^2 = 1$. Let H_z ($z \in K$) be a coordinate subgroup in B , and $t \in H_z$; since S has defect 2 in SB , we have $[t, {}_2bx] \in S$ and, by expanding the commutator using the facts that $[H_z, H_{zx}] = 1$ and $x^2 = 1$, we get:

$$S \ni [t^{-1} b^{bx}, bx] = t(t^{-2})^{bx} t^{bx}.$$

Let now $f, g, h \in H_z$ and $w = [h, g]$, then $w \in S_1$. Since H/H' is divisible, there exist $t \in H_{zx}$, $u \in H'_{zx}$ such that $ut^2 = h^{b^{-1}}$. Since $[t, {}_2bx] \in S$, we have:

$$S \ni [w, [t, {}_2bx]] = [w, t t^{b^2} (t^{-2})^{bx}] = [w, (t^{-2})^{bx}]$$

as $H_{zx} = H_z$ commutes with H_{zx} . Hence:

$$(+)\quad S \ni [w, h^{-1} u^{bx}] = [w, u^{bx}] [w, h^{-1}]^{u^{bx}}.$$

Since $u \in B' \leq S_1 \leq \mathcal{N}(S)$, we have also:

$$S \ni [w, [u, bx]] = [w, u^{-1}u^{bx}] = [w, u^{bx}]$$

and, by comparing with $(+)$, we get: $[w, h^{-1}]^{u^{bx}} \in S$, yielding

$$[w, h^{-1}] = [f, g, h^{-1}] \in S.$$

Hence $S \geq \gamma_3(H_z) = H'_z$, and so $S \geq B'$. $\square$

Let G be a group; we denote by $D(G) = D_1(G)$ the maximal normal divisible periodic abelian subgroup of G (see e.g. Robinson [10], Lemma 2.2) and we put: $D_n(G)/D_{n-1}(G) = D(G/D_{n-1}(G))$ for any $n > 1$. If $G \in PQ$ then there is a minimal integer n such that $D_n(G) = G$. Such n is called the Q -length of G , and it is equal to the derived length of G , according to a result of McDougall ([8], Th. 4.5).

PROOF OF THEOREM 1. By induction on n , we construct a p -group in $PQ \cap \mathfrak{B}_4$ with derived length $2n$ (to get a group of the required type and odd derived length we can take the derived subgroup of one of the groups we will construct). More precisely, we obtain p -groups satisfying the following conditions:

- (i) G is a PQ -group;
- (ii) $C_o(D(G)) = D(G)$;
- (iii) if S is subnormal in G , then either $S \leq D_2(G)$ or $S \geq D_1(G)$;
- (iv) G is in $\mathfrak{B}_4$.

Let Q be a divisible abelian p -group and $C \cong C_{p^\infty}$. Let V be the base group of Q wr C , considering the latter as the semidirect product $[V]C$, let $H = [V, C]C$. Then H satisfies (i)-(iv) and it has derived length 2. Moreover $H' = [V, C] = D_1(H) = C_H(D_1(H))$ (this follows easily from Lemma 2.2 in [10]).

Assume now that we have already constructed a group G_0 which satisfies (i)-(iv) and has derived length $2(n-1)$. Let B be the base group of $W = H$ wr G_0 , where again we consider W as the semidirect product $[B]G_0$. Let $D_1 = D_1(G_0) \triangleleft G_0$. The action of G_0 on B induces an action of G_0 on $\bar{B} = B/B'$; with respect to this action, let $\bar{R} = R/B' = [\bar{B}, D_1]$. Since $D_1 \triangleleft G_0$, we have $R \triangleleft W$. We put

$G = RG_0$ and verify that G satisfies (i)-(iv) and has derived length (= Q -length) $2n$.

(i) Since H is a metabelian PQ -group, such is B ; therefore $\bar{B}$ is divisible and so $\bar{R}$ is divisible. Hence R is a PQ -group of derived length 2. Since the class PQ is closed under extensions, we get $G \in PQ$.

(ii) Let $K = D_1(G)$; then $K \cap R = D_1(R)$ and, by the structure of H , we have $B' = D_1(B)$, in particular $B' = D_1(R) = K \cap R$. It is $C_G(B') = G \cap C_W(B')$ and, since K is abelian, $K \leq C_G(B')$. Let $C = C_W(B')$, then $C \triangleleft W$ and so

$$[C, B] \leq C \cap B = B', \quad \text{since } C_H(H') = H'.$$

But $W/B' \cong (H/H') \text{ wr } G_0$ and, since C centralizes B/B' , we have $C \leq B$, that is $C = B'$. In particular $C_G(B') = B'$, yielding $K = B'$ and $C_G(K) = K$, thus proving (ii).

We observe now that $D_2(G) = R$. Clearly, in fact, $D_2(G) \geq R$. Assume $D_2(G) \neq R$. Then $(D_2(G) \cap G_0)B'/B'$ is infinite, and therefore, its centralizer in the base group B/B' of the restricted wreath product W/B' is trivial (see Robinson [13], Lemma 6.28), contradicting the fact that $D_2(G)$ centralizes $R/B' \leq B/B'$. Hence $D_2(G) = R$. This proves that the Q -length, and so the derived length, of G is equal to the Q -length of G_0 plus two, that is $2(n-1) + 2 = 2n$.

(iii) Let S be a subnormal subgroup of G and assume that $S \not\leq D_2(G) = R$. We use « $\sim$ » to denote subgroups modulo R . We have $\tilde{S} = SR/R \neq 1$ and $\tilde{S} \triangleleft \triangleleft \tilde{G}$, hence $\tilde{S}$ is normalized by $\widetilde{D_1(G_0)}$. If $\tilde{S} \cap \widetilde{D_1(G_0)} = 1$, then $\tilde{S}$ centralizes $\widetilde{D_1(G_0)}$ and, since G_0 satisfies (ii), we get $\tilde{S} = 1$, contradicting $S \not\leq R$. Hence $L/R = \tilde{S} \cap \widetilde{D_1(G_0)} \neq 1$; let $T = L \cap S$, then $T \triangleleft \triangleleft G$ and $TR = (L \cap S)R = L \cap SR = L$, and so $T \not\leq R$. It is therefore sufficient to prove that $T \geq D_1(G) = B'$. Looking at W , we have: $T \triangleleft \triangleleft L \triangleleft RD_1(G_0) \triangleleft BD_1(G_0) \triangleleft W$, yielding $T \triangleleft \triangleleft W$ and $T \not\leq B$, since $T \not\leq R$. Now, by applying Lemma 2, we get $T \geq B' = D_1(G)$, as we wanted.

(iv) Let S be a subnormal subgroup of G .

Now, R is in $\mathfrak{B}_2$, since it is a PQ -group of length 2 (McDougall [8], Th. 4.7); hence if $S \leq D_2(G) = R$, then S has defect at most 2 in R and, since R is normal in G , it has defect at most 3 in G . Other-

wise $S \geq D_1(G) = B'$. Since R/B' is a normal abelian divisible subgroup of G/B' , it normalizes every subnormal subgroup, in particular: $S/B' \trianglelefteq SR/B'$ and so $S \triangleleft SR \triangleleft G$. Since G satisfies (iii) we have two cases:

a) $SR \geq D_1(G_0)R$, in this case, working modulo B' , we get:

$$\bar{R} = [\overline{D_1(G_0)R}, \bar{R}] \leq [\overline{SR}, \bar{R}] = [\bar{S}, \bar{R}],$$

and, since $\bar{S}$ is subnormal in $\bar{G}$, $\bar{S} \geq \bar{R}$, yielding $S \geq R$ as $S \geq B'$. It follows from the property (iv) of G_0 , that S has defect at most 4 in G .

b) $SR \leq D_2(G_0)R'$, then we have $S \trianglelefteq SR \triangleleft D_2(G_0)R \trianglelefteq G$, with SR of defect at most 2 in $D_2(G_0)R$; hence S has defect at most 4 in G .

This completes the proof of Theorem 1. $\square$

REMARK. In the notation used in the previous proof, $G > B'G_0 \cong \cong H' \text{ wr } G_0$ and H' is an arbitrary divisible abelian p -group; it is therefore easy to verify that Theorem 1 can be restated in the following way: every finite p -group is isomorphic to a subgroup of a p -group in $\mathfrak{B}_4$, which is a PQ -group (in particular it is soluble).

4. Before giving the proofs of Theorems 2 and 3, we state in a lemma a well known fact concerning the classes $\mathfrak{B}_n$ (see e.g. Robinson [12], Ex. 1, pag. 127).

LEMMA 3. *For each positive integer n , $\mathfrak{B}_n$ is a local class.*

That is, $G \in \mathfrak{B}_n$ if every finite subset of G is contained in a subgroup of G , which is a $\mathfrak{B}_n$ -group.

Throughout this section G_n will denote the group constructed as in the proof of Theorem 1, of derived length $2n$, for any fixed prime p .

PROOF OF THEOREM 2. For every prime p , we may take the group $G = \text{Dir}_{n \in \mathbb{N}} G_n$, and apply the following result:

PROPOSITION. *If, for a fixed n , $H_\lambda \in \mathfrak{B}_n$ for every $\lambda \in \Lambda$, then $\text{Dir}_{\lambda \in \Lambda} H \in \mathfrak{B}_{\sigma(n)}$, where $\sigma(n)$ is a function depending only on n .*

PROOF. Let S be a subnormal subgroup of $H = \text{Dir}_{\lambda \in \Lambda} H_\lambda$, of defect say d , and let Π_λ be the projection on H_λ for each $\lambda \in \Lambda$. Then: $\gamma_{d+1}(S\Pi_\lambda) \leq [H_\lambda, {}_a S\Pi_\lambda] = [H_\lambda, {}_a S] \leq S$, hence $S\Pi_\lambda / (H_\lambda \cap S)$ is nilpotent,

for every $\lambda \in \mathcal{A}$. Now $S\Pi_\lambda$ is subnormal in H_λ , which is a $\mathfrak{B}_n$ -group; therefore all the subgroups of $S\Pi_\lambda/(H_\lambda \cap S)$ have defect at most n . By Roseblade's Theorem [14], there exists a function $\eta(n)$ such that the class of $S\Pi_\lambda/(H_\lambda \cap S)$ is at most $\eta(n)$, for every $\lambda \in \mathcal{A}$. Let

$$K = \langle S\Pi_\lambda : \lambda \in \mathcal{A} \rangle \quad \text{and} \quad T = \langle S \cap H_\lambda : \lambda \in \mathcal{A} \rangle$$

Then $T \leq S \leq K$ and K is subnormal in H of defect at most n . But K/T is nilpotent of class at most $\eta(n)$, hence S has defect at most $\eta(n)$ in K , and defect at most $\eta(n) + n$ in H . Therefore $H \in \mathfrak{B}_{\sigma(n)}$, where $\sigma(n) = \eta(n) + n$. $\square$

(For this property, see also Smith [15], Th. 4.16.)

We now show that $G = \text{Dir}_{n \in \mathbb{N}} G_n$ actually is in $\mathfrak{B}_5$. We first observe that the groups G_n , besides of (i)-(iv) in the proof of Theorem 1, satisfy the following property:

$$(+) \quad \text{if } S \triangleleft \triangleleft G_n \quad \text{and} \quad S \not\leq D_4(G_n), \quad \text{then } S \geq D_2(G_n);$$

in fact, (by repeating an argomentation used in the proof of Theorem 1) if $S \not\leq D_4(G_n)$ then $SD_2(G_n) \geq D_3(G_n)$ by (iii), since $G_n/D_2(G_n) \cong \cong G_{n-1}$. Hence

$$D_2(G_n) = [D_3(G_n), D_2(G_n)] \leq [SD_2(G_n), D_2(G_n)] \leq [S, D_2(G_n)]$$

since $S \geq D_1(G_n)$ by (iii). Then, S subnormal yields $S \geq D_2(G_n)$.

Since $\mathfrak{B}_5$ is a local property, in order to show that G is in $\mathfrak{B}_5$, we need only to show that, for a finite number of integers $n_1, \dots, n_r$, the group $\text{Dir}_{i=1, \dots, r} G_{n_i}$ is in $\mathfrak{B}_5$. We make induction on $\sum n_i$. The case $\sum n_i = 1$ is trivial, hence let $\sum n_i > 1$ and $H = \text{Dir}_{i=1, \dots, r} G_{n_i}$, and let T be a subnormal subgroup of H . If $T \leq D_4(H)$ then, by McDougall ([8], Th. 4.7), T has defect at most 4 in $D_4(H)$ and so it has defect at most 5 in H . Otherwise, for some index $i \in \{1, \dots, r\}$ which we may assume to be 1, the projection $T\Pi_1$ of T in G_{n_1} is not contained in $D_4(G_{n_1})$, hence by (+), $T\Pi_1 \geq D_2(G_{n_1})$. Now, for some integer d :

$$T \geq [G_{n_1}, {}_d T] = [G_{n_1}, {}_d T\Pi_1] \geq [G_{n_1}, {}_d D_2(G_{n_1})] = D_2(G_{n_1})$$

(observe that, since $T\Pi_1$ is not contained in $D_4(G_{n_1})$, it is $n_1 > 2$,

and so $[G_{n_1}, D_2(G_{n_1})] = D_2(G_{n_1})$. Hence $T \geq K = D_2(G_{n_1}) \trianglelefteq H$. Now $H/K \cong G_{n_1}/K \times \dots \times G_{n_r} \cong G_{n_1-1} \times G_{n_2} \times \dots \times G_{n_r}$, and by inductive hypothesis, we have that $T = TK$ has defect at most 5 in H , concluding the proof that H (and therefore G) is in $\mathfrak{B}_5$.

PROOF OF THEOREM 3. Let H be the group of PQ -length 2, as defined in the first part of the proof of Theorem 1. Then G_{n+1} can be viewed as a subgroup of $H \text{ wr } G_n$, containing a subgroup isomorphic to G_n . We have therefore the sequence of inclusions:

$$G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \dots$$

Let G be the direct limit of it. Then $G = \bigcup_{n \in \mathbb{N}} \bar{G}_n$, where $\bar{G}_n \cong G_n$ and $\bar{G}_n \leq \bar{G}_{n+1}$ for every $n \in \mathbb{N}$. Clearly G is a locally finite p -group and, since $\mathfrak{B}_4$ is a local class, G is in $\mathfrak{B}_4$. It is also easy to check that G is a SD -group.

We have only to show now that the Baer radical of G is trivial. Let $\langle x \rangle$ be a cyclic subnormal subgroup of G ; then $x \in \bar{G}_k$ for some $k \in \mathbb{N}$. Now $\bar{G}_{k+1} = R\bar{G}_k$ (where R is the same as in the proof of Theorem 1), with $R \cap \bar{G}_k = 1$, and the subnormal subgroups of $\bar{G}_{k+1}$ which are not contained in R contain $D_1(\bar{G}_{k+1})$. In particular $\langle x \rangle$ cannot contain $D_1(\bar{G}_{k+1})$, therefore $x \in R \cap \bar{G}_k = 1$, proving that the Baer radical of G is trivial. $\square$

5. Since the proof of Theorem 4 is similar to the proof of Theorem 1, we give only a sketch of it, leaving the details to the reader. We first need an analogous of Lemma 2 for finite groups.

LEMMA 4. *Let H and K be finite groups, with $\gamma_3(H) = H'$ and $(|H/H'|, |K|) = 1$, and let B be the base group of $H \text{ wr } K$. If $S \leq H \text{ wr } K$ is subnormal in SB , then either $S \leq B$ or $S \geq B'$.*

PROOF. Omitted. $\square$

PROOF OF THEOREM 4. For a given finite group G let $F_i(G)$ be the i -th term of the Fitting series of G .

By induction on n , one constructs a finite soluble group, with derived length $2n$, satisfying the following conditions:

- (i*) $F_i(G)/F_{i-1}(G)$ is an elementary abelian p_i -group, where the p_i 's are given primes with $p_i \neq p_{i-1}$ for $i = 1, 2, \dots, 2n$. In particular if the p_i 's are all distinct, G is an A -group;

(ii*) if S is a subnormal subgroup of G , then either $S \leq F_2(G)$
or $S \geq F_1(G)$;

(iii*) G is in $\mathfrak{B}_4$.

Let $p_1, \dots, p_{2n}$ be given primes with $p_i \neq p_{i-1}$ for $i = 2, \dots, 2n$. Suppose we have already constructed a group G_0 of derived length $2(n-1)$, satisfying (i*)-(iii*) with involved primes $p_3, \dots, p_{2n}$. Let H be a group such that $F_1(H) = H' = \gamma_3(H)$ is an elementary abelian p_1 -group and H/H' is an elementary abelian p_2 -group. Let B be the base group of $H \text{ wr } G_0$: we take $G = B'[B, F_1(G_0)] G_0$ where G_0 is viewed in a natural way as a subgroup of $H \text{ wr } G_0$. Then G satisfies (i*)-(iii*) and has derived length $2n$. $\square$

REMARK. With a similar construction it is possible to embed any finite soluble group of derived length n in a finite soluble group in $\mathfrak{B}_4$, of derived length $2n$.

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